CALCULUS II, SPRING 2014, MIDTERM EXAM 1

Name: ....

Student No:...

PART A-1	PART A-2	PART A-3	PART B	TOTAL
/20	/15	/15	/50	/100

PART A

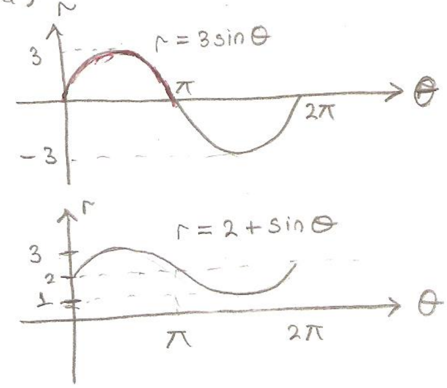
PROBLEM 1

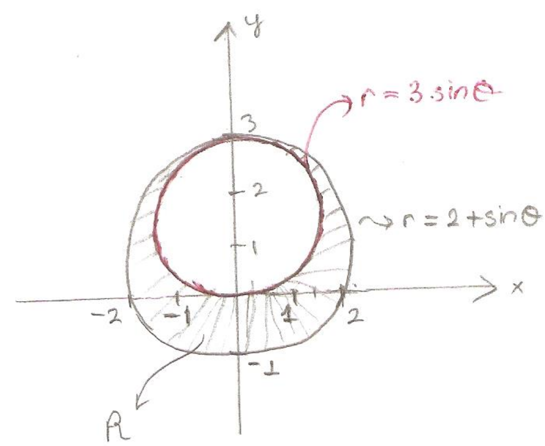
Consder the region R inside the polar curve $r = 2 + \sin\theta$ and outside the polar curve $r = 3\sin\theta$.

(a) Sketch the polar curves on the same polar coordinate frame, and show the region R.

(b) Calculate the area of the region R.

(a)





(b)
$$\underbrace{\text{Area of } R}_{A} = \underbrace{\frac{1}{2}\int_0^{2\pi} (2+\sin\theta)^2\,d\theta}_{\text{area inside } r=2+\sin\theta} - \underbrace{\frac{1}{2}\int_0^{\pi} (3\sin\theta)^2\,d\theta}_{\text{area inside } r=3\sin\theta}$$

$$A = \int_0^{2\pi} \frac{1}{2}(4 + 4\sin\theta + \sin^2\theta)\,d\theta - \int_0^{\pi} \frac{9}{2}\sin^2\theta\,d\theta$$

$$A = \int_0^{2\pi} \left(2 + 2\sin\theta + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\right) d\theta - \int_0^{\pi} \frac{9}{2}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$A = \left(2\theta - 2\cos\theta + \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta\right)\Big|_0^{2\pi} - \left(\frac{9}{4}\theta - \frac{9}{8}\sin 2\theta\right)\Big|_0^{\pi}$$

$$A = \frac{9}{2}\pi - \frac{9}{4}\pi = \frac{9}{4}\pi \quad \text{square units}$$

PROBLEM 2

Consider the parametric curve $x = \frac{2}{3}t^3 + 1, \quad y = 3 - t^2, \quad 0 \le t \le 1.$

(a) Calculate the total arc length of the curve.
(b) Find the area between the curve and the x-axis.

(a) $$\text{Arc Length} = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(2t^2)^2 + (-2t)^2}\, dt$$

$$= \int_0^1 \sqrt{4t^4 + 4t^2}\, dt = \int_0^1 2t\sqrt{t^2+1}\, dt = \int_1^2 \sqrt{u}\, du = \frac{2}{3} u^{3/2}\Big|_1^2$$

$\boxed{u = t^2+1}$

$$= \frac{2}{3}(2\sqrt{2} - 1) \text{ units}$$

(b) $\frac{dx}{dt} = 2t^2 > 0 \qquad y = 3 - t^2 > 0$ for $0 \le t \le 1$

$$\text{Area} = \int_0^1 y \cdot \frac{dx}{dt}\, dt = \int_0^1 (3-t^2)2t^2 dt = \int_0^1 (6t^2 - 2t^4)\, dt$$

$$= \left(\frac{6}{3}t^3 - \frac{2}{5}t^5\right)\Big|_0^1 = 2 - \frac{2}{5} = \frac{8}{5} \text{ square units}$$

PROBLEM 3

Consider the parametric curve $x = t^2, \quad y = t^3 - 3t, \quad -\infty < t < \infty$

(a) Compute the slope of the tangent line to the curve at the point $(4, 2)$.
(b) Find the points where the tangent line to the curve is horizontal.

