SOLUTION KEY

CALCULUS II, SPRING 2013, FINAL EXAM

Name: ....

Student No:...

1	2	3	4	5	6	7	8	TOTAL
/10	/10	/10	/10	/15	/15	/20	/10	/100

1. (10 pts) Determine the interval of convergence (including the endpoints) for the following power series. In other words, state for what values of x the series converges.

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{(n+1)^2}(x+2)^n$$

Apply the ratio test

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} 3^{n+1} (x+2)^{n+1}}{(n+2)^2}}{\frac{(-1)^n 3^n (x+2)^n}{(n+1)^2}} \right| = 3|x+2| \underbrace{\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^2}_{=1}$$

Series converges absolutely when:

$\rho = 3|x+2| < 1$ or $|x+2| < \frac{1}{3} \Rightarrow -\frac{7}{3} < x < -\frac{5}{3}$

At the end point $x = -\frac{5}{3}$ the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$. Converges..

At the end point $x = -\frac{7}{3}$ the series becomes $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$. converges...

So the interval of convergence is $[-\frac{7}{3}, -\frac{5}{3}]$.

The series converges for $|x+2| \leq \frac{1}{3}$.

2. (10 pts) Find the directional derivative of $f(x,y,z) = z^2 + ye^{-xy}$ at the point $(0,2,5)$ in the direction of vector $\vec{v} = \langle 2,1,-2\rangle$.

Unit vector in the direction of $\vec{v}$ $(\vec{u})$: $\vec{u} = \frac{\langle 2,1,-2\rangle}{\sqrt{2^2+1^2+(-2)^2}}$

$\vec{u} = \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$

$\nabla f = \langle -y^2 e^{-xy}, (1-xy)e^{-xy}, 2z \rangle$

$\nabla f(0,2,5) = \langle -4, 1, 10 \rangle$

$D_{\vec{u}} f(0,2,5) = \nabla f(0,2,5) \cdot \vec{u} = \langle -4,1,10\rangle \langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \rangle$

$= -9$

3. (10 pts) Find a parametric equation for the line passing through the point $(1, 2, 3)$, and in the direction of a vector $\vec{u}$ which is perpendicular to both $\vec{v}$ and $\vec{w}$ where $\vec{v} = \langle 2, 1, -2 \rangle$ and $\vec{w} = \langle 1, 3, 1 \rangle$.

$$\vec{u} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{vmatrix} = 7\vec{i} - 4\vec{j} + 5\vec{k} = \langle 7, -4, 5 \rangle$$

The parametric equation of the line

$$\vec{r} = \vec{r}_0 + t\vec{u} \qquad \vec{r} = \langle 1, 2, 3 \rangle + t \langle 7, -4, 5 \rangle$$

$x = 1 + 7t$

$y = 2 - 4t$

$z = 3 + 5t$

4. (10 pts) Find all critical points of the function $f(x, y) = x^3 + 3xy + y^3$. Classify the critical points as local maximums, local minimums, or saddle points.

Critical points satisfy

$$\left.\begin{aligned} f_x &= 3x^2 + 3y = 0 \\ f_y &= 3x + 3y^2 = 0 \end{aligned}\right\} \quad \left.\begin{aligned} y &= -x^2 \\ x &= -y^2 \end{aligned}\right\} \quad x = -x^4 \quad \begin{pmatrix} x = 0 \\ y = 0 \end{pmatrix} \text{ or } \begin{pmatrix} x = -1 \\ y = -1 \end{pmatrix}$$

Critical points : $(0, 0)$ and $(-1, -1)$

Second derivatives :

$$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = 3$$

At $(0, 0)$

$$f_{xx} f_{yy} - f_{xy}^2 = 0 \cdot 0 - 3^2 = -9 < 0 \quad \Rightarrow \text{saddle point}$$

At $(-1, -1)$

$$f_{xx} f_{yy} - f_{xy}^2 = (-6)\cdot(-6) - 3^2 = 27 > 0 \quad \text{and } f_{xx} = -6 < 0$$

so, this is a local maximum.

5. (15 pts) Find the shortest distance from the origin to the surface $xyz^2 = 2$. Use the method of Lagrange multipliers.

