CALCULUS II – EXERCISE SET – 3 – SOLUTIONS

1. $$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots=\frac{1}{3}\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots\right)$$
$$=\frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}}=\frac{1}{2}.$$

2. $$3-\frac{3}{4}+\frac{3}{16}-\frac{3}{64}+\cdots=\sum_{n=1}^{\infty}3\left(-\frac{1}{4}\right)^{n-1}=\frac{3}{1+\frac{1}{4}}=\frac{12}{5}.$$

3. $$\sum_{n=5}^{\infty}\frac{1}{(2+\pi)^{2n}}$$
$$=\frac{1}{(2+\pi)^{10}}+\frac{1}{(2+\pi)^{12}}+\frac{1}{(2+\pi)^{14}}+\cdots$$
$$=\frac{1}{(2+\pi)^{10}}\left[1+\frac{1}{(2+\pi)^2}+\frac{1}{(2+\pi)^4}+\cdots\right]$$
$$=\frac{1}{(2+\pi)^{10}}\cdot\frac{1}{1-\frac{1}{(2+\pi)^2}}=\frac{1}{(2+\pi)^8\left[(2+\pi)^2-1\right]}.$$

4. $$\sum_{n=0}^{\infty}\frac{5}{10^{3n}}=5\left[1+\frac{1}{1000}+\left(\frac{1}{1000}\right)^2+\cdots\right]$$
$$=\frac{5}{1-\frac{1}{1000}}=\frac{5000}{999}.$$

5. $$\sum_{n=2}^{\infty}\frac{(-5)^n}{8^{2n}}=\frac{(-5)^2}{8^4}+\frac{(-5)^3}{8^6}+\frac{(-5)^4}{8^8}+\cdots$$
$$=\frac{25}{8^4}\left[1-\frac{5}{64}+\frac{5^2}{64^2}-\cdots\right]$$
$$=\frac{25}{8^4}\cdot\frac{1}{1+\frac{5}{64}}=\frac{25}{64\times 69}=\frac{25}{4416}.$$

6. $$\sum_{n=0}^{\infty}\frac{1}{e^n}=1+\frac{1}{e}+\left(\frac{1}{e}\right)^2+\cdots=\frac{1}{1-\frac{1}{e}}=\frac{e}{e-1}.$$

7. $$\sum_{k=0}^{\infty}\frac{2^{k+3}}{e^{k-3}}=8e^3\sum_{k=0}^{\infty}\left(\frac{2}{e}\right)^k=\frac{8e^3}{1-\frac{2}{e}}=\frac{8e^4}{e-2}.$$

8. $\sum_{j=1}^{\infty} \pi^{j/2}\cos(j\pi) = \sum_{j=2}^{\infty}(-1)^j \pi^{j/2}$ diverges because $\lim_{j\to\infty}(-1)^j\pi^{j/2}$ does not exist.

9. $\sum_{n=1}^{\infty} \dfrac{3+2^n}{2^{n+2}}$ diverges to ∞ because

$$\lim_{n\to\infty} \frac{3+2^n}{2^{n+2}} = \lim_{n\to\infty} \frac{\frac{3}{2^n}+1}{4} = \frac{1}{4} > 0.$$

10.
$$\begin{aligned}\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}} &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n + \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \\ &= \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}} + \frac{1}{9}\cdot\frac{1}{1-\frac{2}{3}} = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}.\end{aligned}$$

11. Since $\dfrac{1}{n(n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$, therefore

$$\begin{aligned}s_n &= \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots + \frac{1}{n(n+2)} \\ &= \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots \right. \\ &\qquad \left. + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}\right] \\ &= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right].\end{aligned}$$

Thus $\lim s_n = \dfrac{3}{4}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$.

