CALCULUS II – EXERCISE SET – 1 – SOLUTIONS

CONIC SECTIONS

1. The ellipse with foci $(0, \pm 2)$ has major axis along the y-axis and $c = 2$. If $a = 3$, then $b^2 = 9 - 4 = 5$. The ellipse has equation

$$\frac{x^2}{5} + \frac{y^2}{9} = 1.$$

2. The ellipse with foci $(0, 1)$ and $(4, 1)$ has $c = 2$, centre $(2, 1)$, and major axis along $y = 1$. If $\epsilon = 1/2$, then $a = c/\epsilon = 4$ and $b^2 = 16 - 4 = 12$. The ellipse has equation

$$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{12} = 1.$$

3. A parabola with focus $(2, 3)$ and vertex $(2, 4)$ has $a = -1$ and principal axis $x = 2$. Its equation is $(x-2)^2 = -4(y-4) = 16 - 4y$.

4. A parabola with focus at $(0, -1)$ and principal axis along $y = -1$ will have vertex at a point of the form $(v, -1)$. Its equation will then be of the form $(y+1)^2 = \pm 4v(x - v)$. The origin lies on this curve if
$1 = \pm 4(-v^2)$. Only the $-$ sign is possible, and in this case $v = \pm 1/2$. The possible equations for the parabola are $(y+1)^2 = 1 \pm 2x$.

5. The hyperbola with semi-transverse axis $a = 1$ and foci $(0, \pm 2)$ has transverse axis along the y-axis, $c = 2$, and $b^2 = c^2 - a^2 = 3$. The equation is

$$y^2 - \frac{x^2}{3} = 1.$$

6. The hyperbola with foci at $(\pm 5, 1)$ and asymptotes $x = \pm(y - 1)$ is rectangular, has centre at $(0, 1)$ and has transverse axis along the line $y = 1$. Since $c = 5$ and $a = b$ (because the asymptotes are perpendicular to each other) we have $a^2 = b^2 = 25/2$. The equation of the hyperbola is

$$x^2 - (y-1)^2 = \frac{25}{2}.$$

7. If $x^2 + y^2 + 2x = -1$, then $(x + 1)^2 + y^2 = 0$. This represents the single point $(-1, 0)$.

8. If $x^2 + 4y^2 - 4y = 0$, then

$$x^2 + 4\left(y^2 - y + \frac{1}{4}\right) = 1, \quad \text{or} \quad \frac{x^2}{1} + \frac{(y - \frac{1}{2})^2}{\frac{1}{4}} = 1.$$

This represents an ellipse with centre at $\left(0, \frac{1}{2}\right)$, semi-major axis 1, semi-minor axis $\frac{1}{2}$, and foci at $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

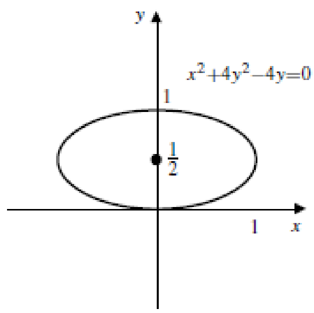

Fig. 8.1.8

9. If $4x^2 + y^2 - 4y = 0$, then

$$4x^2 + y^2 - 4y + 4 = 4$$
$$4x^2 + (y - 2)^2 = 4$$
$$x^2 + \frac{(y - 2)^2}{4} = 1$$

This is an ellipse with semi-axes 1 and 2, centred at $(0, 2)$.

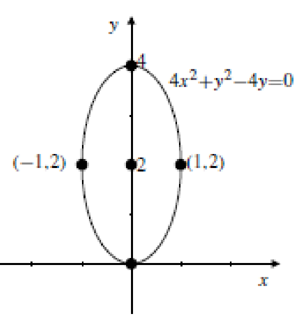

Fig. 8.1.9

10. If $4x^2 - y^2 - 4y = 0$, then

$$4x^2 - (y^2 + 4y + 4) = -4, \quad \text{or} \quad \frac{x^2}{1} - \frac{(y + 2)^2}{4} = -1.$$

This represents a hyperbola with centre at $(0, -2)$, semi-transverse axis 2, semi-conjugate axis 1, and foci at $(0, -2 \pm \sqrt{5})$. The asymptotes are $y = \pm 2x - 2$.

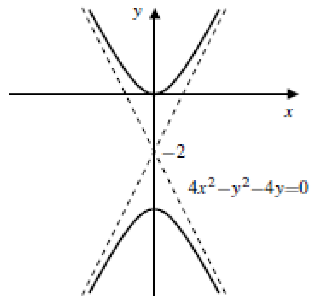

Fig. 8.1.10

11. If $x^2 + 2x - y = 3$, then $(x+1)^2 - y = 4$.
Thus $y = (x+1)^2 - 4$. This is a parabola with vertex $(-1, -4)$, opening upward.

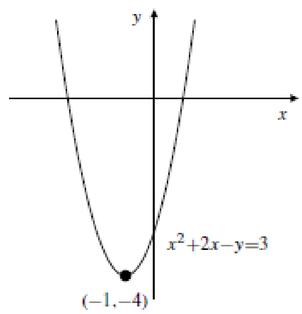

Fig. 8.1.11

12. If $x + 2y + 2y^2 = 1$, then

$$2\left(y^2 + y + \frac{1}{4}\right) = \frac{3}{2} - x$$
$$\Leftrightarrow \quad x = \frac{3}{2} - 2\left(y + \frac{1}{2}\right)^2.$$

This represents a parabola with vertex at $(\frac{3}{2}, -\frac{1}{2})$, focus at $(\frac{11}{8}, -\frac{1}{2})$ and directrix $x = \frac{13}{8}$.

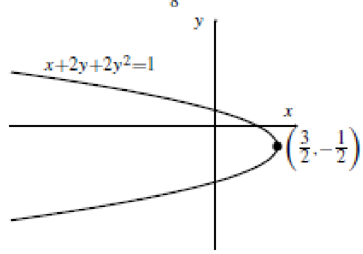

Fig. 8.1.12

13. If $x^2 - 2y^2 + 3x + 4y = 2$, then

$$\left(x + \frac{3}{2}\right)^2 - 2(y-1)^2 = \frac{9}{4}$$
$$\frac{(x+\frac{3}{2})^2}{\frac{9}{4}} - \frac{(y-1)^2}{\frac{9}{8}} = 1$$

This is a hyperbola with centre $(-\frac{3}{2}, 1)$, and asymptotes the straight lines $2x + 3 = \pm 2\sqrt{2}(y-1)$.

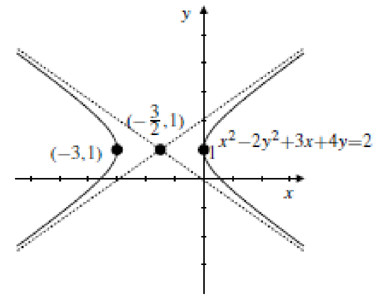

Fig. 8.1.13

14. If $9x^2 + 4y^2 - 18x + 8y = -13$, then

$$9(x^2 - 2x + 1) + 4(y^2 + 2y + 1) = 0$$
$$\Leftrightarrow 9(x-1)^2 + 4(y+1)^2 = 0.$$

This represents the single point $(1, -1)$.

15. If $9x^2 + 4y^2 - 18x + 8y = 23$, then

$$9(x^2 - 2x + 1) + 4(y^2 + 2y + 1) = 23 + 9 + 4 = 36$$
$$9(x-1)^2 + 4(y+1)^2 = 36$$
$$\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1.$$

This is an ellipse with centre $(1, -1)$, and semi-axes 2 and 3.