$L = x^2 + y^2 + z^2 + \lambda(xyz^2 - 2)$

$L_x = 2x + \lambda y z^2 = 0 \longrightarrow \lambda z^2 = \frac{-2x}{y} \qquad \lambda z = \frac{-2x}{yz}$

$L_y = 2y + \lambda x z^2 = 0 \longrightarrow 2y + x\left(-\frac{2x}{y}\right) = 0 \Rightarrow x^2 = y^2$

$L_z = 2z + 2\lambda xyz = 0 \longrightarrow 2z + 2xy\left(-\frac{2x}{yz}\right) = 0 \Rightarrow$

$L_\lambda = xyz^2 - 2 = 0$

$2z = \frac{4x^2y}{yz} \Rightarrow z^2 = 2x^2$

$xyz^2 = 2$

$x^2y^2z^4 = 2 \Rightarrow x^8 = 1 \qquad \begin{pmatrix} x = 1 \\ y = 1 \\ z = \sqrt{2} \end{pmatrix} \qquad \begin{pmatrix} x = -1 \\ y = -1 \\ z = -\sqrt{2} \end{pmatrix}$

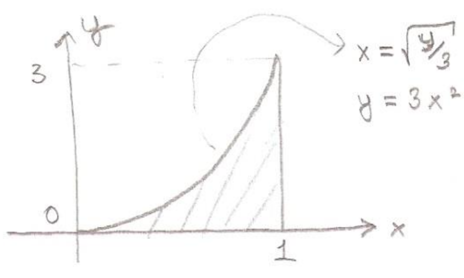
$\lambda = -1 \qquad \lambda = -1$

$D = \sqrt{1 + 1 + 2} = 2$ units

minimum distance from the origin to the surface $xyz^2 = 2$

6. (15 pts) Sketch the region of the following integral and calculate it by reversing the order of integration. Be careful with the limits of the integration.

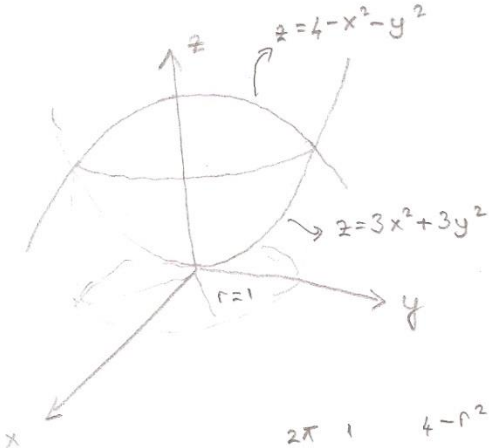
$$\int_0^3 \int_{\sqrt{y/3}}^1 e^{x^3}\, dx\, dy$$



$\int_0^3 \left(\int_{\sqrt{y/3}}^1 e^{x^3} dx \right) dy = \int_0^1 \left(\int_0^{3x^2} e^{x^3} dy \right) dx = \int_0^1 \left(y e^{x^3} \Big|_{y=0}^{y=3x^2} \right) dx$

$= \int_0^1 3x^2 e^{x^3} dx = e^{x^3} \Big|_0^1 = e - 1$

7. (20 pts) Find the volume of the solid bounded by the two paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$. Use cylindrical coordinates.



Intersection

$$4 - x^2 - y^2 = 3x^2 + 3y^2$$

$$4 - r^2 = 3r^2$$

$$4 = 4r^2 \Rightarrow r = 1$$

$$0 \leq r \leq 1$$

$$0 \leq \theta \leq 2\pi$$

$$3r^2 \leq z \leq 4 - r^2$$

$dV = r\,dr\,d\theta\,dz$

$$V = \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{3r^2}^{4-r^2} dz = \int_0^{2\pi} d\theta \int_0^1 r\,dr\,(4 - r^2 - 3r^2)$$

$$= \int_0^{2\pi} d\theta \int_0^1 (4r - 4r^3)\,dr = \int_0^{2\pi} d\theta \left[2r^2 - r^4 \Big|_0^1\right]$$

$$= \int_0^{2\pi} d\theta = 2\pi$$

8. (10 pts) Evaluate the following integral.

$$\int_{-2}^{4} \int_{x-1}^{x+1} \int_0^{\sqrt{2y/x}} 3xyz\,dz\,dy\,dx$$

$$\int_{-2}^{4} \int_{x-1}^{x+1} \left(3xy\frac{z^2}{2} \Big|_{z=0}^{z=\sqrt{2y/x}}\right) dy\,dx = \int_{-2}^{4} \int_{x-1}^{x+1} \frac{3}{2}xy\,\frac{2y}{x}\,dy\,dx$$

$$= \int_{-2}^{4} \int_{x-1}^{x+1} 3y^2\,dy\,dx = \int_{-2}^{4} \left(y^3 \Big|_{y=x-1}^{y=x+1}\right) dx = \int_{-2}^{4} \left((x+1)^3 - (x-1)^3\right) dx$$

$$= \left(\frac{(x+1)^4}{4} - \frac{(x-1)^4}{4} \Big|_{-2}^{4}\right) = \frac{5^4}{4} - \frac{3^4}{4} - \frac{1^4}{4} + \frac{3^4}{4} = 156$$