12. Let

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots.$$

Since $\dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}\left(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right)$, the partial sum is

$$\begin{aligned}s_n &= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \\ &\qquad + \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right).\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim s_n = \frac{1}{2}.$$

13. Since $\dfrac{1}{(3n-2)(3n+1)} = \dfrac{1}{3}\left(\dfrac{1}{3n-2} - \dfrac{1}{3n+1}\right)$, therefore

$$\begin{aligned} s_n &= \frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \cdots + \frac{1}{(3n-2)(3n+1)} \\ &= \frac{1}{3}\left[\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots \right. \\ &\quad \left. + \frac{1}{3n-5} - \frac{1}{3n-2} + \frac{1}{3n-2} - \frac{1}{3n+1}\right] \\ &= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) \quad \to \quad \frac{1}{3}. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \dfrac{1}{(3n-2)(3n+1)} = \dfrac{1}{3}$.

14. Since

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right],$$

the partial sum is

$$\begin{aligned} s_n &= \frac{1}{2}\left(1 - \frac{2}{2} + \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \cdots \\ &\quad + \frac{1}{2}\left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim s_n = \frac{1}{4}.$$

15. Since $\dfrac{1}{2n-1} > \dfrac{1}{2n} = \dfrac{1}{2}\cdot\dfrac{1}{n}$, therefore the partial sums of the given series exceed half those of the divergent harmonic series $\sum(1/2n)$. Hence the given series diverges to infinity.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$ diverges to infinity since $\displaystyle\lim \frac{n}{n+2} = 1 > 0$.

17. Since $n^{-1/2} = \dfrac{1}{\sqrt{n}} \geq \dfrac{1}{n}$ for $n \geq 1$, we have

$$\sum_{k=1}^{n} k^{-1/2} \geq \sum_{k=1}^{n} \frac{1}{k} \to \infty,$$

as $n \to \infty$ (harmonic series). Thus $\sum n^{-1/2}$ diverges to infinity.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n+1} = 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$ diverges to infinity since it is just twice the harmonic series with the first term omitted.

1. $\sum \frac{1}{n^2+1}$ converges by comparison with $\sum \frac{1}{n^2}$ since $0 < \frac{1}{n^2+1} < \frac{1}{n^2}$.

2. $\sum_{n=1}^{\infty} \frac{n}{n^4-2}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ since

$$\lim \frac{\left(\frac{n}{n^4-2}\right)}{\left(\frac{1}{n^3}\right)} = 1, \quad \text{and} \quad 0 < 1 < \infty.$$

3. $\sum \frac{n^2+1}{n^3+1}$ diverges to infinity by comparison with $\sum \frac{1}{n}$, since $\frac{n^2+1}{n^3+1} > \frac{1}{n}$.

4. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ since

$$\lim \frac{\left(\frac{\sqrt{n}}{n^2+n+1}\right)}{\left(\frac{1}{n^{3/2}}\right)} = 1, \quad \text{and} \quad 0 < 1 < \infty.$$

5. Since $\sin x \le x$ for $x \ge 0$, we have

$$\left|\sin \frac{1}{n^2}\right| = \sin \frac{1}{n^2} \le \frac{1}{n^2},$$

so $\sum \left|\sin \frac{1}{n^2}\right|$ converges by comparison with $\sum \frac{1}{n^2}$.

6. $\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$ converges by comparison with the geometric series $\sum_{n=8}^{\infty} \left(\frac{1}{\pi}\right)^n$ since $0 < \frac{1}{\pi^n+5} < \frac{1}{\pi^n}$.

7. Since $(\ln n)^3 < n$ for large n, $\sum \frac{1}{(\ln n)^3}$ diverges to infinity by comparison with $\sum \frac{1}{n}$.

8. $\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$ diverges to infinity by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{3n}$ since $\frac{1}{\ln(3n)} > \frac{1}{3n}$ for $n \ge 1$.

9. Since $\lim_{n\to\infty} \frac{\pi^n}{\pi^n - n^\pi} = \lim \frac{1}{1 - \frac{n^\pi}{\pi^n}} = 1$, the series $\sum \frac{1}{\pi^n - n^\pi}$ converges by comparison with the geometric series $\sum \frac{1}{\pi^n}$.