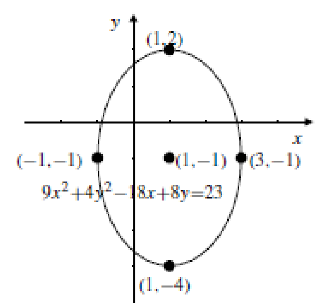

Fig. 8.1.15

16. The equation $(x-y)^2 - (x+y)^2 = 1$ simplifies to $4xy = -1$ and hence represents a rectangular hyperbola with centre at the origin, asymptotes along the coordinate axes, transverse axis along $y = -x$, conjugate axis along $y = x$, vertices at $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$, semi-transverse and semi-conjugate axes equal to $1/\sqrt{2}$, semi-focal separation equal to $\sqrt{\frac{1}{2} + \frac{1}{2}} = 1$, and hence foci at the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The eccentricity is $\sqrt{2}$.

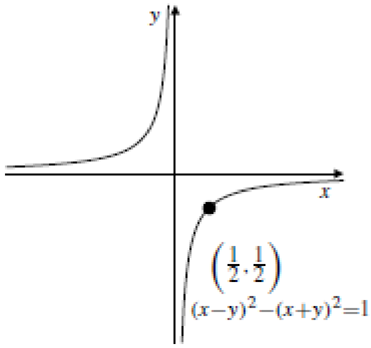

Fig. 8.1.16

PARAMETRIC CURVES

1. If $x = t$, $y = 1 - t$, $(0 \le t \le 1)$ then $x + y = 1$. This is a straight line segment.

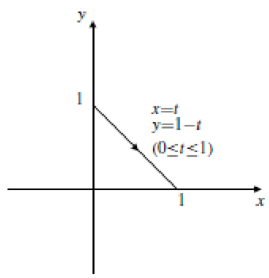

Fig. 8.2.1

2. If $x = 2 - t$ and $y = t + 1$ for $0 \le t < \infty$, then $y = 2 - x + 1 = 3 - x$ for $-\infty < x \le 2$, which is a half line.

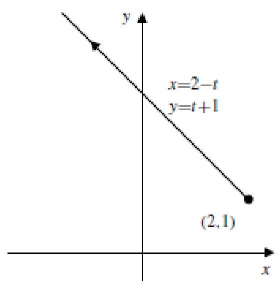

Fig. 8.2.2

3. If $x = 1/t$, $y = t - 1$, $(0 < t < 4)$, then $y = \dfrac{1}{x} - 1$. This is part of a hyperbola.

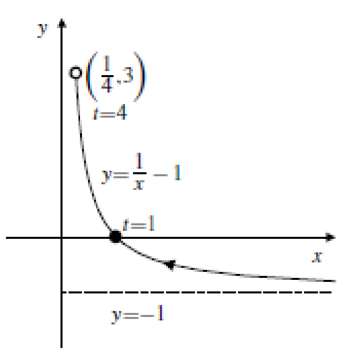

Fig. 8.2.3

4. If $x = \dfrac{1}{1+t^2}$ and $y = \dfrac{t}{1+t^2}$ for $-\infty < t < \infty$, then

$$x^2 + y^2 = \frac{1+t^2}{(1+t^2)^2} = \frac{1}{1+t^2} = x$$
$$\Leftrightarrow \quad \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

This curve consists of all points of the circle with centre at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$ except the origin $(0, 0)$.

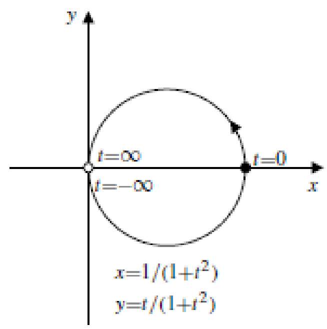

Fig. 8.2.4

5. If $x = 3\sin 2t$, $y = 3\cos 2t$, $(0 \le t \le \pi/3)$, then $x^2 + y^2 = 9$. This is part of a circle.

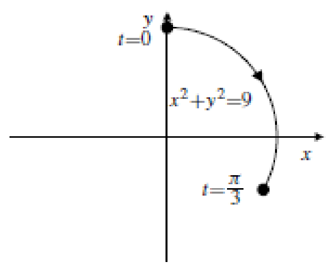

Fig. 8.2.5

6. If $x = a\sec t$ and $y = b\tan t$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$, then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1.$$

The curve is one arch of this hyperbola.

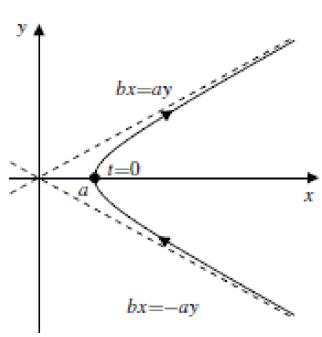

Fig. 8.2.6

7. If $x = 3\sin \pi t$, $y = 4\cos \pi t$, $(-1 \le t \le 1)$, then $\frac{x^2}{9} + \frac{y^2}{16} = 1$. This is an ellipse.

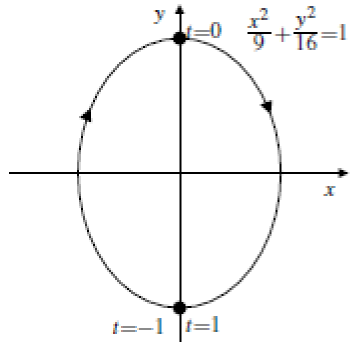

Fig. 8.2.7

8. If $x = \cos \sin s$ and $y = \sin \sin s$ for $-\infty < s < \infty$, then $x^2 + y^2 = 1$. The curve consists of the arc of this circle extending from $(a, -b)$ through $(1, 0)$ to (a, b) where $a = \cos(1)$ and $b = \sin(1)$, traversed infinitely often back and forth.

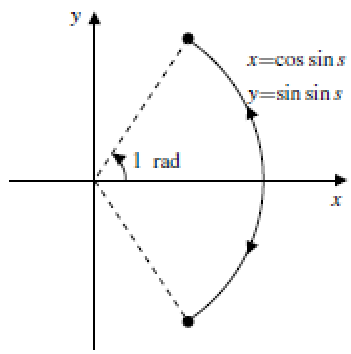

Fig. 8.2.8

9. If $x = \cos^3 t$, $y = \sin^3 t$, $(0 \le t \le 2\pi)$, then $x^{2/3} + y^{2/3} = 1$. This is an astroid.

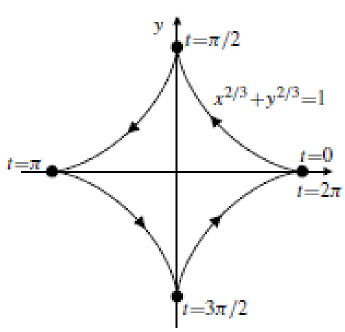

Fig. 8.2.9

10. If $x = 1 - \sqrt{4 - t^2}$ and $y = 2 + t$ for $-2 \le t \le 2$ then

$$(x-1)^2 = 4 - t^2 = 4 - (y-2)^2.$$

The parametric curve is the left half of the circle of radius 4 centred at $(1, 2)$, and is traced in the direction of increasing y.