(a) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t}$$

$4 = t^2 \Rightarrow t = \pm 2$
$2 = t^3 - 3t \Rightarrow \boxed{t = 2}$

$$\frac{dy}{dx}\Big|_{t=2} = \frac{9}{4}$$

(b) $\left.\begin{array}{l} \frac{dy}{dt} = 0 \\ \frac{dx}{dt} \neq 0 \end{array}\right\} \Rightarrow 3t^2 - 3 = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

$\underline{t = 1}$: $x = 1, \ y = -2$ — $P_1 : (1, -2)$

$\underline{t = -1}$: $x = 1, \ y = 2$ — $P_2 : (1, 2)$

points with horizontal tangents

PART B – MULTIPLE CHOICE QUESTIONS

1. The area of the closed region bounded by the polar graph of $r = \sqrt{3 + cos\theta}$ is given by the integral

   (A) $\int_0^{2\pi} \sqrt{3 + cos\theta}\, d\theta$ (B) $\int_0^{\pi} \sqrt{3 + cos\theta}\, d\theta$ (C) $2\int_0^{\pi/2} (3 + cos\theta) d\theta$

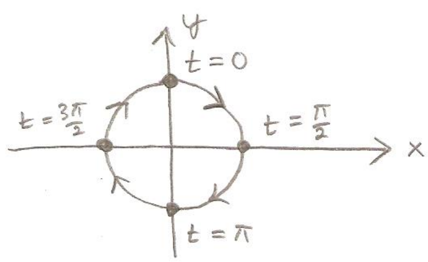
   (D) $\int_0^{\pi} (3 + cos\theta) d\theta$ (E) $2\int_0^{\pi/2} \sqrt{3 + cos\theta}\, d\theta$

$$A = \frac{1}{2}\int_0^{2\pi} \left(\sqrt{3+\cos\theta}\right)^2 d\theta = 2\cdot\frac{1}{2}\int_0^{\pi} \left(\sqrt{3+\cos\theta}\right)^2 d\theta$$
$$= \int_0^{\pi} (3+\cos\theta)\, d\theta$$

2. For which of the following parametrizations of the unit circle will the circle be traversed **clockwise**?

   (A) $x = \cos t, \quad y = \sin t, \qquad 0 \le t \le 2\pi$
   (B) $x = \sin t, \quad y = \cos t, \qquad 0 \le t \le 2\pi$
   (C) $x = -\cos t, \quad y = -\sin t, \qquad 0 \le t \le 2\pi$
   (D) $x = -\sin t, \quad y = \cos t, \qquad 0 \le t \le 2\pi$
   (E) $x = \sin t, \quad y = -\cos t, \qquad 0 \le t \le 2\pi$

$x = \sin t$
$y = \cos t$

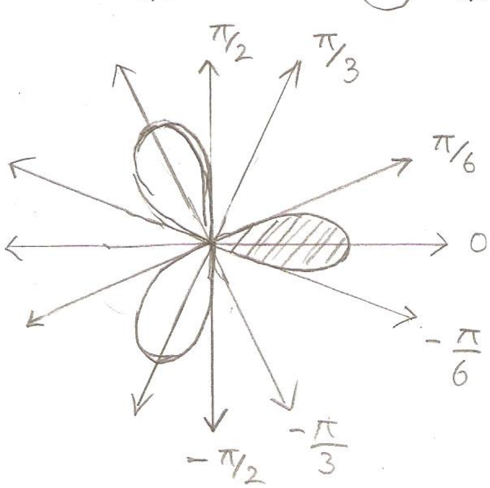


$t = 0 \Rightarrow (x, y) = (0, 1)$
$t = \frac{\pi}{2} \Rightarrow (x, y) = (1, 0)$
$t = \pi \Rightarrow (x, y) = (0, -1)$
$t = \frac{3\pi}{2} \Rightarrow (x, y) = (-1, 0)$

3. The area enclosed by **one leaf** of the 3-leaved rose $r = 4\cos(3\theta)$ is given by which integral?

   (A) $16\int_{-\pi/3}^{\pi/3} \cos(3\theta)\, d\theta$ (B) $8\int_{-\pi/6}^{\pi/6} \cos(3\theta)\, d\theta$ (C) $8\int_{-\pi/3}^{\pi/3} (\cos(3\theta))^2 d\theta$

   (D) $16\int_{-\pi/6}^{\pi/6} (\cos(3\theta))^2 d\theta$ (E) $8\int_{-\pi/6}^{\pi/6} (\cos(3\theta))^2 d\theta$

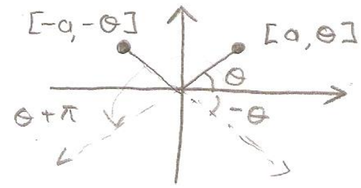


$$A = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\pi/6} (4\cos 3\theta)^2 d\theta$$

$$A = 8\int_{-\frac{\pi}{6}}^{\pi/6} (\cos(3\theta))^2 d\theta$$

4. If $a \neq 0$ and $\theta \neq 0$, all of the following must represent the same point in polar coordinates **except** which ordered pair?