10. $\sum_{n=0}^{\infty} \frac{1+n}{2+n}$ diverges to infinity since $\lim \frac{1+n}{2+n} = 1 > 0$.

11. $\sum \frac{1+n^{4/3}}{2+n^{5/3}}$ diverges to infinity by comparison with the divergent p-series $\sum \frac{1}{n^{1/3}}$, since

$$\lim_{n\to\infty} \frac{1+n^{4/3}}{2+n^{5/3}} \Big/ \frac{1}{n^{1/3}} = \lim \frac{n^{1/3}+n^{5/3}}{2+n^{5/3}} = 1.$$

12. $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$ diverges to infinity since $\lim \frac{n^2}{1+n\sqrt{n}} = \infty$.

13. $\sum_{n=3}^{\infty} \frac{1}{n \ln n \sqrt{\ln \ln n}}$ diverges to infinity by the integral test, since

$$\int_3^\infty \frac{dt}{t \ln t \sqrt{\ln \ln t}} = \int_{\ln \ln 3}^\infty \frac{du}{\sqrt{u}} = \infty.$$

14. $\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln \ln n)^2}$ converges by the integral test:

$$\int_a^\infty \frac{dt}{t \ln t (\ln \ln t)^2} = \int_{\ln \ln a}^\infty \frac{du}{u^2} < \infty \quad \text{if } \ln \ln a > 0.$$

15. $\sum \frac{1-(-1)^n}{n^4}$ converges by comparison with $\sum \frac{1}{n^4}$, since $0 \le \frac{1-(-1)^n}{n^4} \le \frac{2}{n^4}$.

16. The series

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}} = 0 + \frac{2}{\sqrt{2}} + 0 + \frac{2}{\sqrt{4}} + 0 + \frac{2}{\sqrt{6}} + \cdots$$
$$= 2\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} = \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

diverges to infinity.

17. Since $\frac{1}{2^n(n+1)} < \frac{1}{2^n}$, the series $\sum \frac{1}{2^n(n+1)}$ converges by comparison with the geometric series $\sum \frac{1}{2^n}$.

18. $\sum_{n=1}^{\infty} \frac{n^4}{n!}$ converges by the ratio test since

$$\lim \frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}} = \lim \left(\frac{n+1}{n}\right)^4 \frac{1}{n+1} = 0.$$

19. $\sum \frac{n!}{n^2 e^n}$ diverges to infinity by the ratio test, since

$$\rho = \lim \frac{(n+1)!}{(n+1)^2 e^{n+1}} \cdot \frac{n^2 e^n}{n!} = \frac{1}{e} \lim \frac{n^2}{n+1} = \infty.$$

20. $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$ converges by the ratio test since

$$\lim \frac{(2n+2)!6^{n+1}}{(3n+3)!} \Big/ \frac{(2n)!6^n}{(3n)!}$$
$$= \lim \frac{(2n+2)(2n+1)6}{(3n+3)(3n+2)(3n+1)} = 0.$$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$ converges by the ratio test, since

$$\begin{aligned}\rho &= \lim \frac{\sqrt{n+1}}{3^{n+1}\ln(n+1)} \cdot \frac{3^n \ln n}{\sqrt{n}} \\ &= \frac{1}{3} \lim \sqrt{\frac{n+1}{n}} \cdot \lim \frac{\ln n}{\ln(n+1)} = \frac{1}{3} < 1.\end{aligned}$$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{100}\, 2^n}{\sqrt{n!}}$ converges by the ratio test since

$$\begin{aligned}&\lim \frac{(n+1)^{100}\, 2^{n+1}}{\sqrt{(n+1)!}} \Big/ \frac{n^{100}\, 2^n}{\sqrt{n!}} \\ &= \lim 2 \left(\frac{n+1}{n}\right)^{100} \frac{1}{\sqrt{n+1}} = 0.\end{aligned}$$

23. $\displaystyle\sum \frac{(2n)!}{(n!)^3}$ converges by the ratio test, since

$$\rho = \lim \frac{(2n+2)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)!} = \lim \frac{(2n+2)(2n+1)}{(n+1)^3} = 0 < 1.$$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$ diverges by comparison with the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ since $\displaystyle\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{n+1}$.