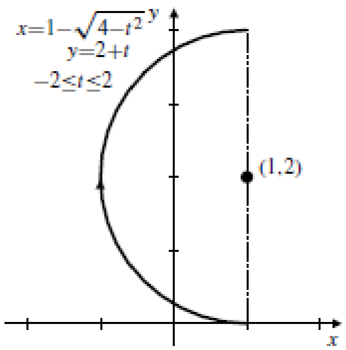

Fig. 8.2.10

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1. $x = t^2 + 1 \qquad y = 2t - 4$

$\dfrac{dx}{dt} = 2t \qquad \dfrac{dy}{dt} = 2$

No horizontal tangents. Vertical tangent at $t = 0$, i.e., at $(1, -4)$.

2. $x = t^2 - 2t \qquad y = t^2 + 2t$

$\dfrac{dx}{dt} = 2t - 2 \qquad \dfrac{dy}{dt} = 2t + 2$

Horizontal tangent at $t = -1$, i.e., at $(3, -1)$.
Vertical tangent at $t = 1$, i.e., at $(-1, 3)$.

3. $x = t^2 - 2t \qquad y = t^3 - 12t$

$\dfrac{dx}{dt} = 2(t-1) \qquad \dfrac{dy}{dt} = 3(t^2 - 4)$

Horizontal tangent at $t = \pm 2$, i.e., at $(0, -16)$ and $(8, 16)$.
Vertical tangent at $t = 1$, i.e., at $(-1, -11)$.

4. $x = t^3 - 3t \qquad y = 2t^3 + 3t^2$

$\dfrac{dx}{dt} = 3(t^2 - 1) \qquad \dfrac{dy}{dt} = 6t(t+1)$

Horizontal tangent at $t = 0$, i.e., at $(0, 0)$.
Vertical tangent at $t = 1$, i.e., at $(-2, 5)$.
At $t = -1$ (i.e., at $(2, 1)$) both dx/dt and dy/dt change sign, so the curve is not smooth there. (It has a cusp.)

5. $x = te^{-t^2/2}$ $\quad y = e^{-t^2}$

$\dfrac{dx}{dt} = (1-t^2)e^{-t^2/2}$ $\quad \dfrac{dy}{dt} = -2te^{-t^2}$

Horizontal tangent at $t = 0$, i.e., at $(0, 1)$.
Vertical tangent at $t = \pm 1$, i.e. at $(\pm e^{-1/2}, e^{-1})$.

6. $x = \sin t$ $\quad y = \sin t - t\cos t$

$\dfrac{dx}{dt} = \cos t$ $\quad \dfrac{dy}{dt} = t\sin t$

Horizontal tangent at $t = n\pi$, i.e., at $(0, -(-1)^n n\pi)$ (for integers n).
Vertical tangent at $t = (n+\frac{1}{2})\pi$, i.e. at $(1, 1)$ and $(-1, -1)$.

7. $x = \sin(2t)$ $\quad y = \sin t$

$\dfrac{dx}{dt} = 2\cos(2t)$ $\quad \dfrac{dy}{dt} = \cos t$

Horizontal tangent at $t = (n+\frac{1}{2})\pi$, i.e., at $(0, \pm 1)$.
Vertical tangent at $t = \frac{1}{2}(n+\frac{1}{2})\pi$, i.e., at $(\pm 1, 1/\sqrt{2})$ and $(\pm 1, -1/\sqrt{2})$.

8. $x = \dfrac{3t}{1+t^3}$ $\quad y = \dfrac{3t^2}{1+t^3}$

$\dfrac{dx}{dt} = \dfrac{3(1-2t^3)}{(1+t^3)^2}$ $\quad \dfrac{dy}{dt} = \dfrac{3t(2-t^3)}{(1+t^3)^2}$

Horizontal tangent at $t = 0$ and $t = 2^{1/3}$, i.e., at $(0, 0)$ and $(2^{1/3}, 2^{2/3})$.
Vertical tangent at $t = 2^{-1/3}$, i.e., at $(2^{2/3}, 2^{1/3})$. The curve also approaches $(0, 0)$ vertically as $t \to \pm\infty$.

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9. $x = t^3 + t$ $\quad y = 1 - t^3$

$\dfrac{dx}{dt} = 3t^2 + 1$ $\quad \dfrac{dy}{dt} = -3t^2$

At $t = 1$; $\dfrac{dy}{dx} = \dfrac{-3(1)^2}{3(1)^2+1} = -\dfrac{3}{4}$.

10. $x = t^4 - t^2$ $\quad y = t^3 + 2t$

$\dfrac{dx}{dt} = 4t^3 - 2t$ $\quad \dfrac{dy}{dt} = 3t^2 + 2$

At $t = -1$; $\dfrac{dy}{dx} = \dfrac{3(-1)^2+2}{4(-1)^3 - 2(-1)} = -\dfrac{5}{2}$.

11. $x = \cos(2t)$ $\quad y = \sin t$

$\dfrac{dx}{dt} = -2\sin(2t)$ $\quad \dfrac{dy}{dt} = \cos t$

At $t = \dfrac{\pi}{6}$; $\dfrac{dy}{dx} = \dfrac{\cos(\pi/6)}{-2\sin(\pi/3)} = -\dfrac{1}{2}$.

12. $x = e^{2t}$ $\quad y = te^{2t}$

$\dfrac{dx}{dt} = 2e^{2t}$ $\quad \dfrac{dy}{dt} = e^{2t}(1+2t)$

At $t = -2$; $\dfrac{dy}{dx} = \dfrac{e^{-4}(1-4)}{2e^{-4}} = -\dfrac{3}{2}$.

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15. $x = t^3 - t$, $y = t^2$ is at $(0, 1)$ at $t = -1$ and $t = 1$. Since

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 1} = \frac{\pm 2}{2} = \pm 1,$$

the tangents at $(0, 1)$ at $t = \pm 1$ have slopes ± 1.

16. $x = \sin t$, $y = \sin(2t)$ is at $(0, 0)$ at $t = 0$ and $t = \pi$. Since

$$\frac{dy}{dx} = \frac{2\cos(2t)}{\cos t} = \begin{cases} 2 & \text{if } t = 0 \\ -2 & \text{if } t = \pi, \end{cases}$$

the tangents at $(0, 0)$ at $t = 0$ and $t = \pi$ have slopes 2 and -2, respectively.

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1. $x = 3t^2 \qquad y = 2t^3 \quad (0 \le t \le 1)$

$\dfrac{dx}{dt} = 6t \qquad \dfrac{dy}{dt} = 6t^2$

$$\begin{aligned} \text{Length } &= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt \\ &= 6\int_0^1 t\sqrt{1 + t^2}\, dt \quad \text{Let } u = 1 + t^2 \\ &\qquad\qquad\qquad\qquad du = 2t\, dt \\ &= 3\int_1^2 \sqrt{u}\, du = 2u^{3/2}\Big|_1^2 = 4\sqrt{2} - 2 \text{ units} \end{aligned}$$

2. If $x = 1 + t^3$ and $y = 1 - t^2$ for $-1 \le t \le 2$, then the arc length is

$$\begin{aligned} s &= \int_{-1}^2 \sqrt{(3t^2)^2 + (-2t)^2}\, dt \\ &= \int_{-1}^2 |t|\sqrt{9t^2 + 4}\, dt \\ &= \left(\int_0^1 + \int_0^2\right) t\sqrt{9t^2 + 4}\, dt \quad \text{Let } u = 9t^2 + 4 \\ &\qquad\qquad\qquad\qquad du = 18t\, dt \\ &= \frac{1}{18}\left(\int_4^{13} + \int_4^{40}\right)\sqrt{u}\, du \\ &= \frac{1}{27}\left(13\sqrt{13} + 40\sqrt{40} - 16\right) \text{ units.} \end{aligned}$$