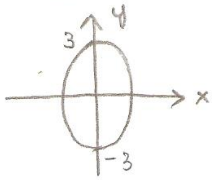
(A) $[a, \theta]$ (B) $[-a, -\theta]$ (C) $[-a, \theta - \pi]$ (D) $[-a, \theta + \pi]$ (E) $[a, \theta - 2\pi]$



$[a, \theta]$, $[-a, \theta - \pi]$, $[-a, \theta + \pi]$, $[a, \theta - 2\pi]$

5. Find an equation of an ellipse containing the point $(-\frac{1}{2}, \frac{3\sqrt{3}}{2})$ and with vertices $(0, -3)$ and $(0,3)$.

(A) $x^2 + \frac{y^2}{9} = 1$ (B) $x^2 + \frac{y^2}{3} = 1$ (C) $\frac{x^2}{4} + \frac{y^2}{9} = 1$ (D) $x^2 - \frac{y^2}{9} = 1$ (E) $\frac{x^2}{2} + \frac{y^2}{3\sqrt{3}} = 1$



$$\frac{x^2}{b^2} + \frac{y^2}{3^2} = 1$$

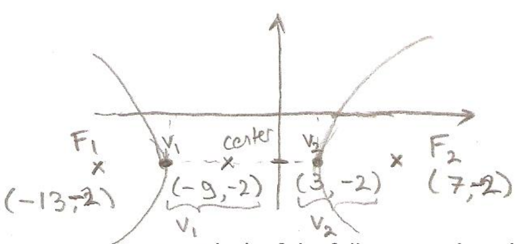
$$\frac{(-\frac{1}{2})^2}{b^2} + \frac{(\frac{3\sqrt{3}}{2})^2}{9} = 1$$

$$\frac{1}{4b^2} + \frac{9 \cdot 3}{4 \cdot 9} = 1 \Rightarrow b = 1$$

6. Write an equation of a hyperbola with vertices $(3, -2)$ and $(-9, -2)$, and foci $(7, -2)$ and $(-13, 2)$.

(A) $\frac{(x-3)^2}{36} - \frac{(y-2)^2}{64} = 1$ (B) $\frac{(x-3)^2}{12} - \frac{(y-2)^2}{16} = 1$

(C) $\frac{(x-3)^2}{12} - \frac{(y+2)^2}{16} = 1$ (D) $\frac{(x+3)^2}{36} - \frac{(y+2)^2}{64} = 1$



center : $(\frac{-9+3}{2}, -2)$: $(-3, -2)$

$a = 6$

$c = 10$

$b^2 = c^2 - a^2 = 100 - 36 = 64$

$b = 8$

7. Which of the following is described by the equation $4x^2 + 7y^2 + 32x - 56y + 148 = 0$?

(A) Ellipse with center $(4, -4)$, and foci at $(4 \pm \sqrt{3}, -4)$.

(B) Hyperbola with center $(-4, -4)$, and foci at $(4, -4 \pm \sqrt{3})$.

(C) Ellipse with center $(-4, 4)$, and foci at $(-4 \pm \sqrt{3}, 4)$.

(D) Hyperbola with center $(4, 4)$, and foci at $(4, 4 \pm \sqrt{3})$.

$$4(x^2 + 8x + 16) + 7(y^2 - 8y + 16) = 28$$

$$\frac{(x+4)^2}{7} + \frac{(y-4)^2}{4} = 1$$

$$c = \sqrt{a^2 - b^2} = \sqrt{3}$$

$a^2 = 7$

$b^2 = 4$

8. Which of the following sets of parametric equations constitute a parametrization of the **whole parabola** $y = x^2$?

✓ (i) $x = t, \quad y = t^2, \quad -\infty < t < \infty$

(ii) $x = t^2, \quad y = t^4, \quad -\infty < t < \infty$ (x is always positive)

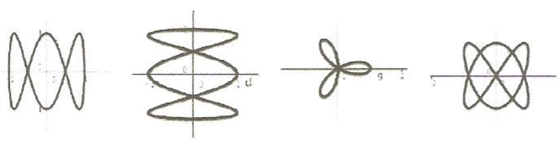
✓ (iii) $x = t^3, \quad y = t^6, \quad -\infty < t < \infty$

(iv) $x = \sin(t), \quad y = (\sin(t))^2, \quad -\infty < t < \infty$ $(-1 \leq x \leq 1, \ 0 \leq y \leq 1)$

(A) (i), (ii), and (ii) only
(B) (i) only
(C) (i) and (iii) only
(D) (ii) and (iii) only
(E) All four of them

9. The graphs of the parametric equations $x = f(t)$ and $g = f(t)$ are given below. Which is the corresponding parametric curve?

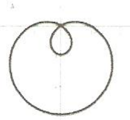
$x = f(t)$ $\qquad\qquad$ $y = g(t)$



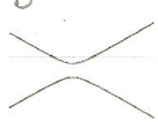
A $\qquad$ B $\qquad$ C $\qquad$ D

10. Identify the parametric curve represented by the polar equation $r^2 = 1/\theta$.

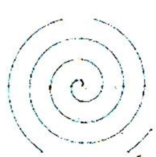
A



B



C



D