CALCULUS II – EXERCISE SET – 4 – SOLUTIONS

1. The distance between $(0, 0, 0)$ and $(2, -1, -2)$ is

$$\sqrt{2^2 + (-1)^2 + (-2)^2} = 3 \text{ units.}$$

2. The distance between $(-1, -1, -1)$ and $(1, 1, 1)$ is

$$\sqrt{(1+1)^2 + (1+1)^2 + (1+1)^2} = 2\sqrt{3} \text{ units.}$$

3. The distance between $(1, 1, 0)$ and $(0, 2, -2)$ is

$$\sqrt{(0-1)^2 + (2-1)^2 + (-2-0)^2} = \sqrt{6} \text{ units.}$$

4. The distance between $(3, 8, -1)$ and $(-2, 3, -6)$ is

$$\sqrt{(-2-3)^2 + (3-8)^2 + (-6+1)^2} = 5\sqrt{3} \text{ units.}$$

5. a) The shortest distance from (x, y, z) to the xy-plane is $|z|$ units.

b) The shortest distance from (x, y, z) to the x-axis is $\sqrt{y^2 + z^2}$ units.

6. If $A = (1, 2, 3)$, $B = (4, 0, 5)$, and $C = (3, 6, 4)$, then

$$|AB| = \sqrt{3^2 + (-2)^2 + 2^2} = \sqrt{17}$$
$$|AC| = \sqrt{2^2 + 4^2 + 1^2} = \sqrt{21}$$
$$|BC| = \sqrt{(-1)^2 + 6^2 + (-1)^2} = \sqrt{38}.$$

Since $|AB|^2 + |AC|^2 = 17 + 21 = 38 = |BC|^2$, the triangle ABC has a right angle at A.

7. If $A = (2, -1, -1)$, $B = (0, 1, -2)$, and $C = (1, -3, 1)$, then

$$c = |AB| = \sqrt{(0-2)^2 + (1+1)^2 + (-2+1)^2} = 3$$
$$b = |AC| = \sqrt{(1-2)^2 + (-3+1)^2 + (1+1)^2} = 3$$
$$a = |BC| = \sqrt{(1-0)^2 + (-3-1)^2 + (1+2)^2} = \sqrt{26}.$$

By the Cosine Law,

$$a^2 = b^2 + c^2 - 2bc\cos\angle A$$
$$26 = 9 + 9 - 18\cos\angle A$$
$$\angle A = \cos^{-1}\frac{26-18}{-18} \approx 116.4°.$$

8. If $A = (1, 2, 3)$, $B = (1, 3, 4)$, and $C = (0, 3, 3)$, then

$$|AB| = \sqrt{(1-1)^2 + (3-2)^2 + (4-3)^2} = \sqrt{2}$$
$$|AC| = \sqrt{(0-1)^2 + (3-2)^2 + (3-3)^2} = \sqrt{2}$$
$$|BC| = \sqrt{(0-1)^2 + (3-3)^2 + (3-4)^2} = \sqrt{2}.$$

All three sides being equal, the triangle is equilateral.

9. If $A = (1, 1, 0)$, $B = (1, 0, 1)$, and $C = (0, 1, 1)$, then

$$|AB| = |AC| = |BC| = \sqrt{2}.$$

Thus the triangle ABC is equilateral with sides $\sqrt{2}$. Its area is, therefore,

$$\frac{1}{2} \times \sqrt{2} \times \sqrt{2 - \frac{1}{2}} = \frac{\sqrt{3}}{2} \text{ sq. units.}$$

10. The distance from the origin to $(1, 1, 1, \dots, 1)$ in $\mathbb{R}^n$ is

$$\sqrt{1^2 + 1^2 + 1^2 + \cdots + 1} = \sqrt{n} \text{ units.}$$

11. The point on the x_1-axis closest to $(1, 1, 1, \dots, 1)$ is $(1, 0, 0, \dots, 0)$. The distance between these points is

$$\sqrt{0^2 + 1^2 + 1^2 + \cdots + 1^2} = \sqrt{n-1} \text{ units.}$$

12. $z = 2$ is a plane, perpendicular to the z-axis at $(0, 0, 2)$.

