# Eğri Uzunluğu, Yüzey Alanı, Genelleştirilmiş İntegraller

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1–6 alıştırmalarındaki eğrilerin uzunluklarını bulun.

**2.** $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$

$$\ell = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = -\sin t \quad , \quad \frac{dy}{dt} = 1+\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \sin^2 t + (1+\cos t)^2 = \overbrace{\sin^2 t + 1 + 2\cos t + \cos^2 t}^{1}$$

$$= 2 + 2\cos t$$

$$= 2(1+\cos t)$$

$$= 2 \cdot 2 \cdot \cos^2 \frac{t}{2} = 4\cos^2\frac{t}{2}$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$= 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1-\cos 2t}{2} \Rightarrow \sin^2 \frac{t}{2} = \frac{1-\cos t}{2}$$

$$= 2\cos^2 t - 1 \qquad \cos^2 t = \frac{1+\cos 2t}{2} \qquad \cos^2 \frac{t}{2} = \frac{1+\cos t}{2}$$

$$\ell = \int_0^\pi \sqrt{4\cos^2\frac{t}{2}}\, dt = 2\int_0^\pi \cos\frac{t}{2}\, dt = 4\left(\sin\frac{t}{2}\right)\Big|_0^\pi = 4\left(\overset{1}{\cancel{\sin\frac{\pi}{2}}} - \overset{0}{\cancel{\sin 0}}\right)$$

$$\underline{\underline{\ell = 4 \text{ br}}}$$

7–16 alıştırmalarındaki eğrilerin uzunluklarını bulun.

**7.** $y = (1/3)(x^2 + 2)^{3/2}$, $x = 0$'dan $x = 3$'e kadar

$$l = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y' = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot (x^2+2)^{\frac{1}{2}} \cdot \cancel{2}x$$

$$(y')^2 = x^2(x^2+2)$$

$$1 + (y')^2 = 1 + x^4 + 2x^2 = (x^2+1)^2$$

$$l = \int_0^3 \sqrt{(x^2+1)^2}\, dx = \int_0^3 (x^2+1)dx = \frac{x^3}{3} + x \Big|_0^3 = 12 \text{ br}$$

**10.** $x = (y^{3/2}/3) - y^{1/2}$, $y = 1$'den $y = 9$'a kadar

$$\ell = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{1}{3}\cdot\frac{3}{2}\cdot y^{1/2} - \frac{1}{2} y^{-1/2} = \frac{1}{2}\left(y^{1/2} - y^{-1/2}\right)$$

$$\left(\frac{dx}{dy}\right)^2 = \frac{1}{4}\left(y - 2 + \frac{1}{y}\right) = \frac{1}{4}\left(\frac{y^2-2y+1}{y}\right) = \frac{(y-1)^2}{4y}$$

$$1+\left(\frac{dx}{dy}\right)^2 = 1+\frac{(y-1)^2}{4y} = \frac{y^2-2y+1+4y}{4y} = \frac{y^2+2y+1}{4y} = \frac{(y+1)^2}{4y}$$

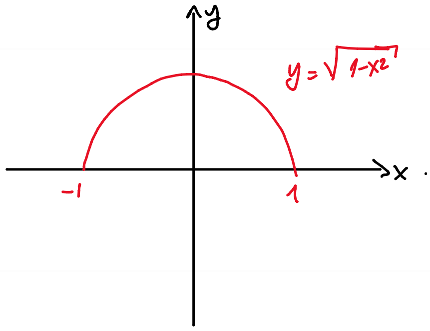
$$\ell = \int_1^9 \sqrt{\frac{(y+1)^2}{4y}}\, dy$$

$$= \int_1^9 \frac{y+1}{2\sqrt{y}}\, dy = \frac{1}{2}\int_1^9 \left(y^{1/2} + y^{-1/2}\right) dy$$

$$= \frac{1}{2}\left(\frac{2}{3} y^{\frac{3}{2}} + 2 y^{\frac{1}{2}}\right)\Bigg|_1^9$$

$$= \frac{1}{2}\left[(18+6) - \left(\frac{2}{3}+2\right)\right] = \frac{32}{3}$$

**31.** $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$




$$\ell = \int_{-1}^{1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \sqrt{1-x^2}$$

$$y' = \frac{1}{2} \cdot (1-x^2)^{-1/2} \cdot (-2x)$$

$$y' = \frac{-x}{\sqrt{1-x^2}}$$

$$(y')^2 = \frac{x^2}{1-x^2}$$

$$1 + (y')^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$\ell = \int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\, dx$$

$$= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

$$= \arcsin x \Big|_{-1}^{1}$$

$$= \arcsin 1 - \arcsin(-1)$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \underline{\underline{\pi}}$$

yüzey alanı, genelliştiril...

