CALCULUS II – EXERCISE SET – 2 – SOLUTIONS

Section 9.1 Sequences and Convergence (page 478)

1. $\left\{\frac{2n^2}{n^2+1}\right\} = \left\{2 - \frac{2}{n^2+1}\right\} = \left\{1, \frac{8}{5}, \frac{9}{5}, \ldots\right\}$ is bounded, positive, increasing, and converges to 2.

2. $\left\{\frac{2n}{n^2+1}\right\} = \left\{1, \frac{4}{5}, \frac{3}{5}, \frac{8}{17}, \ldots\right\}$ is bounded, positive, decreasing, and converges to 0.

3. $\left\{4 - \frac{(-1)^n}{n}\right\} = \left\{5, \frac{7}{2}, \frac{13}{3}, \ldots\right\}$ is bounded, positive, and converges to 4.

4. $\left\{\sin\frac{1}{n}\right\} = \left\{\sin 1, \sin\left(\frac{1}{2}\right), \sin\left(\frac{1}{3}\right), \ldots\right\}$ is bounded, positive, decreasing, and converges to 0.

5. $\left\{\frac{n^2-1}{n}\right\} = \left\{n - \frac{1}{n}\right\} = \left\{0, \frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \ldots\right\}$ is bounded below, positive, increasing, and diverges to infinity.

6. $\left\{\frac{e^n}{\pi^n}\right\} = \left\{\frac{e}{\pi}, \left(\frac{e}{\pi}\right)^2, \left(\frac{e}{\pi}\right)^3, \ldots\right\}$ is bounded, positive, decreasing, and converges to 0, since $e < \pi$.

7. $\left\{\frac{e^n}{\pi^{n/2}}\right\} = \left\{\left(\frac{e}{\sqrt{\pi}}\right)^n\right\}$. Since $e/\sqrt{\pi} > 1$, the sequence is bounded below, positive, increasing, and diverges to infinity.

8. $\left\{\frac{(-1)^n n}{e^n}\right\} = \left\{\frac{-1}{e}, \frac{2}{e^2}, \frac{-3}{e^3}, \ldots\right\}$ is bounded, alternating, and converges to 0.

9. $\{2^n/n^n\}$ is bounded, positive, decreasing, and converges to 0.

10. $\frac{(n!)^2}{(2n)!} = \frac{1}{n+1}\,\frac{2}{n+2}\,\frac{3}{n+3}\cdots\frac{n}{2n} \le \left(\frac{1}{2}\right)^n$.
Also, $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} < \frac{1}{2}$. Thus the sequence $\left\{\frac{(n!)^2}{(2n)!}\right\}$ is positive, decreasing, bounded, and convergent to 0.

11. $\{n\cos(n\pi/2)\} = \{0, -2, 0, 4, 0, -6, \ldots\}$ is divergent.

12. $\left\{\frac{\sin n}{n}\right\} = \left\{\sin 1, \frac{\sin 2}{2}, \frac{\sin 3}{3}, \ldots\right\}$ is bounded and converges to 0.

13. $\{1, 1, -2, 3, 3, -4, 5, 5, -6, \ldots\}$ is divergent.

14. $\lim \frac{5-2n}{3n-7} = \lim \frac{\frac{5}{n} - 2}{3 - \frac{7}{n}} = -\frac{2}{3}$.

15. $\lim \frac{n^2-4}{n+5} = \lim \frac{n - \frac{4}{n}}{1 + \frac{5}{n}} = \infty$.

16. $\lim \frac{n^2}{n^3+1} = \lim \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$.

17. $\lim (-1)^n \frac{n}{n^3+1} = 0$.

18. $\lim \frac{n^2 - 2\sqrt{n} + 1}{1 - n - 3n^2} = \lim \frac{1 - \frac{2}{n\sqrt{n}} + \frac{1}{n^2}}{\frac{1}{n^2} - \frac{1}{n} - 3} = -\frac{1}{3}$.

19. $\lim \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim \frac{1 - e^{-2n}}{1 + e^{-2n}} = 1$.

20. $\lim n \sin\frac{1}{n} = \lim_{x\to 0+} \frac{\sin x}{x} = \lim_{x\to 0+} \frac{\cos x}{1} = 1$.

21. $\lim \left(\frac{n-3}{n}\right)^n = \lim \left(1 + \frac{-3}{n}\right)^n = e^{-3}$ by l'Hôpital's Rule.

22. $\lim \frac{n}{\ln(n+1)} = \lim_{x\to\infty} \frac{x}{\ln(x+1)}$
$= \lim_{x\to\infty} \frac{1}{\left(\frac{1}{x+1}\right)} = \lim_{x\to\infty} x + 1 = \infty$.

23. $\lim(\sqrt{n+1} - \sqrt{n}) = \lim \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = 0$.

24. $\lim\left(n - \sqrt{n^2-4n}\right) = \lim \frac{n^2 - (n^2 - 4n)}{n + \sqrt{n^2-4n}}$
$= \lim \frac{4n}{n + \sqrt{n^2-4n}} = \lim \frac{4}{1 + \sqrt{1 - \frac{4}{n}}} = 2$.

25. $\lim(\sqrt{n^2+n} - \sqrt{n^2-1})$
$= \lim \frac{n^2 + n - (n^2 - 1)}{\sqrt{n^2+n} + \sqrt{n^2-1}}$
$= \lim \frac{n+1}{n\left(\sqrt{1 + \frac{1}{n}} + \sqrt{1 - \frac{1}{n^2}}\right)}$
$= \lim \frac{1 + \frac{1}{n}}{\sqrt{1 + \frac{1}{n}} + \sqrt{1 - \frac{1}{n^2}}} = \frac{1}{2}$.

26. If $a_n = \left(\frac{n-1}{n+1}\right)^n$, then

$$\begin{aligned}\lim a_n &= \lim \left(\frac{n-1}{n}\right)^n \left(\frac{n}{n+1}\right)^n \\ &= \lim \left(1 - \frac{1}{n}\right)^n \Big/ \lim \left(1 + \frac{1}{n}\right)^n \\ &= \frac{e^{-1}}{e} = e^{-2} \quad \text{(by Theorem 6 of Section 3.4).}\end{aligned}$$

27. $$a_n = \frac{(n!)^2}{(2n)!} = \frac{(1 \cdot 2 \cdot 3 \cdots n)(1 \cdot 2 \cdot 3 \cdots n)}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1) \cdot (n+2) \cdots 2n}$$
$$= \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \frac{3}{n+3} \cdots \frac{n}{n+n} \le \left(\frac{1}{2}\right)^n.$$
Thus $\lim a_n = 0$.

28. We have $\lim \frac{n^2}{2^n} = 0$ since 2^n grows much faster than n^2 and $\lim \frac{4^n}{n!} = 0$ by Theorem 3(b). Hence,

$$\lim \frac{n^2 \, 2^n}{n!} = \lim \frac{n^2}{2^n} \cdot \frac{2^{2n}}{n!} = \left(\lim \frac{n^2}{2^n}\right)\left(\lim \frac{4^n}{n!}\right) = 0.$$

29. $a_n = \frac{\pi^n}{1 + 2^{2n}} \quad \Rightarrow \quad 0 < a_n < (\pi/4)^n$. Since $\pi/4 < 1$, therefore $(\pi/4)^n \to 0$ as $n \to \infty$. Thus $\lim a_n = 0$.