3. $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \le t \le 2\pi)$. The length is

$$\begin{aligned} &\int_0^{2\pi} \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\, dt \\ &= 3a\int_0^{2\pi} |\sin t \cos t|\, dt \\ &= 12a\int_0^{\pi/2} \frac{1}{2}\sin 2t\, dt \\ &= 6a\left(-\frac{\cos 2t}{2}\right)\Big|_0^{\pi/2} = 6a \text{ units.} \end{aligned}$$

4. If $x = \ln(1+t^2)$ and $y = 2\tan^{-1} t$ for $0 \le t \le 1$, then

$$\frac{dx}{dt} = \frac{2t}{1+t^2}; \qquad \frac{dy}{dt} = \frac{2}{1+t^2}.$$

The arc length is

$$\begin{aligned}
s &= \int_0^1 \sqrt{\frac{4t^2+4}{(1+t^2)^2}}\,dt \\
&= 2\int_0^1 \frac{dt}{\sqrt{1+t^2}} \quad \text{Let } t = \tan\theta \\
&\qquad\qquad\qquad\quad\; dt = \sec^2\theta\,d\theta \\
&= 2\int_0^{\pi/4} \sec\theta\,d\theta \\
&= 2\ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} = 2\ln(1+\sqrt{2}) \text{ units.}
\end{aligned}$$

5. $x = t^2 \sin t$, $y = t^2 \cos t$, $(0 \le t \le 2\pi)$.

$$\begin{aligned}
\frac{dx}{dt} &= 2t\sin t + t^2\cos t \\
\frac{dy}{dt} &= 2t\cos t - t^2\sin t \\
\left(\frac{ds}{dt}\right)^2 &= t^2\Big[4\sin^2 t + 4t\sin t\cos t + t^2\cos^2 t \\
&\qquad + 4\cos^2 t - 4t\sin t\cos t + t^2\sin^2 t\Big] \\
&= t^2(4+t^2).
\end{aligned}$$

The length of the curve is

$$\begin{aligned}
&\int_0^{2\pi} t\sqrt{4+t^2}\,dt \quad \text{Let } u = 4+t^2 \\
&\qquad\qquad\qquad\qquad\; du = 2t\,dt \\
=&\frac{1}{2}\int_4^{4+4\pi^2} u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_4^{4+4\pi^2} \\
=&\frac{8}{3}\left((1+\pi^2)^{3/2} - 1\right) \text{ units.}
\end{aligned}$$

6. $x = \cos t + t\sin t \qquad y = \sin t - t\cos t \quad (0 \le t \le 2\pi)$

$$\frac{dx}{dt} = t\cos t \qquad\qquad \frac{dy}{dt} = t\sin t$$

$$\begin{aligned}
\text{Length } &= \int_0^{2\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt \\
&= \int_0^{2\pi} t\,dt = \frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2 \text{ units.}
\end{aligned}$$

7. $x = t + \sin t \qquad y = \cos t \quad (0 \le t \le \pi)$

$$\frac{dx}{dt} = 1 + \cos t \qquad \frac{dy}{dt} = -\sin t$$

$$\begin{aligned}\text{Length} &= \int_0^{\pi} \sqrt{1 + 2\cos t + \cos^2 t + \sin^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{4\cos^2(t/2)}\, dt = 2\int_0^{\pi} \cos\frac{t}{2}\, dt \\ &= 4\sin\frac{t}{2}\Big|_0^{\pi} = 4 \text{ units.}\end{aligned}$$

8. $x = \sin^2 t \qquad y = 2\cos t \quad (0 \le t \le \pi/2)$

$\frac{dx}{dt} = 2\sin t\cos t \qquad \frac{dy}{dt} = -2\sin t$

Length

$$\begin{aligned} &= \int_0^{\pi/2} \sqrt{4\sin^2 t\cos^2 t + 4\sin^2 t}\, dt \\ &= 2\int_0^{\pi/2} \sin t\sqrt{1 + \cos^2 t}\, dt \quad \text{Let } \cos t = \tan u \\ &\qquad\qquad\qquad\qquad\qquad\quad -\sin t\, dt = \sec^2 u\, du \\ &= 2\int_0^{\pi/4} \sec^3 u\, du \\ &= \Big(\sec u\tan u + \ln(\sec u + \tan u)\Big)\Big|_0^{\pi/4} \\ &= \sqrt{2} + \ln(1 + \sqrt{2}) \text{ units.}\end{aligned}$$

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11. $x = e^t\cos t \qquad y = e^t\sin t \quad (0 \le t \le \pi/2)$

$\frac{dx}{dt} = e^t(\cos t - \sin t) \qquad \frac{dy}{dt} = e^t(\sin t + \cos t)$

Arc length element:

$$\begin{aligned} ds &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2}\, dt \\ &= \sqrt{2}e^t\, dt.\end{aligned}$$

The area of revolution about the x-axis is

$$\begin{aligned}\int_{t=0}^{t=\pi/2} 2\pi y\, ds &= 2\sqrt{2}\pi\int_0^{\pi/2} e^{2t}\sin t\, dt \\ &= 2\sqrt{2}\pi\frac{e^{2t}}{5}(2\sin t - \cos t)\Big|_0^{\pi/2} \\ &= \frac{2\sqrt{2}\pi}{5}(2e^{\pi} + 1) \text{ sq. units.}\end{aligned}$$

12. The area of revolution of the curve in Exercise 11 about the y-axis is

$$\begin{aligned}\int_{t=0}^{t=\pi/2} 2\pi x\, ds &= 2\sqrt{2}\pi\int_0^{\pi/2} e^{2t}\cos t\, dt \\ &= 2\sqrt{2}\pi\frac{e^{2t}}{5}(2\cos t + \sin t)\Big|_0^{\pi/2} \\ &= \frac{2\sqrt{2}\pi}{5}(e^{\pi} - 2) \text{ sq. units.}\end{aligned}$$

---

15. $x = t^3 - 4t,\ y = t^2,\ (-2 \le t \le 2).$

$$\begin{aligned}\text{Area } &= \int_{-2}^{2} t^2(3t^2 - 4)\,dt \\ &= 2\int_0^2 (3t^4 - 4t^2)\,dt \\ &= 2\left(\frac{3t^5}{5} - \frac{4t^3}{3}\right)\Bigg|_0^2 = \frac{256}{15} \text{ sq. units.}\end{aligned}$$

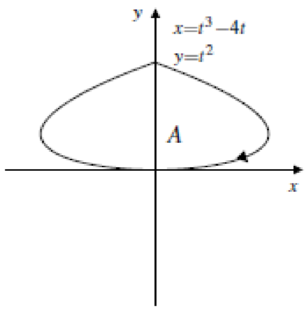

Fig. 8.4.15

16. Area of $R = 4 \times \int_{\pi/2}^{0} (a \sin^3 t)(-3a \sin t \cos^2 t)\,dt$

$$\begin{aligned}&= -12a^2 \int_{\pi/2}^{0} \sin^4 t \cos^2 t\,dt \\ &= 12a^2\left[\frac{t}{16} - \frac{\sin(4t)}{64} - \frac{\sin^3(2t)}{48}\right]\Bigg|_0^{\pi/2}\end{aligned}$$

(See Exercise 34 of Section 6.4.)

