Name: KEY

Student No:

1	2	3	4	5	6	TOTAL
/15	/20	/15	/20	/15	/15	/100

1. (15 pts) Identify and sketch the set of points in the plane satisfying the equations in (a) and (b). Label the focus (or foci), specify the asymptotes if there are any. Find the eccentricity for the curves.

(a) $16x^2 + 4y^2 + 160x - 16y + 352 = 0$

(b) $25y^2 - 9x^2 + 50y + 54x - 281 = 0$

SOLUTION

(a) $16x^2 + 160x + 4y^2 - 16y = -352$

$16(x^2 + 10x + 25) + 4(y^2 - 4y + 4) = -352 + 400 + 16$

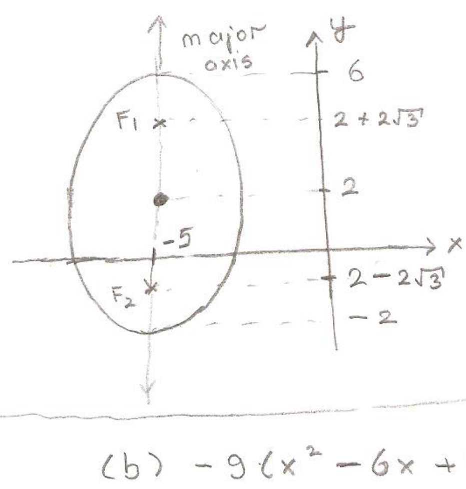
$16(x+5)^2 + 4(y-2)^2 = 64$

$\dfrac{(x+5)^2}{4} + \dfrac{(y-2)^2}{16} = 1$

Ellipse with center $(-5, 2)$

semi major axis $= 4$

semi minor axis $= 2$



$c = \sqrt{b^2 - a^2}$

$c = \sqrt{12} = 2\sqrt{3}$

foci at $(-5, 2+2\sqrt{3})$ and $(-5, 2-2\sqrt{3})$

$\varepsilon = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2}$

(b) $-9(x^2 - 6x + 9) + 25(y^2 + 2y + 1) = 281 + 25 - 81$

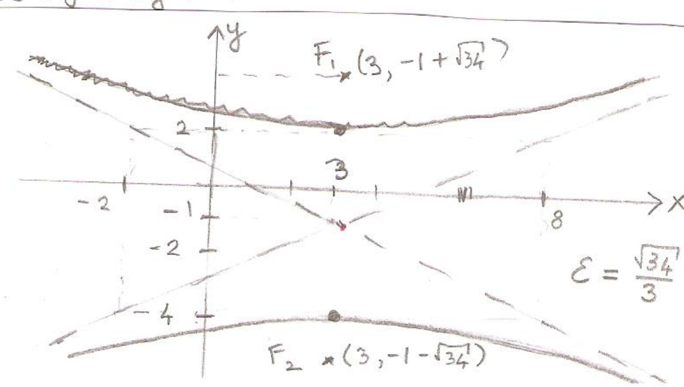
$\dfrac{(y+1)^2}{9} - \dfrac{(x-3)^2}{25} = 1$

Hyperbola with center $(3, -1)$

semi transverse axis $= 3$

semi conjugate axis $= 5$

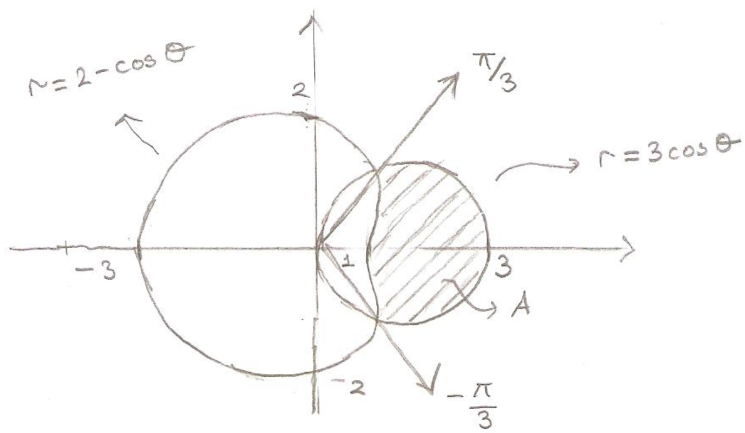
semi focal separation
$c = \sqrt{9+25}$
$c = \sqrt{34}$



$\varepsilon = \dfrac{\sqrt{34}}{3}$

2. (20 pts) R is defined as the region inside the polar curve $r = 3cos\theta$ and outside the polar curve $r = 2 - cos\theta$. Sketch the region R and calculate its area.

