CALCULUS II, SPRING 2014, MIDTERM EXAM 2

Name: ……………………………………………………………………………

Student No:…………………………………………………………………….

1	2	3	4	5	6	7	8	9	10	11	12	TOTAL
/5	/5	/10	/5	/10	/7	/10	/5	/8	/10	/10	/15	/100

1. **(5 pts)** Determine whether the following sequence is (a) bounded above, (b) bounded below, (c) convergent. Find the limit if it is convergent.

$$\{a_n\} = \left\{\frac{e^n}{\pi^{n/2}}\right\}$$

SOLUTION

$\left\{\frac{e^n}{\pi^{n/2}}\right\} = \left\{\left(\frac{e}{\sqrt{\pi}}\right)^n\right\}$. Since $e/\sqrt{\pi} > 1$, the sequence is bounded below, positive, increasing, and diverges to infinity.

2. **(5 pts)** Determine whether the following sequence is convergent. Find the limit if it is convergent.

$$\{a_n\} = \left\{\frac{(n!)^2}{(2n)!}\right\}$$

SOLUTION

$$a_n = \frac{(n!)^2}{(2n)!} = \frac{(1\cdot 2\cdot 3\cdots n)(1\cdot 2\cdot 3\cdots n)}{1\cdot 2\cdot 3\cdots n\cdot (n+1)\cdot (n+2)\cdots 2n}$$
$$= \frac{1}{n+1}\cdot\frac{2}{n+2}\cdot\frac{3}{n+3}\cdots\frac{n}{n+n} \le \left(\frac{1}{2}\right)^n.$$

Thus $\lim a_n = 0$.

3. **(10 pts)** Determine whether the following series converge absolutely, converge conditionally, or diverge. Use alternating series test for parts (a) and (b). For part (b) you can use comparison test with $\sum_{n=1}^{\infty}\frac{1}{n}$.

$(a)\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

$(b)\quad \sum_{n=10}^{\infty}\frac{\sin(n+\frac{1}{2})\pi}{\ln(\ln(n))}$

SOLUTION ►

$\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test (since the terms alternate in sign, decrease in size, and approach 0). However, the convergence is only conditional, since $\sum \frac{1}{\sqrt{n}}$ diverges to infinity.

$\sum_{n=10}^{\infty}\frac{\sin(n+\frac{1}{2})\pi}{\ln\ln n} = \sum_{n=10}^{\infty}\frac{(-1)^n}{\ln\ln n}$ converges by the alternating series test but only conditionally since $\sum_{n=10}^{\infty}\frac{1}{\ln\ln n}$ diverges to infinity by comparison with $\sum_{n=10}^{\infty}\frac{1}{n}$. ($\ln\ln n < n$ for $n \ge 10$.)

4. (5 pts) Find the scalar and vector projections of $\mathbf{v}$ along $\mathbf{u}$ if $\mathbf{u} = \langle 3, 4, -5 \rangle$ and $\mathbf{v} = \langle 3, -4, -5 \rangle$.

SOLUTION

The scalar projection of $\mathbf{v}$ along $\mathbf{u}$ is

$$\frac{\mathbf{v} \bullet \mathbf{u}}{|\mathbf{u}|} = \frac{18}{5\sqrt{2}}$$

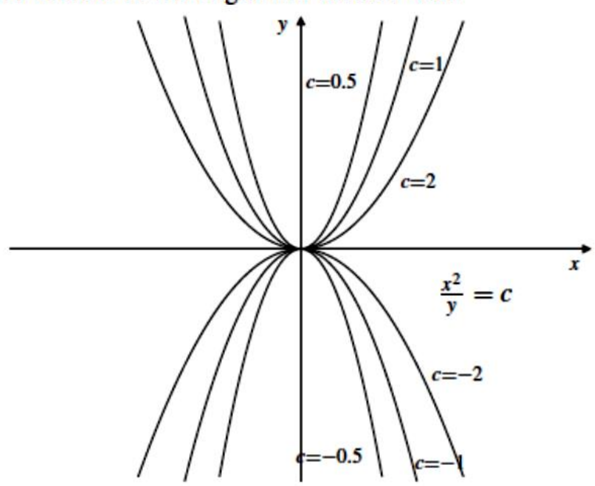
The vector projection of $\mathbf{v}$ along $\mathbf{u}$ is

$$\frac{(\mathbf{v} \bullet \mathbf{u})\mathbf{u}}{|\mathbf{u}|^2} = \frac{9}{25}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}).$$

5. (10 pts) Sketch the contour plot (level curves) of the function $f(x, y) = x^2/y$ for $k = -1, -2, 0, 1, 2$.

SOLUTION

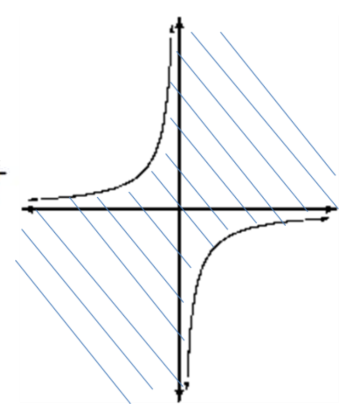
$f(x, y) = \dfrac{x^2}{y} = C$, a family of parabolas, $y = x^2/C$, with vertices at the origin and vertical axes.



6. (7 pts) Specify and sketch the domain of the function $f(x, y) = \ln(1 + xy)$.

SOLUTION

$f(x, y) = \ln(1 + xy)$.
The domain consists of all points satisfying $xy > -1$, that is, points lying between the two branches of the hyperbola $xy = -1$.