## Yüzey Alanlarını Bulmak

**9.** $y = x/2, 0 \leq x \leq 4$, doğru parçasının x-ekseni etrafında döndürülmesiyle elde edilen koninin yanal yüzey alanını bulun.

$$S = 2\pi \int_a^b y \cdot \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \frac{x}{2}, \quad y' = \frac{1}{2}, \quad (y')^2 = \frac{1}{4}, \quad 1+(y')^2 = 1+\frac{1}{4} = \frac{5}{4}$$

$$S = 2\pi \int_0^4 \left(\frac{x}{2}\right) \cdot \sqrt{\frac{5}{4}}\, dx$$

$$= \pi \int_0^4 \frac{\sqrt{5}}{2} \cdot x\, dx$$

$$= \frac{\sqrt{5}\pi}{2} \int_0^4 x dx = \frac{\sqrt{5}\pi}{2} \left( \frac{x^2}{2} \Big|_0^4 = 4\sqrt{5}\,\pi \quad br^2$$

13–22 alıştırmalarındaki eğrilerin belirtilen eksenler etrafında döndürülmeleriyle elde edilen yüzeylerin alanlarını bulun. Bir grafik programınız varsa, eğrileri çizebilirsiniz.

**15.** $y = \sqrt{2x - x^2}, \quad 0.5 \leq x \leq 1.5; \quad x\text{-ekseni etrafında}$

$$y' = \frac{1-x}{\sqrt{2x-x^2}}$$

$$(y')^2 = \frac{x^2-2x+1}{2x-x^2}$$

$$1+(y')^2 = 1+\frac{x^2-2x+1}{2x-x^2} = \frac{2x-x^2+x^2-2x+1}{2x-x^2} = \frac{1}{2x-x^2}$$

$$S = 2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\cancel{2x-x^2}} \cdot \sqrt{\frac{1}{\cancel{2x-x^2}}}\, dx$$

$$= 2\pi \int_{\frac{1}{2}}^{3/2} dx = 2\pi \left( x \Big|_{\frac{1}{2}}^{3/2} = \underline{\underline{2\pi}} \quad br^2\right.$$

**18.** $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \leq y \leq 3;$ $y$-ekseni etrafında

$$S = 2\pi \int_c^d x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{1}{3}\cdot\frac{3}{2}\cdot y^{1/2} - \frac{1}{2}y^{-1/2} = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$1+\left(\frac{dx}{dy}\right)^2 = \frac{(y+1)^2}{4y}$$

$$S = 2\pi \int_1^3 \left(\frac{1}{3}y^{3/2} - y^{1/2}\right)\cdot\sqrt{\frac{(y+1)^2}{4y}}\, dy$$

$$= \cancel{2}\pi \int_1^3 \left(\frac{1}{3}\cancel{y}^{3/2} - y^{\cancel{1/2}}\right)\cdot \frac{y+1}{\cancel{2}\sqrt{\cancel{y}}}\, dy$$

$$= \pi \int_1^3 \left(\frac{1}{3}y - 1\right)(y+1)\, dy$$

$$= \pi \int_1^3 \left(\frac{1}{3}y^2 + \frac{1}{3}y - y - 1\right) dy$$

$$= \pi \int_1^3 \left(\frac{1}{3}y^2 - \frac{2}{3}y - 1\right) dy$$

## Genelleştirilmiş İntegralleri Hesaplamak

**20.** $\displaystyle\int_0^{\infty} \frac{16 \tan^{-1} x}{1 + x^2}\, dx$

$$= \lim_{b \to \infty} \int_0^b \frac{16 \arctan x}{1+x^2}\, dx \quad ,$$

$\arctan x = u$

$\frac{dx}{1+x^2} = du$

$= \int 16u\, du$

$= 8u^2$

$= 8(\arctan x)^2$

$$= 8 \lim_{b \to \infty} \left( (\arctan x)^2 \Big|_0^b \right)$$

$$= 8 \lim_{b \to \infty} \left( (\underbrace{\arctan b}_{\frac{\pi}{2}})^2 - (\underbrace{\arctan 0}_{0})^2 \right)$$

$$= 8 \left( \frac{\pi^2}{4} - 0 \right) = 2\pi^2 \quad , \quad \underline{\underline{\text{yakınsak}}}$$

