SOLUTION KEY

CALCULUS II, SPRING 2013, MIDTERM EXAM 2

Name: ....

Student No:..

1	2	3	4	5	6	7	8	9	10	11	TOTAL
/15	/5	/10	/10	/10	/5	/5	/5	/15	/10	/10	/100

1. (15 pts) Find the Maclaurin series representation of $f(x) = \ln(1-x)$. For what values of x is the representation valid?

$$\ln(1-x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \cancel{\frac{f(0)}{0!}}^{\,0} x^0 + \sum_{n=1}^{\infty} \frac{-(n-1)!\,x^n}{n!} = \sum_{n=1}^{\infty} -\frac{x^n}{n}$$

$$f'(x) = -\frac{1}{1-x}$$

$$f''(x) = -\frac{1}{(1-x)^2}$$

$$f'''(x) = -\frac{1\cdot 2}{(1-x)^3}$$

$$f^{(4)}(x) = -\frac{1\cdot 2\cdot 3}{(1-x)^4}$$

$$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^4}$$

$$\boxed{\begin{array}{l} f^{(n)}(0) = -(n-1)! \quad n \geq 1 \\ f(0) = \ln(1-0) = 0 \end{array}}$$

Use ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \left|\frac{x}{1+1/n}\right| \to |x| \text{ as } n \to \infty$$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x| < 1 \text{ for convergence}$$

At $x = 1$:

$\sum_{n=1}^{\infty} -\frac{1^n}{n}$: the series diverges

At $x = -1$

$\sum_{n=1}^{\infty} -\frac{(-1)^n}{n}$: the series converges.

hence $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $x \in [-1, 1)$

2. (5 pts) Find the vector projection of **a** onto **b** if $\mathbf{a} = \langle 4, 2, 0\rangle$ and $\mathbf{b} = \langle 1, 1, 1\rangle$.

$$\vec{a}_{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{6}{3}\langle 1,1,1\rangle = \langle 2, 2, 2\rangle$$

$$\vec{a}\cdot\vec{b} = \langle 4,2,0\rangle \cdot \langle 1,1,1\rangle = 4\cdot 1 + 2\cdot 1 + 0\cdot 1 = 6$$

$$|\vec{b}|^2 = 1^2 + 1^2 + 1^2 = 3$$

3. **(10 pts)** Sketch the contour plot (level curves) of the function $f(x,y) = 3x^2 + y$ for $k = 0, 1, 2$.

$3x^2 + y = 0$

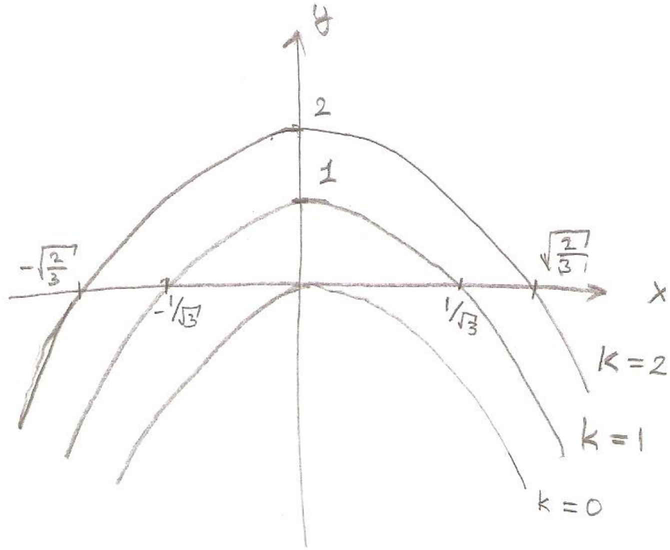
$y = -3x^2$

$3x^2 + y = 1$

$y = 1 - 3x^2$

$3x^2 + y = 2$

$y = 2 - 3x^2$



4. **(10 pts)** Find the length of the curve represented by $\mathbf{r}(t) = \langle e^t, e^t sint, e^t cost\rangle$ between the points (1, 0, 1) and $(e^{2\pi},\ 0,\ e^{2\pi})$.

$$L = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt$$

$(1, 0, 1) \Rightarrow t = 0$

$(e^{2\pi}, 0, e^{2\pi}) \Rightarrow t = 2\pi$

$$\vec{r}\,'(t) = \langle e^t,\ e^t \sin t + e^t \cos t,\ e^t \cos t - e^t \sin t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{e^{2t} + e^{2t}(\sin t + \cos t)^2 + e^{2t}(\sin t - \cos t)^2}$$

$$= \sqrt{e^{2t} + e^{2t}(\sin^2 t + \cancel{2\sin t \cos t} + \cos^2 t + \sin^2 t - \cancel{2 \sin t \cos t} + \cos^2 t)}$$

$$= e^t\sqrt{3}$$

$$L = \int_0^{2\pi} e^t \sqrt{3}\, dt = \sqrt{3}(e^{2\pi} - e^0) = \sqrt{3}(e^{2\pi} - 1) \text{ units}$$

5. **(10 pts)** Write the parametric equations of the tangent line to the curve represented by the vector function $\mathbf{r}(t) = \langle \sqrt{t}\,,\ 1\,,\ t^4 \rangle$ at the point (1, 1, 1).