25. $\displaystyle\sum \frac{2^n}{3^n - n^3}$ converges by the ratio test since

$$\begin{aligned}\rho &= \lim \frac{2^{n+1}}{3^{n+1} - (n+1)^3} \cdot \frac{3^n - n^3}{2^n} \\ &= \frac{2}{3} \lim \frac{3^n - n^3}{3^n - \dfrac{(n+1)^3}{3}} = \frac{2}{3} \lim \frac{1 - \dfrac{n^3}{3^n}}{1 - \dfrac{(n+1)^3}{3^{n+1}}} = \frac{2}{3} < 1.\end{aligned}$$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$ converges by the ratio test since

$$\lim \frac{(n+1)^{n+1}}{\pi^{(n+1)}(n+1)!} \Big/ \frac{n^n}{\pi^n n!} = \frac{1}{\pi} \lim \left(1 + \frac{1}{n}\right)^n = \frac{e}{\pi} < 1.$$

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1. $\displaystyle\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test (since the terms alternate in sign, decrease in size, and approach 0). However, the convergence is only conditional, since $\displaystyle\sum \frac{1}{\sqrt{n}}$ diverges to infinity.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}$ converges absolutely since $\displaystyle\left|\frac{(-1)^n}{n^2 + \ln n}\right| \le \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

3. $\displaystyle\sum \frac{\cos(n\pi)}{(n+1)\ln(n+1)} = \sum \frac{(-1)^n}{(n+1)\ln(n+1)}$ converges by the alternating series test, but only conditionally since $\displaystyle\sum \frac{1}{(n+1)\ln(n+1)}$ diverges to infinity (by the integral test).

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2^n}=\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a positive, convergent geometric series so must converge absolutely.

5. $\displaystyle\sum\frac{(-1)^n(n^2-1)}{n^2+1}$ diverges since its terms do not approach zero.

6. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$ converges absolutely by the ratio test since

$$\lim\left|\frac{(-2)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-2)^n}\right|=2\lim\frac{1}{n+1}=0.$$

7. $\displaystyle\sum\frac{(-1)^n}{n\pi^n}$ converges absolutely, since, for $n\ge 1$,

$$\left|\frac{(-1)^n}{n\pi^n}\right|\le\frac{1}{\pi^n},$$

and $\displaystyle\sum\frac{1}{\pi^n}$ is a convergent geometric series.

8. $\displaystyle\sum_{n=0}^{\infty}\frac{-n}{n^2+1}$ diverges to $-\infty$ since all terms are negative and $\displaystyle\sum_{n=0}^{\infty}\frac{n}{n^2+1}$ diverges to infinity by comparison with $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n}$.

9. $\displaystyle\sum(-1)^n\frac{20n^2-n-1}{n^3+n^2+33}$ converges by the alternating series test (the terms are ultimately decreasing in size, and approach zero), but the convergence is only conditional since $\displaystyle\sum\frac{20n^2-n-1}{n^3+n^2+33}$ diverges to infinity by comparison with $\displaystyle\sum\frac{1}{n}$.

10. $\displaystyle\sum_{n=1}^{\infty}\frac{100\cos(n\pi)}{2n+3}=\sum_{n=1}^{\infty}\frac{100(-1)^n}{2n+3}$ converges by the alternating series test but only conditionally since

$$\left|\frac{100(-1)^n}{2n+3}\right|=\frac{100}{2n+3}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{100}{2n+3}$ diverges to infinity.

11. $\displaystyle\sum\frac{n!}{(-100)^n}$ diverges since $\displaystyle\lim\frac{n!}{100^n}=\infty$.