SOLUTION



points of intersection

$3\cos\theta = 2 - \cos\theta$

$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \pm\frac{\pi}{3}$

Area of shaded region = Area between the curve $r = 3\cos\theta$ and $\theta = -\frac{\pi}{3}$, $\theta = \frac{\pi}{3}$

− Area between the curve $r = 2 - \cos\theta$ and $\theta = -\frac{\pi}{3}$, $\theta = \frac{\pi}{3}$

$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(3\cos\theta)^2\,d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2}(2-\cos\theta)^2\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} \left[\frac{9}{2}\cos^2\theta - \frac{1}{2}(4 - 4\cos\theta + \cos^2\theta)\right] d\theta$$

$$= \int_{-\pi/3}^{\pi/3} (4\cos^2\theta + 2\cos\theta - 2)\,d\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \int_{-\pi/3}^{\pi/3} (2 + 2\cos 2\theta + 2\cos\theta - 2)\,d\theta$$

$$= \int_{-\pi/3}^{\pi/3} (2\cos 2\theta + 2\cos\theta)\,d\theta = (\sin 2\theta + 2\sin\theta)\Big|_{-\pi/3}^{\pi/3}$$

$$= \sin\frac{2\pi}{3} + 2\sin\frac{\pi}{3} - \sin\left(-\frac{2\pi}{3}\right) - 2\sin\left(-\frac{\pi}{3}\right)$$

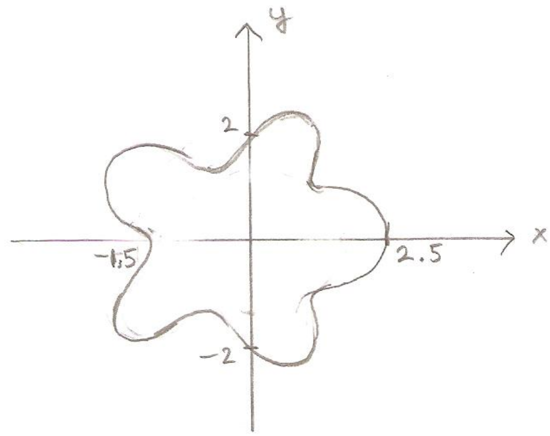
$$\boxed{A = 3\sqrt{3}}$$

3. (15 pts) Sketch the polar curve given by the equation

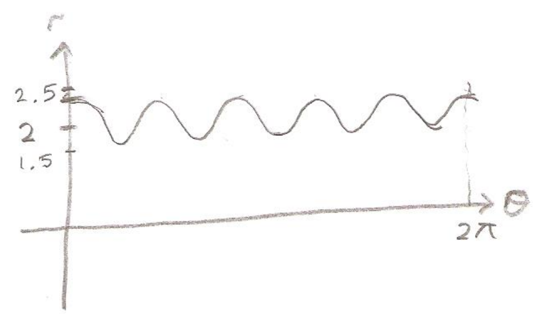
$$r = 2 + \frac{1}{2}\cos(5\theta), \qquad 0 \le \theta \le 2\pi\ .$$

Compute the area enclosed by the curve.

SOLUTION



$$\underbrace{r = 2}_{\text{a circe}} + \underbrace{\frac{1}{2}\cos(5\theta)}_{\text{a 5-leave rose}}$$



$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(2 + \frac{1}{2}\cos(5\theta)\right)^2 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 + 2\cos(5\theta) + \frac{1}{4}\cos^2(5\theta)\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(4 + 2\cos(5\theta) + \frac{1}{8}(1 + \cos(10\theta)\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left(\frac{33}{8} + 2\cos(5\theta) + \frac{1}{8}\cos(10\theta)\right) d\theta$$

$$= \frac{1}{2} \times 2\pi \times \frac{33}{8}$$

$$A = \frac{33\pi}{8}$$

4. (20 pts) Consider a circle with radius 2 centered at the point (3, 4).