$\vec{r}\,'(t) = \langle \frac{1}{2\sqrt{t}}, 0, 4t^3 \rangle$ At $(1,1,1)$, $t = 1$.

$\vec{r}\,'(1) = \langle \frac{1}{2}, 0, 4 \rangle$: vector in the same direction with the tangent line passing through $(1,1,1)$

The parametric equations of the tangent line :

$$x = \frac{t}{2} + 1$$
$$y = 1 \qquad -\infty < t < \infty$$
$$z = 4t + 1$$

6. **(5 pts)** Find cosine of the angle between the two planes:
$2x + y + z = 0$ and $3x - y + 2z = 0$

The angle between two planes is the angle between their normals :

$\vec{n}_1 = \langle 2, 1, 1 \rangle \qquad \vec{n}_2 = \langle 3, -1, 2 \rangle$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{2\cdot 3 + 1\cdot(-1) + 1\cdot 2}{\sqrt{4+1+1}\,\sqrt{9+1+4}} = \frac{7}{\sqrt{6}\,\sqrt{14}} = \frac{\sqrt{21}}{6}$$

7. **(5 pts)** Find the partial derivative f_{xxy} for the function $f(x,y) = e^{xy^2}$.

$$f_x = y^2 e^{xy^2}$$
$$f_{xx} = y^4 e^{xy^2}$$
$$f_{xxy} = 4y^3 e^{xy^2} + 2xy^5 e^{xy^2}$$

8. **(5 pts)** Let $z = \ln(u^3 + v^4)$, where $u = t^4$ and $v = t^2$. Compute dz/dt.

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ u \arrow[d, dash] & & v \arrow[d, dash] \\ t & & t \end{tikzcd}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}$$

$$= \frac{3u^2}{u^3 + v^4} \cdot 4t^3 + \frac{4v^3}{u^3 + v^4} \cdot 2t$$

$$= \frac{3t^8 \cdot 4t^3 + 4t^6 \cdot 2t}{t^{12} + t^8}$$

$$\frac{dz}{dt} = \frac{12t^{11} + 8t^7}{t^{12} + t^8} = \frac{12t^4 + 8}{t^5 + t}$$

9. **(15 pts)** Let $f(x, y, z) = xz + e^{y-x^2}$.

(a) Compute the gradient ∇f.

(b) Find the directional derivative at the point $(x, y, z) = (0, 0, 1)$ and in the direction of $\mathbf{u} = \langle 0, \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

(c) Find the direction for which $f(x, y, z)$ increases most rapidly at the point $(0, 0, 1)$.

(a) $\nabla f = \langle z - 2x e^{y-x^2}, e^{y-x^2}, x \rangle$

(b) $\nabla f(0,0,1) = \langle 1, 1, 0 \rangle$

$D_{\vec{u}} f(0,0,1) = \nabla f(0,0,1) \cdot \langle 0, \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \langle 1,1,0 \rangle \cdot \langle 0, \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$= \frac{1}{2}$

(c) The direction of the largest increase is

$\nabla f(0,0,1) = \langle 1,1,0 \rangle$

the unit vector in this direction is $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle$

10. **(10 pts)** Find an equation of the tangent plane at $(x, y) = (1, 3)$ to the graph of $f(x, y) = xy^2 - xy + 3x^3y$.

The tangent plane : $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

$f_x = y^2 - y + 9x^2y$ $\quad f_x(1,3) = 9 - 3 + 9 \cdot 3 = 33$

$f_y = 2xy - x + 3x^3$ $\quad f_y(1,3) = 6 - 1 + 3 = 8$

$z_0 = f(x_0, y_0) = f(1,3) = 9 - 3 + 9 = 15$

The equation of the tangent plane :

$z - 15 = 33(x-1) + 8(y-3)$

$33x + 8y - z = 42$

11. **(10 pts)** Find the critical points of $f(x, y) = x^2 + y^2 + xy + 9x$. Identify them as a local minimum, local maximum, or a saddle point.

$f_x = 2x + y + 9 = 0$

$f_y = 2y + x = 0$

$x = -2y$

$-4y + y = -9 \Rightarrow y = 3, \ x = -6$

$(-6, 3)$ is the only critical point

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 1$

$D(-6,3) = f_{xx}(-6,3) f_{yy}(-6,3) - (f_{xy}(-6,3))^2$

$= 4 - 1 = 3 > 0$

$f_{xx}(-6,3) = 2 > 0$

$(-6,3)$ is a local minimum.