**28.** $\displaystyle\int_0^1 \frac{4r\,dr}{\sqrt{1 - r^4}}$ $\rightarrow$ ข้อนี้

**16.** $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

$4-s^2=t^2$

$-2s\,ds = 2t\,dt$

$s\,ds = -t\,dt$

$= \int \frac{-t\,dt}{t}$

$= -t$

$$\int \frac{s+1}{\sqrt{4-s^2}}\,ds = \int \frac{s}{\sqrt{4-s^2}}\,ds + \int \frac{1}{\sqrt{4-s^2}}\,ds$$

$$= -\sqrt{4-s^2} + \arcsin\frac{s}{2}$$

$$= \lim_{b\to 2^-} \int_0^b \frac{s+1}{\sqrt{4-s^2}}\,ds,$$

$$= \lim_{b\to 2^-} \left(-\sqrt{4-s^2} + \arcsin\left(\frac{s}{2}\right)\right)\Bigg|_0^b$$

$$= \lim_{b\to 2^-} \left[\left(-\sqrt{\cancel{4-b^2}}^{\,0} + \underbrace{\arcsin\left(\frac{b}{2}\right)}_{\frac{\pi}{2}}\right) - \left(-\sqrt{\cancel{4-0^2}}^{\,2} + \cancel{\arcsin\left(\frac{0}{2}\right)}^{\,0}\right)\right]$$

$$= \frac{\pi}{2} + 2,$$ yakınsak

**32.** $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

$$= \int_0^1 \frac{dx}{\sqrt{|x-1|}} + \int_1^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{|x-1|}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{|x-1|}}$$

$$= \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{-x+1}} + \lim_{a\to 1^+} \int_a^2 \frac{dx}{\sqrt{x-1}}$$

$$= \lim_{b\to 1^-} \left(-2\sqrt{-x+1}\,\Big|_0^b\right) + \lim_{a\to 1^+} \left(2\sqrt{x-1}\,\Big|_a^2\right)$$

$$= \lim_{b\to 1^-} \left(-2\sqrt{1-\cancel{b}^{\,0}} + 2\sqrt{1-0}\right) + \lim_{a\to 1^+} \left(2\sqrt{2-1} - 2\sqrt{\cancel{a}-1}^{\,0}\right)$$

$= 4$, yakınsak

$$\int \frac{dx}{\sqrt{x-1}}$$

$x-1=t$

$dx=dt$

$$\int \frac{dt}{\sqrt{t}} = 2\sqrt{t}$$

$$= 2\sqrt{x-1}$$

**139.** $\displaystyle\int_3^{\infty} \frac{2\,du}{u^2 - 2u}$

$$= \lim_{b\to+\infty} \int_3^b \frac{2du}{u^2-2u},$$

$$\int \frac{2du}{u^2-2u}, \quad \frac{2}{u(u-2)} = \frac{A}{u} + \frac{B}{u-2}$$
$$(u-2) \quad (u)$$

$$2 = Au - 2A + Bu$$
$$-2A = 2, \quad A+B=0$$
$$A=-1 \quad B=1$$

$$= -\int \frac{du}{u} + \int \frac{du}{u-2}$$
$$= -\ln u + \ln(u-2)$$

$$= \lim_{b\to+\infty} \left(-\ln u + \ln(u-2)\Big|_3^b\right)$$

$$= \lim_{b\to+\infty} \left(\left(-\ln b + \ln(b-2)\right) - \left(-\ln 3 + \cancel{\ln 1}^{\,0}\right)\right)$$

$$= \lim_{b\to+\infty} \left(\ln\left(\frac{b-2}{b}\right) + \ln 3\right)$$

$$\ln\Big(\underbrace{\lim_{b\to+\infty} \frac{b-2}{b}}_{1}\Big)$$
$$0$$

$$= \ln 3, \quad \underline{\underline{\text{yakınsak}}}$$

**137.** $\displaystyle\int_{-1}^{1} \frac{dy}{y^{2/3}}$