$$= \frac{3}{8}\pi a^2 \text{ sq. units.}$$

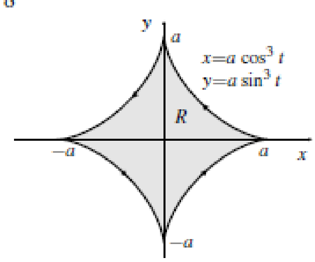

Fig. 8.4.16

17. $x = \sin^4 t,\ y = \cos^4 t,\ \left(0 \le t \le \frac{\pi}{2}\right).$

$$\begin{aligned}\text{Area } &= \int_0^{\pi/2} (\cos^4 t)(4\sin^3 t \cos t)\,dt \\ &= 4\int_0^{\pi/2} \cos^5 t(1 - \cos^2 t)\sin t\,dt \qquad \text{Let } u = \cos t \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad du = -\sin t\,dt \\ &= 4\int_0^1 (u^5 - u^7)\,du = 6\left(\frac{1}{6} - \frac{1}{8}\right) = \frac{1}{6} \text{ sq. units.}\end{aligned}$$

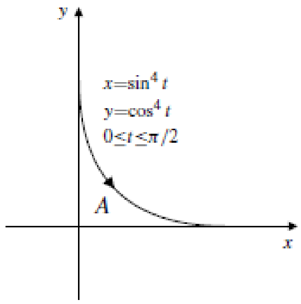

Fig. 8.4.17

18. If $x = \cos s \, \sin s = \frac{1}{2}\sin 2s$ and $y = \sin^2 s = \frac{1}{2} - \frac{1}{2}\cos 2s$ for $0 \le s \le \frac{1}{2}\pi$, then

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}\sin^2 2s + \frac{1}{4}\cos^2 2s = \frac{1}{4}$$

which is the right half of the circle with radius $\frac{1}{2}$ and centre at $(0, \frac{1}{2})$. Hence, the area of R is

$$\frac{1}{2}\left[\pi\left(\frac{1}{2}\right)^2\right] = \frac{\pi}{8} \text{ sq. units.}$$

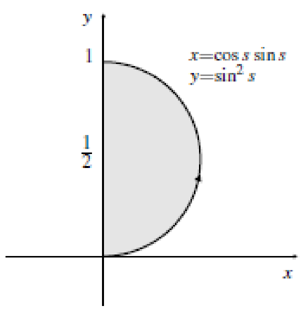

Fig. 8.4.18

19. $x = (2 + \sin t)\cos t, \; y = (2 + \sin t)\sin t, \; (0 \le t \le 2\pi)$. This is just the polar curve $r = 2 + \sin\theta$.

$$\begin{aligned}
\text{Area } &= -\int_0^{2\pi} (2 + \sin t)\sin t \frac{d}{dt}\Big((2 + \sin t)\cos t\Big)\, dt \\
&= -\int_0^{2\pi} (2\sin t + \sin^2 t)(\cos^2 t - 2\sin t - \sin^2 t)\, dt \\
&= \int_0^{2\pi} \Big[4\sin^2 t + 4\sin^3 t + \sin^4 t \\
&\qquad - 2\sin t\cos^2 t - \sin^2 t\cos^2 t\Big]\, dt \\
&= \int_0^{2\pi} \left[2(1 - \cos 2t) + \frac{1 - \cos 2t}{2}(-\cos 2t)\right] dt \\
&\qquad + \int_0^{2\pi} \sin t\Big[4 - 6\cos^2 t\Big]\, dt \\
&= 4\pi + \frac{\pi}{2} + 0 = \frac{9\pi}{2} \text{ sq. units.}
\end{aligned}$$

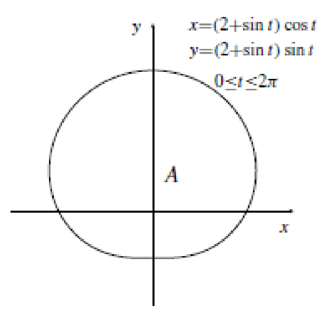

Fig. 8.4.19

POLAR COORDINATES AND POLAR CURVES

1. $r = 3\sec\theta$

$r\cos\theta = 3$

$x = 3$ vertical straight line.

2. $r = -2\csc\theta \Rightarrow r\sin\theta = -2$

$\Leftrightarrow \quad y = -2$ a horizontal line.

3. $r = 5/(3\sin\theta - 4\cos\theta)$

$3r\sin\theta - 4r\cos\theta = 5$

$3y - 4x = 5$ straight line.

4. $r = \sin\theta + \cos\theta$

$r^2 = r\sin\theta + r\cos\theta$

$x^2 + y^2 = y + x$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

a circle with centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$.

5. $r^2 = \csc 2\theta$

$r^2 \sin 2\theta = 1$

$2r^2 \sin\theta\cos\theta = 1$

$2xy = 1$ a rectangular hyperbola.

6. $r = \sec\theta\tan\theta \Rightarrow r\cos\theta = \frac{r\sin\theta}{r\cos\theta}$

$x^2 = y$ a parabola.

7. $r = \sec\theta(1 + \tan\theta)$

$r\cos\theta = 1 + \tan\theta$

$x = 1 + \frac{y}{x}$

$x^2 - x - y = 0$ a parabola.

8. $r = \frac{2}{\sqrt{\cos^2\theta + 4\sin^2\theta}}$

$r^2\cos^2\theta + 4r^2\sin^2\theta = 4$

$x^2 + 4y^2 = 4$ an ellipse.

9. $r = \frac{1}{1 - \cos\theta}$

$r - x = 1$

$r^2 = (1 + x)^2$

$x^2 + y^2 = 1 + 2x + x^2$

$y^2 = 1 + 2x$ a parabola.

10. $r = \dfrac{2}{2-\cos\theta}$

$2r - r\cos\theta = 2$

$4r^2 = (2+x)^2$

$4x^2 + 4y^2 = 4 + 4x + x^2$

$3x^2 + 4y^2 - 4x = 4$ an ellipse.

11. $r = \dfrac{2}{1-2\sin\theta}$

$r - 2y = 2$

$x^2 + y^2 = r^2 = 4(1+y)^2 = 4 + 8y + 4y^2$

$x^2 - 3y^2 - 8y = 4$ a hyperbola.

12. $r = \dfrac{2}{1+\sin\theta}$

$r + r\sin\theta = 2$

$r^2 = (2-y)^2$

$x^2 + y^2 = 4 - 4y + y^2$

$x^2 = 4 - 4y$ a parabola.

13. $r = 1 + \sin\theta$ (cardioid)

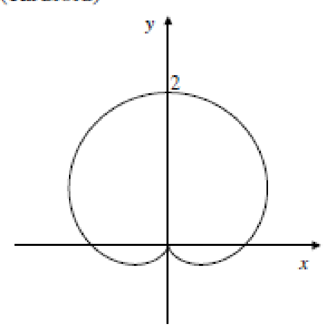

Fig. 8.5.13

14. If $r = 1 - \cos\left(\theta + \dfrac{\pi}{4}\right)$, then $r = 0$ at $\theta = -\dfrac{\pi}{4}$ and $\dfrac{7\pi}{4}$.
This is a cardioid.

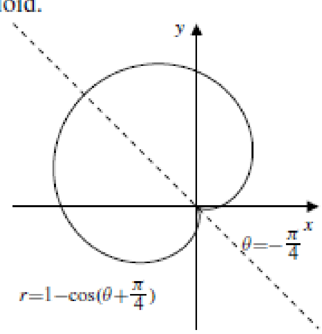

Fig. 8.5.14

15. $r = 1 + 2\cos\theta$
$r = 0$ if $\theta = \pm 2\pi/3$.