12. $\displaystyle\sum_{n=10}^{\infty}\frac{\sin(n+\frac{1}{2})\pi}{\ln\ln n}=\sum_{n=10}^{\infty}\frac{(-1)^n}{\ln\ln n}$ converges by the alternating series test but only conditionally since $\displaystyle\sum_{n=10}^{\infty}\frac{1}{\ln\ln n}$ diverges to infinity by comparison with $\displaystyle\sum_{n=10}^{\infty}\frac{1}{n}$. ($\ln\ln n<n$ for $n\ge 10$.)

1. For $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$ we have $R = \lim \left| \dfrac{\sqrt{n+2}}{\sqrt{n+1}} \right| = 1$. The radius of convergence is 1; the centre of convergence is 0; the interval of convergence is $(-1, 1)$. (The series does not converge at $x = -1$ or $x = 1$.)

2. We have $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$. The centre of convergence is $x = -1$. The radius of convergence is

$$R = \lim \frac{3n}{3(n+1)} = 1.$$

The series converges absolutely on $(-2, 0)$ and diverges on $(-\infty, -2)$ and $(0, \infty)$. At $x = -2$, the series is $\displaystyle\sum_{n=0}^{\infty} 3n(-1)^n$, which diverges. At $x = 0$, the series is $\displaystyle\sum_{n=0}^{\infty} 3n$, which diverges to infinity. Hence, the interval of convergence is $(-2, 0)$.

3. For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x+2}{2}\right)^n$ we have $R = \lim \dfrac{2^{n+1}(n+1)}{2^n n} = 2$. The radius of convergence is 2; the centre of convergence is -2. For $x = -4$ the series is an alternating harmonic series, so converges. For $x = 0$, the series is a divergent harmonic series. Therefore the interval of convergence is $[-4, 0)$.

4. We have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$. The centre of convergence is $x = 0$. The radius of convergence is

$$R = \lim \left| \frac{(-1)^n}{n^4 2^{2n}} \cdot \frac{(n+1)^4 \, 2^{2n+2}}{(-1)^{n+1}} \right| = \lim \left| \left(\frac{n+1}{n}\right)^4 \cdot 4 \right| = 4.$$

At $x = 4$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$, which converges.

At $x = -4$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$, which also converges.

Hence, the interval of convergence is $[-4, 4]$.

5. $\displaystyle\sum_{n=0}^{\infty} n^3(2x-3)^n = \sum_{n=0}^{\infty} 2^n n^3 \left(x - \tfrac{3}{2}\right)^n$. Here $R = \lim \dfrac{2^n n^3}{2^{n+1}(n+1)^3} = \dfrac{1}{2}$. The radius of convergence is $1/2$; the centre of convergence is $3/2$; the interval of convergence is $(1, 2)$.

6. We have $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$. The centre of convergence is $x = 4$. The radius of convergence is

$$R = \lim \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} = \frac{1}{e}.$$

At $x = 4 + \dfrac{1}{e}$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$, which converges.

At $x = 4 - \dfrac{1}{e}$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, which also converges.

Hence, the interval of convergence is $\left[4 - \dfrac{1}{e}, 4 + \dfrac{1}{e}\right]$.

7. For $\sum_{n=0}^{\infty} \dfrac{1+5^n}{n!} x^n$ we have $R = \lim \dfrac{1+5^n}{n!} \cdot \dfrac{(n+1)!}{1+5^{n+1}} = \infty$. The radius of convergence is infinite; the centre of convergence is 0; the interval of convergence is the whole real line $(-\infty, \infty)$.

8. We have $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n \left(x - \frac{1}{4}\right)^n$. The centre of convergence is $x = \frac{1}{4}$. The radius of convergence is

$$R = \lim \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} = \frac{1}{4} \lim \left(\frac{n+1}{n}\right)^n (n+1) = \infty.$$

Hence, the interval of convergence is $(-\infty, \infty)$.