7. (10 pts) Find the length of the curve represented by $\mathbf{r}(t) = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = 0$ to $t = 1$.

SOLUTION

$$\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad (0 \le t \le 1)$$
$$v = \sqrt{(2t)^2 + (2t)^2 + (3t^2)^2} = t\sqrt{8 + 9t^2}$$
$$\text{Length} = \int_0^1 t\sqrt{8 + 9t^2}\, dt \quad \text{Let } u = 8 + 9t^2$$
$$du = 18t\, dt$$
$$= \frac{1}{18}\frac{2}{3}u^{3/2}\Big|_8^{17} = \frac{17\sqrt{17} - 16\sqrt{2}}{27} \text{ units.}$$

8. (5 pts) Find the unit tangent vector $\hat{\mathbf{T}}(t)$ for the curve

$\mathbf{r}(t) = \cos t \sin t\, \mathbf{i} + \sin^2 t\, \mathbf{j} + \cos t\, \mathbf{k}$.

SOLUTION

$$\mathbf{r} = \cos t \sin t\mathbf{i} + \sin^2 t + \cos t\mathbf{k}$$
$$= \frac{1}{2}\sin 2t\mathbf{i} + \frac{1}{2}(1 - \cos 2t)\mathbf{j} + \cos t\mathbf{k}$$
$$\mathbf{v} = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k}$$
$$v = |\mathbf{v}| = \sqrt{1 + \sin^2 t}$$
$$\hat{\mathbf{T}} = \frac{1}{\sqrt{1 + \sin^2 t}}\left(\cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k}\right).$$

9. (8 pts) Let $u = \sqrt{x^2 + y^2}$, where $x = e^{st}$ and $y = 1 + s^2 \cos t$. Compute $\partial u/\partial t$ using the chain method.

SOLUTION

$$\frac{\partial u}{\partial t} = \frac{x}{\sqrt{x^2 + y^2}}se^{st} + \frac{y}{\sqrt{x^2 + y^2}}(-s^2 \sin t)$$
$$= \frac{xse^{st} - ys^2 \sin t}{\sqrt{x^2 + y^2}}.$$

10. (10 pts) Let $f(x, y) = x^2 y$.

(a) Compute the gradient ∇f.

(b) Find the directional derivative at the point $(x, y) = (-1, -1)$ in the direction of $\mathbf{u} = \langle 1,\ 2 \rangle$

(c) Find the direction for which $f(x, y)$ increases most rapidly at the point $(-1, -1)$.

SOLUTION

$f(x, y) = x^2 y, \quad \nabla f = 2xy\mathbf{i} + x^2\mathbf{j},$
$\nabla f(-1, -1) = 2\mathbf{i} + \mathbf{j}.$
Rate of change of f at $(-1, -1)$ in the direction of $\mathbf{i} + 2\mathbf{j}$ is

$$\frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} \bullet (2\mathbf{i} + \mathbf{j}) = \frac{4}{\sqrt{5}}.$$

$f(x, y)$ increases most rapidly along the direction of the gradient.

11. (10 pts) Find the critical points of $f(x, y) = x^3 + y^3 - 3xy$. Identify them as a local minimum, local maximum, or a saddle point.

SOLUTION

$f(x, y) = x^3 + y^3 - 3xy$

$f_1(x, y) = 3(x^2 - y), \qquad f_2(x, y) = 3(y^2 - x).$

For critical points: $x^2 = y$ and $y^2 = x$. Thus $x^4 - x = 0$, that is, $x(x-1)(x^2 + x + 1) = 0$. Thus $x = 0$ or $x = 1$. The critical points are $(0, 0)$ and $(1, 1)$. We have

$$A = f_{11}(x, y) = 6x, \qquad B = f_{12}(x, y) = -3,$$
$$C = f_{22}(x, y) = 6y.$$

At $(0, 0)$: $A = C = 0$, $B = -3$. Thus $AC < B^2$, and $(0, 0)$ is a saddle point of f.

At $(1, 1)$: $A = C = 6$, $B = -3$, so $AC > B^2$. Thus f has a local minimum value at $(1, 1)$.

12. (15 pts) Consider the ellipse with equation: $3x^2 + 2xy + 3y^2 = 16$

The center of the ellipse is the origin. Using **Lagrange Multipliers method**, find the ends of the major and minor axes of the ellipse; i.e. the points on the ellipse that are nearest to and farthest from the center. **(No partial credit if you don't use Lagrange Multipliers method).**

SOLUTION

Let $L = x^2 + y^2 + \lambda(3x^2 + 2xy + 3y^2 - 16)$. We have

$$0 = \frac{\partial L}{\partial x} = 2x + 6\lambda x + 2\lambda y \qquad (A)$$
$$0 = \frac{\partial L}{\partial y} = 2y + 6\lambda y + 2\lambda x. \qquad (B)$$

Multiplying (A) by y and (B) by x and subtracting we get

$$2\lambda(y^2 - x^2) = 0.$$

Thus, either $\lambda = 0$, or $y = x$, or $y = -x$.

$\lambda = 0$ is not possible, since it implies $x = 0$ and $y = 0$, and the point $(0, 0)$ does not lie on the given ellipse.

If $y = x$, then $8x^2 = 16$, so $x = y = \pm\sqrt{2}$.

If $y = -x$, then $4x^2 = 16$, so $x = -y = \pm 2$.

The points on the ellipse nearest the origin are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. The points farthest from the origin are $(2, -2)$ and $(-2, 2)$. The major axis of the ellipse lies along $y = -x$ and has length $4\sqrt{2}$. The minor axis lies along $y = x$ and has length 4.