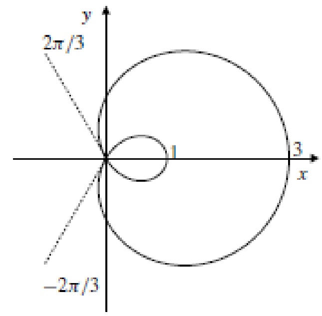

Fig. 8.5.15

16. If $r = 1 - 2\sin\theta$, then $r = 0$ at $\theta = \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$.

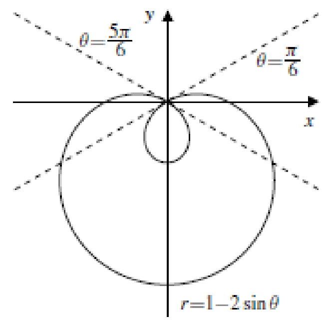

Fig. 8.5.16

17. $r = 2 + \cos\theta$

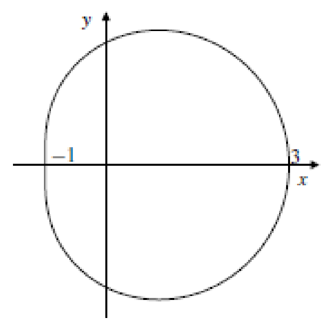

Fig. 8.5.17

18. If $r = 2\sin 2\theta$, then $r = 0$ at $\theta = 0,\ \pm\dfrac{\pi}{2}$ and π.

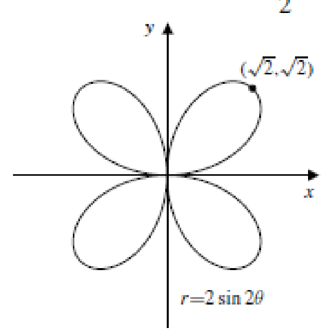

Fig. 8.5.18

19. $r = \cos 3\theta$ (three leaf rosette)
$r = 0$ at $\theta = \pm\pi/6,\ \pm\pi/2,\ \pm 5\pi/6$.

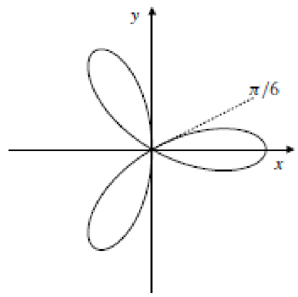

Fig. 8.5.19

20. If $r = 2\cos 4\theta$, then $r = 0$ at $\theta = \pm\dfrac{\pi}{8}$, $\pm\dfrac{3\pi}{8}$, $\pm\dfrac{5\pi}{8}$ and $\pm\dfrac{7\pi}{8}$. (an eight leaf rosette)

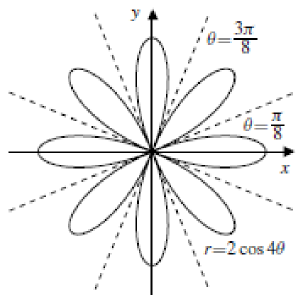

Fig. 8.5.20

21. $r^2 = 4\sin 2\theta$. Thus $r = \pm 2\sqrt{\sin 2\theta}$. This is a lemniscate. $r = 0$ at $\theta = 0$, $\theta = \pm\pi/2$, and $\theta = \pi$.

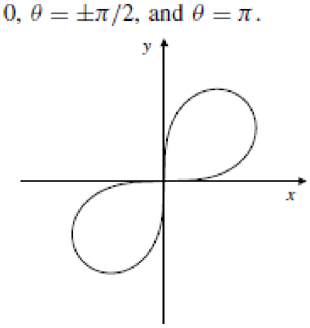

Fig. 8.5.21

22. If $r^2 = 4\cos 3\theta$, then $r = 0$ at $\theta = \pm\dfrac{\pi}{6}$, $\pm\dfrac{\pi}{2}$ and $\pm\dfrac{5\pi}{6}$. This equation defines two functions of r, namely $r = \pm 2\sqrt{\cos 3\theta}$. Each contributes 3 leaves to the graph.

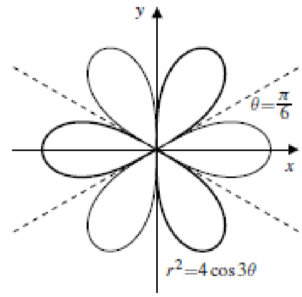

Fig. 8.5.22

23. $r^2 = \sin 3\theta$. Thus $r = \pm\sqrt{\sin 3\theta}$. This is a lemniscate. $r = 0$ at $\theta = 0$, $\pm\pi/3$, $\pm 2\pi/3$, π.

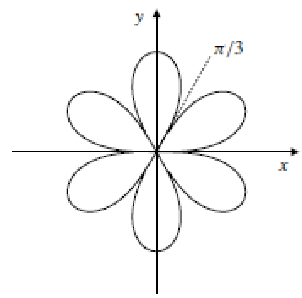

Fig. 8.5.23

24. If $r = \ln\theta$, then $r = 0$ at $\theta = 1$. Note that

$$y = r\sin\theta = \ln\theta\,\sin\theta = (\theta\ln\theta)\left(\frac{\sin\theta}{\theta}\right) \to 0$$

as $\theta \to 0+$. Therefore, the (negative) x-axis is an asymptote of the curve.

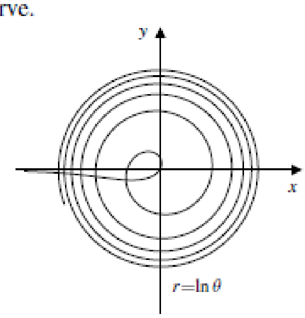

Fig. 8.5.24

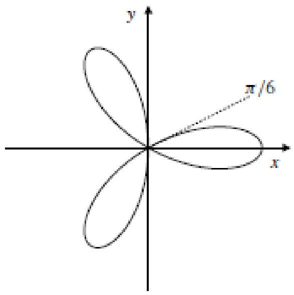

Fig. 8.5.19

20. If $r = 2\cos 4\theta$, then $r = 0$ at $\theta = \pm\frac{\pi}{8}$, $\pm\frac{3\pi}{8}$, $\pm\frac{5\pi}{8}$ and $\pm\frac{7\pi}{8}$. (an eight leaf rosette)

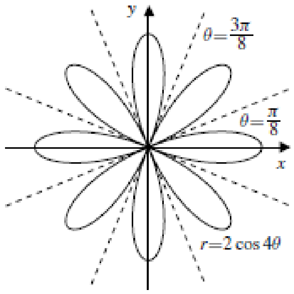

Fig. 8.5.20

21. $r^2 = 4\sin 2\theta$. Thus $r = \pm 2\sqrt{\sin 2\theta}$. This is a lemniscate. $r = 0$ at $\theta = 0$, $\theta = \pm\pi/2$, and $\theta = \pi$.

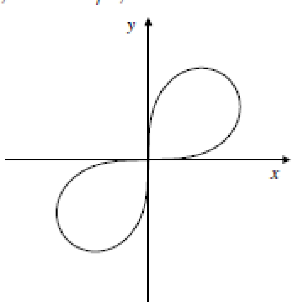

Fig. 8.5.21

22. If $r^2 = 4\cos 3\theta$, then $r = 0$ at $\theta = \pm\frac{\pi}{6}$, $\pm\frac{\pi}{2}$ and $\pm\frac{5\pi}{6}$. This equation defines two functions of r, namely $r = \pm 2\sqrt{\cos 3\theta}$. Each contributes 3 leaves to the graph.