1. $$e^{3x+1} = e \cdot e^{3x} = e\left(\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}\right) = \sum_{n=0}^{\infty}\frac{e3^n x^n}{n!} \qquad \text{(for all } x\text{)}.$$

2. $$\cos(2x^3) = 1 - \frac{(2x^3)^2}{2!} + \frac{(2x^3)^4}{4!} - \frac{(2x^3)^6}{6!} + \cdots = 1 - \frac{2^2x^6}{2!} + \frac{2^4x^{12}}{4!} - \frac{2^6x^{18}}{6!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n 4^n}{(2n)!}x^{6n} \qquad \text{(for all } x\text{)}.$$

3. $$\sin\left(x - \frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} - \cos x \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} - \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\left[-\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right] \qquad \text{(for all } x\text{)}.$$

4. $$\cos(2x - \pi) = -\cos(2x) = -1 + \frac{2^2x^2}{2!} - \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} - \cdots = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}4^n(x)^{2n} \qquad \text{(for all } x\text{)}.$$

5. $$x^2 \sin\frac{x}{3} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{3^{2n+1}(2n+1)!} \qquad \text{(for all } x\text{)}.$$

6. $$\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 + \cos x) = \frac{1}{2}\left(1 + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \qquad \text{(for all } x\text{)}.$$

7. $$\sin x \cos x = \frac{1}{2}\sin(2x) = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n+1}}{(2n+1)!} \qquad \text{(for all } x\text{)}.$$

8. $$\tan^{-1}(5x^2) = (5x^2) - \frac{(5x^2)^3}{3} + \frac{(5x^2)^5}{5} - \frac{(5x^2)^7}{7} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}(5x^2)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{(2n+1)}x^{4n+2} \qquad \left(\text{for } -\frac{1}{\sqrt{5}} \le x \le \frac{1}{\sqrt{5}}\right).$$

9. $$\frac{1+x^3}{1+x^2} = (1+x^3)\left(1 - x^2 + x^4 - x^6 + \cdots\right) = 1 - x^2 + x^3 + x^4 - x^5 - x^6 + x^7 + x^8 - \cdots = 1 - x^2 + \sum_{n=2}^{\infty}(-1)^n\left(x^{2n-1} + x^{2n}\right) \qquad (|x| < 1).$$

10. $$\ln(2 + x^2) = \ln 2\left(1 + \frac{x^2}{2}\right) = \ln 2 + \ln\left(1 + \frac{x^2}{2}\right) = \ln 2 + \left[\frac{x^2}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + \frac{1}{3}\left(\frac{x^2}{2}\right)^3 - \cdots\right] = \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{x^{2n}}{2^n} \qquad (\text{for } -\sqrt{2} \le x \le \sqrt{2}).$$

11. $$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty}\frac{x^n}{n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = 2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1} \qquad (-1 < x < 1).$$

12. $$\frac{e^{2x^2} - 1}{x^2} = \frac{1}{x^2}\left(e^{2x^2} - 1\right) = \frac{1}{x^2}\left(1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \cdots - 1\right) = 2 + \frac{2^2x^2}{2!} + \frac{2^3x^4}{3!} + \frac{2^4x^6}{4!} + \cdots = \sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}x^{2n} \qquad (\text{for all } x \neq 0).$$

13. $$\cosh x - \cos x = \sum_{n=0}^{\infty}\left[1 - (-1)^n\right]\frac{x^{2n}}{(2n)!} = 2\left(\frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{x^{4n+2}}{(4n+2)!} \qquad \text{(for all } x\text{)}.$$

14. $$\sinh x - \sin x = \sum_{n=0}^{\infty}\left[1 - (-1)^n\right]\frac{x^{2n+1}}{(2n+1)!} = 2\left(\frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots\right) = 2\sum_{n=0}^{\infty}\frac{x^{4n+3}}{(4n+3)!} \qquad \text{(for all } x\text{)}.$$

15. Let $t = x + 1$, so $x = t - 1$. We have

$$\begin{aligned} f(x) = e^{-2x} &= e^{-2(t-1)} \\ &= e^2 \sum_{n=0}^{\infty} \frac{(-2)^n t^n}{n!} \\ &= e^2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n (x+1)^n}{n!} \qquad \text{(for all } x\text{)}. \end{aligned}$$