(a) Represent this curve with a set of parametric equations as

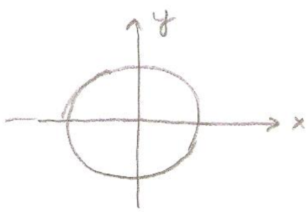
$x = f(t), \quad y = g(t)$

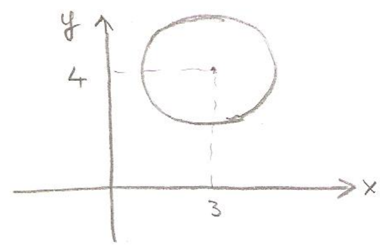
(b) Represent the same curve with a polar equation as

$r = h(\theta)$

SOLUTION

(a) Remember that $\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}\right\}$ represents a circle with radius 2 and centre (0,0)





To shift to center (3,4) we add 3 to x and 4 to y:

$$\boxed{\begin{array}{l} x = 3 + 2\cos t = f(t) \\ y = 4 + 2\sin t = g(t) \end{array}}$$

(b) Use the cartesian - to - polar equations

$x = r\cos\theta \qquad y = r\sin\theta$

$(x-3)^2 + (y-4)^2 = 4$

$(r\cos\theta - 3)^2 + (r\sin\theta - 4)^2 = 4$

$r^2\cos^2\theta - 6r\cos\theta + 9 + r^2\sin^2\theta - 8r\sin\theta + 16 = 4$

$r^2 - (6\cos\theta + 8\sin\theta)\,r + 21 = 0$

To find r, we solve for the quadratic equation:

$$\boxed{\begin{array}{l} r = \frac{1}{2}\left(6\cos\theta + 8\sin\theta \pm \sqrt{(6\cos\theta + 8\sin\theta)^2 - 84}\right) \\ r = h(\theta) \end{array}}$$

5. **(15 pts)** Consider the parametric curve:

$x = t^2 + t + 1$
$y = 4t^3 + 3t^2 + 2$

(a) Find the equation of the line tangent to the curve at the point (1, 1).
(b) Find the intervals of t for which the curve is concave upward or downward.

SOLUTION

(a) $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t^2+6t}{2t+1} = \frac{6t(2t+1)}{2t+1} = 6t$$

$$\left.\frac{dy}{dx}\right|_{t=-1} = -6$$

$$\boxed{\begin{aligned} t^2+t+1 &= 1 \\ 4t^3+3t^2+2 &= 1 \end{aligned}}$$

↳ Find t, corresponding to the point (1,1)

$t(t+1) = 0$ → $t=0$
→ $t=-1$ satisfies the second equation

$y - 1 = -6(x-1)$

$$\underbrace{y = -6x + 7}_{\text{tangent line at } (1,1)}$$

(b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{6}{2t+1}$$

Concave upward for $\frac{6}{2t+1} > 0$ $\quad t > -\frac{1}{2}$

Concave downward for $\frac{6}{2t+1} < 0$ $\quad t < -\frac{1}{2}$

6. (15 pts) For the following sequences, determine if it converges or diverges. Find the limit if the sequence converges.

(a) $\left\{\dfrac{(-1)^n n}{n^3-1}\right\}$

(b) $\left\{n^{1/\ln(n)}\right\}$

SOLUTION

(a)

$$\frac{-n}{n^3-1} \leq \frac{(-1)^n n}{n^3-1} \leq \frac{n}{n^3-1}$$

$$\lim_{n\to\infty} \frac{-n}{n^3-1} = \lim_{n\to\infty} \frac{n}{n^3-1} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{(-1)^n n}{n^3-1} = 0$$

(b)

$$\lim_{n\to\infty} \ln\left(n^{1/\ln(n)}\right) = \lim_{n\to\infty} \frac{1}{\ln(n)} \ln(n) = \lim_{n\to\infty} 1 = 1$$

$$\Downarrow$$

$$\lim_{n\to\infty} n^{1/\ln(n)} = e$$