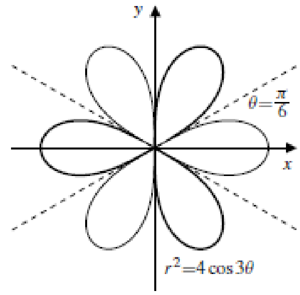

Fig. 8.5.22

23. $r^2 = \sin 3\theta$. Thus $r = \pm\sqrt{\sin 3\theta}$. This is a lemniscate. $r = 0$ at $\theta = 0$, $\pm\pi/3$, $\pm 2\pi/3$, π.

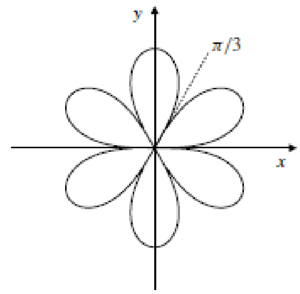

Fig. 8.5.23

24. If $r = \ln\theta$, then $r = 0$ at $\theta = 1$. Note that

$$y = r\sin\theta = \ln\theta\sin\theta = (\theta\ln\theta)\left(\frac{\sin\theta}{\theta}\right) \to 0$$

as $\theta \to 0+$. Therefore, the (negative) x-axis is an asymptote of the curve.

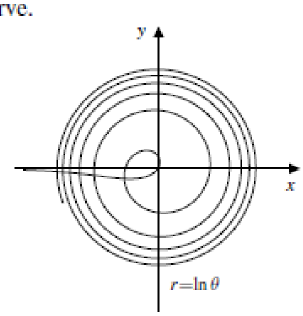

Fig. 8.5.24

25. $r = \sqrt{3}\cos\theta$, and $r = \sin\theta$ both pass through the origin, and so intersect there. Also
$\sin\theta = \sqrt{3}\cos\theta \quad\Rightarrow\quad \tan\theta = \sqrt{3} \quad\Rightarrow\quad \theta = \pi/3, \quad 4\pi/3.$
Both of these give the same point $[\sqrt{3}/2, \pi/3]$.
Intersections: the origin and $[\sqrt{3}/2, \pi/3]$.

26. $r^2 = 2\cos(2\theta)$, $r = 1$.
$\cos(2\theta) = 1/2 \quad\Rightarrow\quad \theta = \pm\pi/6$ or $\theta = \pm 5\pi/6$.
Intersections: $[1, \pm\pi/6]$ and $[1, \pm 5\pi/6]$.

27. $r = 1 + \cos\theta$, $r = 3\cos\theta$. Both curves pass through the origin, so intersect there. Also
$3\cos\theta = 1 + \cos\theta \quad \Rightarrow \quad \cos\theta = 1/2 \quad \Rightarrow \quad \theta = \pm\pi/3$.
Intersections: the origin and $[3/2, \pm\pi/3]$.

28. Let $r_1(\theta) = \theta$ and $r_2(\theta) = \theta + \pi$. Although the equation $r_1(\theta) = r_2(\theta)$ has no solutions, the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ can still intersect if $r_1(\theta_1) = -r_2(\theta_2)$ for two angles θ_1 and θ_2 having the opposite directions in the polar plane. Observe that $\theta_1 = -n\pi$ and $\theta_2 = (n-1)\pi$ are two such angles provided n is any integer. Since

$$r_1(\theta_1) = -n\pi = -r_2((n-1)\pi),$$

the curves intersect at any point of the form $[n\pi, 0]$ or $[n\pi, \pi]$.

---

1. Area $= \dfrac{1}{2}\displaystyle\int_0^{2\pi} \theta\, d\theta = \dfrac{(2\pi)^2}{4} = \pi^2$.

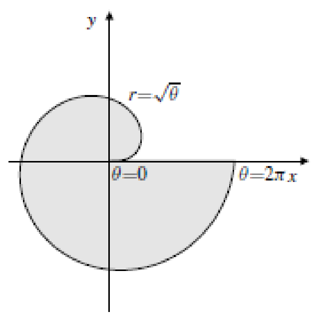

Fig. 8.6.1

2. Area $= \dfrac{1}{2}\displaystyle\int_0^{2\pi} \theta^2\, d\theta = \dfrac{\theta^3}{6}\Big|_0^{2\pi} = \dfrac{4}{3}\pi^3$ sq. units.

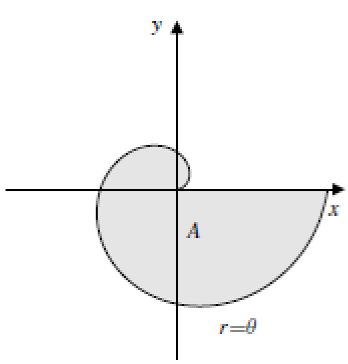

Fig. 8.6.2

3. Area $= 4 \times \frac{1}{2}\int_0^{\pi/4} a^2 \cos 2\theta \, d\theta$

$$= 2a^2 \frac{\sin 2\theta}{2}\bigg|_0^{\pi/4} = a^2 \text{ sq. units.}$$

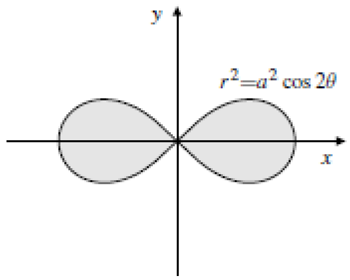

Fig. 8.6.3

4. Area $= \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{1}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$

$$= \frac{1}{4}\left(\theta - \frac{1}{6}\sin 6\theta\right)\bigg|_0^{\pi/3} = \frac{\pi}{12} \text{ sq. units.}$$

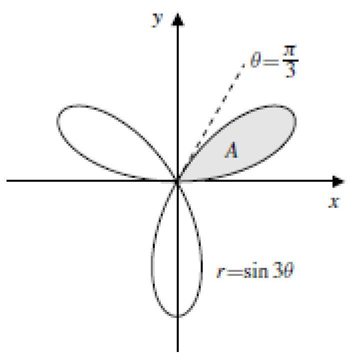

Fig. 8.6.4

5. Total area $= 16 \times \frac{1}{2}\int_0^{\pi/8} \cos^2 4\theta \, d\theta$

$$= 4\int_0^{\pi/8} (1 + \cos 8\theta)\, d\theta$$

$$= 4\left(\theta + \frac{\sin 8\theta}{8}\right)\bigg|_0^{\pi/8} = \frac{\pi}{2} \text{ sq. units.}$$

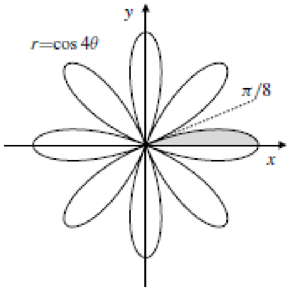

Fig. 8.6.5

6. The circles $r = a$ and $r = 2a\cos\theta$ intersect at $\theta = \pm\pi/3$. By symmetry, the common area is $4\times$(area of sector − area of right triangle) (see the figure), i.e.,

$$4\times\left[\left(\frac{1}{6}\pi a^2\right) - \left(\frac{1}{2}\frac{a}{2}\frac{\sqrt{3}a}{2}\right)\right] = \frac{4\pi - 3\sqrt{3}}{6}a^2 \text{ sq. units.}$$