16. Let $y = x - \frac{\pi}{2}$; then $x = y + \frac{\pi}{2}$. Hence,

$$\begin{aligned} \sin x &= \sin\left(y + \frac{\pi}{2}\right) = \cos y \\ &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots \qquad \text{(for all } y\text{)} \\ &= 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n} \qquad \text{(for all } x\text{)}. \end{aligned}$$

17. Let $t = x - \pi$, so $x = t + \pi$. Then

$$\begin{aligned} f(x) = \cos x = \cos(t + \pi) = -\cos t &= -\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} (x - \pi)^{2n} \qquad \text{(for all } x\text{)}. \end{aligned}$$

18. Let $y = x - 3$; then $x = y + 3$. Hence,

$$\begin{aligned} \ln x &= \ln(y+3) = \ln 3 + \ln\left(1 + \frac{y}{3}\right) \\ &= \ln 3 + \frac{y}{3} - \frac{1}{2}\left(\frac{y}{3}\right)^2 + \frac{1}{3}\left(\frac{y}{3}\right)^3 - \frac{1}{4}\left(\frac{y}{3}\right)^4 + \cdots \\ &= \ln 3 + \frac{(x-3)}{3} - \frac{(x-3)^2}{2 \cdot 3^2} + \frac{(x-3)^3}{3 \cdot 3^3} - \frac{(x-3)^4}{4 \cdot 3^4} + \cdots \\ &= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n} (x-3)^n \qquad (0 < x \le 6). \end{aligned}$$

19.

$$\begin{aligned} \ln(2+x) &= \ln[4 + (x-2)] = \ln\left[4\left(1 + \frac{x-2}{4}\right)\right] \\ &= \ln 4 + \ln\left(1 + \frac{x-2}{4}\right) \\ &= \ln 4 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-2)^n}{n 4^n} \qquad (-2 < x \le 6). \end{aligned}$$

20. Let $t = x + 1$. Then $x = t - 1$, and

$$\begin{aligned} e^{2x+3} &= e^{2t+1} = e\, e^{2t} \\ &= e \sum_{n=0}^{\infty} \frac{2^n t^n}{n!} \qquad \text{(for all } t\text{)} \\ &= \sum_{n=0}^{\infty} \frac{e 2^n (x+1)^n}{n!} \qquad \text{(for all } x\text{)}. \end{aligned}$$

21. Let $t = x - (\pi/4)$, so $x = t + (\pi/4)$. Then

$$\begin{aligned} f(x) &= \sin x - \cos x \\ &= \sin\left(t + \frac{\pi}{4}\right) - \cos\left(t + \frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}\Big[(\sin t + \cos t) - (\cos t - \sin t)\Big] \\ &= \sqrt{2} \sin t = \sqrt{2} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \\ &= \sqrt{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1} \qquad \text{(for all } x\text{)}. \end{aligned}$$

22. Let $y = x - \frac{\pi}{8}$; then $x = y + \frac{\pi}{8}$. Thus,

$$\begin{aligned} \cos^2 x &= \cos^2\left(y + \frac{\pi}{8}\right) \\ &= \frac{1}{2}\left[1 + \cos\left(2y + \frac{\pi}{4}\right)\right] \\ &= \frac{1}{2}\left[1 + \frac{1}{\sqrt{2}}\cos(2y) - \frac{1}{\sqrt{2}}\sin(2y)\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[1 - \frac{(2y)^2}{2!} + \frac{(2y)^4}{4!} - \cdots\right] \\ &\quad - \frac{1}{2\sqrt{2}}\left[2y - \frac{(2y)^3}{3!} + \frac{(2y)^5}{5!} - \cdots\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[1 - 2y - \frac{(2y)^2}{2!} + \frac{(2y)^3}{3!}\right. \\ &\quad \left. + \frac{(2y)^4}{4!} - \frac{(2y)^5}{5!} - \cdots\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[1 - 2\left(x - \frac{\pi}{8}\right) - \frac{2^2}{2!}\left(x - \frac{\pi}{8}\right)^2\right. \\ &\quad \left. + \frac{2^3}{3!}\left(x - \frac{\pi}{8}\right)^3 + \frac{2^4}{4!}\left(x - \frac{\pi}{8}\right)^4 - \frac{2^5}{5!}\left(x - \frac{\pi}{8}\right)^5 - \cdots\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \sum_{n=1}^{\infty} (-1)^n \left[\frac{2^{2n-1}}{(2n-1)!}\left(x - \frac{\pi}{8}\right)^{2n-1}\right. \\ &\quad \left. + \frac{2^{2n}}{(2n)!}\left(x - \frac{\pi}{8}\right)^{2n}\right] \qquad \text{(for all } x\text{)}. \end{aligned}$$