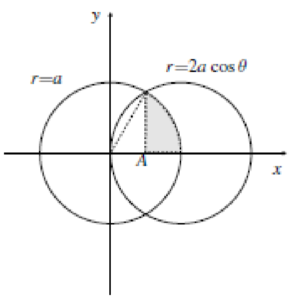

Fig. 8.6.6

7.
$$\begin{aligned}\text{Area } &= 2\times\frac{1}{2}\int_{\pi/2}^{\pi}(1-\cos\theta)^2\,d\theta - \frac{\pi}{2}\\ &= \int_{\pi/2}^{\pi}\left(1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta - \frac{\pi}{2}\\ &= \frac{3}{2}\left(\pi - \frac{\pi}{2}\right) - \left(2\sin\theta - \frac{\sin 2\theta}{4}\right)\Big|_{\pi/2}^{\pi} - \frac{\pi}{2}\\ &= \frac{\pi}{4} + 2 \text{ sq. units.}\end{aligned}$$

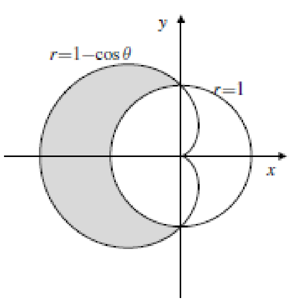

Fig. 8.6.7

8.
$$\begin{aligned}\text{Area} &= \frac{1}{2}\pi a^2 + 2\times\frac{1}{2}\int_0^{\pi/2}a^2(1-\sin\theta)^2\,d\theta\\ &= \frac{\pi a^2}{2} + a^2\int_0^{\pi/2}\left(1 - 2\sin\theta + \frac{1-\cos 2\theta}{2}\right)d\theta\\ &= \frac{\pi a^2}{2} + a^2\left(\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}\\ &= \left(\frac{5\pi}{4} - 2\right)a^2 \text{ sq. units.}\end{aligned}$$

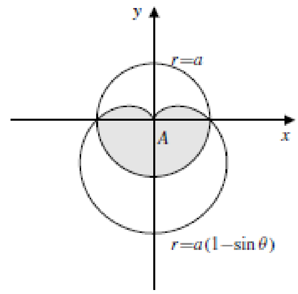

Fig. 8.6.8

9. For intersections: $1+\cos\theta = 3\cos\theta$. Thus $2\cos\theta = 1$ and $\theta = \pm\pi/3$. The shaded area is given by

$$\begin{aligned}&2\times\frac{1}{2}\left[\int_{\pi/3}^{\pi}(1+\cos\theta)^2\,d\theta - 9\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta\right]\\ &\quad= \int_{\pi/3}^{\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta\\ &\qquad - \frac{9}{2}\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta\\ &\quad= \frac{3}{2}\left(\frac{2\pi}{3}\right) + \left(2\sin\theta + \frac{\sin 2\theta}{4}\right)\Big|_{\pi/3}^{\pi}\\ &\qquad - \frac{9}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{\pi/3}^{\pi/2}\\ &\quad= \frac{\pi}{4} - \sqrt{3} - \frac{\sqrt{3}}{8} + \frac{9}{4}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{4} \text{ sq. units.}\end{aligned}$$

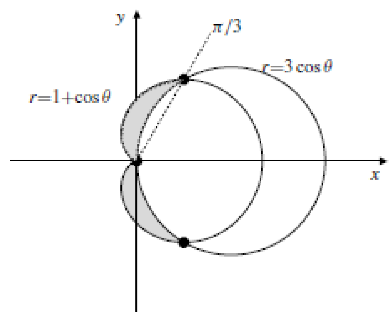

Fig. 8.6.9

10. Since $r^2 = 2\cos 2\theta$ meets $r = 1$ at $\theta = \pm\frac{\pi}{6}$ and $\pm\frac{5\pi}{6}$, the area inside the lemniscate and outside the circle is

$$4 \times \frac{1}{2}\int_0^{\pi/6}\left[2\cos 2\theta - 1^2\right] d\theta$$

$$= 2\sin 2\theta\Big|_0^{\pi/6} - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3} \text{ sq. units.}$$

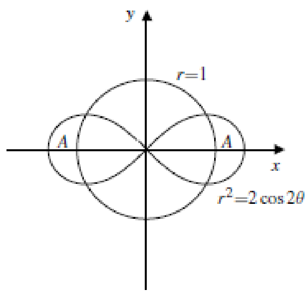

Fig. 8.6.10

11. $r = 0$ at $\theta = \pm 2\pi/3$. The shaded area is

$$2 \times \frac{1}{2}\int_{2\pi/3}^{\pi}(1 + 2\cos\theta)^2\, d\theta$$

$$= \int_{2\pi/3}^{\pi}\Big(1 + 4\cos\theta + 2(1 + \cos 2\theta)\Big)\, d\theta$$

$$= 3\left(\frac{\pi}{3}\right) + 4\sin\theta\Big|_{2\pi/3}^{\pi} + \sin 2\theta\Big|_{2\pi/3}^{\pi}$$

$$= \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = \pi - \frac{3\sqrt{3}}{2} \text{ sq. units.}$$

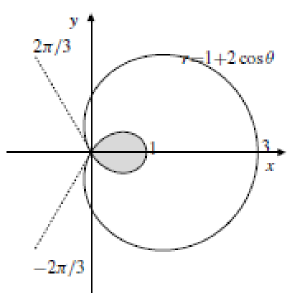

Fig. 8.6.11

12.
$$s = \int_0^{\pi}\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \int_0^{\pi}\sqrt{4\theta^2 + \theta^4}\, d\theta$$

$$= \int_0^{\pi}\theta\sqrt{4 + \theta^2}\, d\theta \quad \text{Let } u = 4 + \theta^2,\ du = 2\theta\, d\theta$$

$$= \frac{1}{2}\int_4^{4+\pi^2}\sqrt{u}\, du = \frac{1}{3}u^{3/2}\Big|_4^{4+\pi^2}$$

$$= \frac{1}{3}\left[(4 + \pi^2)^{3/2} - 8\right] \text{ units.}$$

13. $r = e^{a\theta}$, $(-\pi \le \theta \le \pi)$. $\frac{dr}{d\theta} = ae^{a\theta}$.
$ds = \sqrt{e^{2a\theta} + a^2e^{2a\theta}}\, d\theta = \sqrt{1 + a^2}e^{a\theta}\, d\theta$. The length of the curve is

$$\int_{-\pi}^{\pi}\sqrt{1 + a^2}e^{a\theta}\, d\theta = \frac{\sqrt{1 + a^2}}{a}(e^{a\pi} - e^{-a\pi}) \text{ units.}$$

14.
$$s = \int_0^{2\pi}\sqrt{a^2 + a^2\theta^2}\, d\theta$$

$$= a\int_0^{2\pi}\sqrt{1 + \theta^2}\, d\theta \quad \text{Let } \theta = \tan u,\ d\theta = \sec^2 u\, d\theta$$

$$= a\int_{\theta=0}^{\theta=2\pi}\sec^3 u\, du$$

$$= \frac{a}{2}\Big(\sec u \tan u + \ln|\sec u + \tan u|\Big)\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a}{2}\left[\theta\sqrt{1 + \theta^2} + \ln|\sqrt{1 + \theta^2} + \theta|\right]\Big|_{\theta=0}^{\theta=2\pi}$$

$$= \frac{a}{2}\left[2\pi\sqrt{1 + 4\pi^2} + \ln(2\pi + \sqrt{1 + 4\pi^2})\right] \text{ units.}$$