23. Let $t = x + 2$, so $x = t - 2$. We have

$$\begin{aligned} f(x) = \frac{1}{x^2} = \frac{1}{(t-2)^2} &= \frac{1}{4\left(1 - \frac{t}{2}\right)^2} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} n \frac{t^{n-1}}{2^{n-1}} \qquad (-2 \le t < 2) \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{n(x+2)^{n-1}}{2^{n-1}} \\ &= \frac{1}{4} \sum_{n=0}^{\infty} \frac{(n+1)(x+2)^n}{2^n} \qquad (-4 < x < 0). \end{aligned}$$

24. Let $y = x - 1$; then $x = y + 1$. Thus,

$$\begin{aligned}
\frac{x}{1+x} &= \frac{1+y}{2+y} = 1 - \frac{1}{2\left(1+\frac{y}{2}\right)} \\
&= 1 - \frac{1}{2}\left[1 - \frac{y}{2} + \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^3 + \cdots\right] \\
&= \frac{1}{2}\left[1 + \frac{y}{2} - \frac{y^2}{2^2} + \frac{y^3}{2^3} - \frac{y^4}{2^4} + \cdots\right] \qquad (-1 < y < 1) \\
&= \frac{1}{2} + \frac{1}{2^2}(x-1) - \frac{1}{2^3}(x-1)^2 + \frac{1}{2^4}(x-1)^3 - \cdots \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}}(x-1)^n \qquad (\text{for } 0 < x < 2).
\end{aligned}$$

25. Let $u = x - 1$. Then $x = 1 + u$, and

$$\begin{aligned}
x \ln x &= (1+u)\ln(1+u) \\
&= (1+u)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} \qquad (-1 < u \le 1) \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^{n+1}}{n}.
\end{aligned}$$

Replace n by $n - 1$ in the last sum.

$$\begin{aligned}
x \ln x &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} + \sum_{n=2}^{\infty}(-1)^{n-2}\frac{u^n}{n-1} \\
&= u + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{n} - \frac{1}{n-1}\right)u^n \\
&= (x-1) + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}(x-1)^n \qquad (0 \le x \le 2).
\end{aligned}$$

26. Let $u = x + 2$. Then $x = u - 2$, and

$$\begin{aligned}
xe^x &= (u-2)e^{u-2} \\
&= (u-2)e^{-2}\sum_{n=0}^{\infty}\frac{u^n}{n!} \qquad (\text{for all } u) \\
&= \sum_{n=0}^{\infty}\frac{e^{-2}u^{n+1}}{n!} - \sum_{n=0}^{\infty}\frac{2e^{-2}u^n}{n!}.
\end{aligned}$$

In the first sum replace n by $n - 1$.

$$\begin{aligned}
xe^x &= \sum_{n=1}^{\infty}\frac{e^{-2}u^n}{(n-1)!} - \sum_{n=0}^{\infty}\frac{2e^{-2}u^n}{n!} \\
&= -\frac{2}{e^2} + \sum_{n=1}^{\infty}\frac{1}{e^2}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)u^n \\
&= -\frac{2}{e^2} + \sum_{n=1}^{\infty}\frac{1}{e^2}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)(x+2)^n \quad (\text{for all } x).
\end{aligned}$$