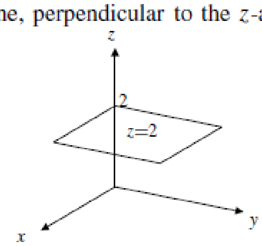

Fig. 10.1.12

13. $y \ge -1$ is the half-space consisting of all points on the plane $y = -1$ (which is perpendicular to the y-axis at $(0, -1, 0)$) and all points on the same side of that plane as the origin.

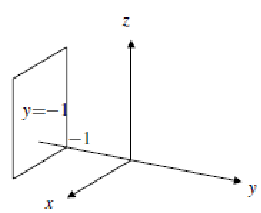

Fig. 10.1.13

14. $z = x$ is a plane containing the y-axis and making 45° angles with the positive directions of the x- and z-axes.

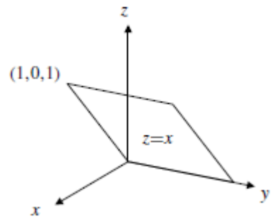

Fig. 10.1.14

15. $x + y = 1$ is a vertical plane (parallel to the z-axis) passing through the points $(1, 0, 0)$ and $(0, 1, 0)$.

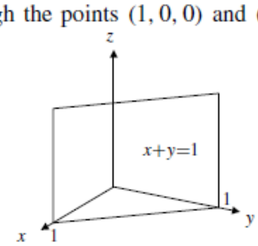

Fig. 10.1.15

16. $x^2 + y^2 + z^2 = 4$ is a sphere centred at the origin and having radius 2 (i.e., all points at distance 2 from the origin).

17. $(x-1)^2 + (y+2)^2 + (z-3)^2 = 4$ is a sphere of radius 2 with centre at the point $(1, -2, 3)$.

18. $x^2 + y^2 + z^2 = 2z$ can be rewritten

$$x^2 + y^2 + (z-1)^2 = 1,$$

and so it represents a sphere with radius 1 and centre at $(0, 0, 1)$. It is tangent to the xy-plane at the origin.

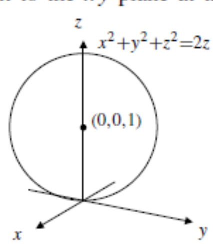

Fig. 10.1.18

19. $y^2+z^2 \le 4$ represents all points inside and on the circular cylinder of radius 2 with central axis along the x-axis (a solid cylinder).

20. $x^2 + z^2 = 4$ is a circular cylindrical surface of radius 2 with axis along the y-axis.

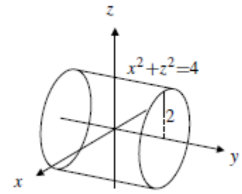

Fig. 10.1.20

21. $z = y^2$ is a "parabolic cylinder" — a surface all of whose cross-sections in planes perpendicular to the x-axis are parabolas.

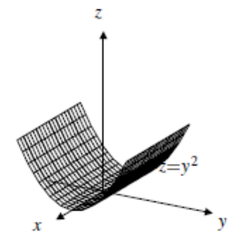

Fig. 10.1.21

22. $z \ge \sqrt{x^2 + y^2}$ represents every point whose distance above the xy-plane is not less than its horizontal distance from the z-axis. It therefore consists of all points inside and on a circular cone with axis along the positive z-axis, vertex at the origin, and semi-vertical angle 45°.

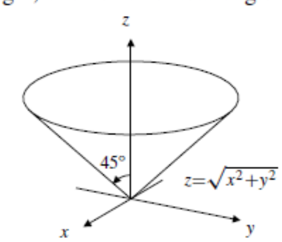

Fig. 10.1.22

23. $x + 2y + 3z = 6$ represents the plane that intersects the coordinate axes at the three points $(6, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$. Only the part of the plane in the first octant is shown in the figure.

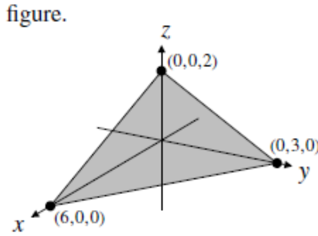

Fig. 10.1.23

1. $A = (-1, 2)$, $B = (2, 0)$, $C = (1, -3)$, $D = (0, 4)$.

(a) $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j}$ (b) $\overrightarrow{BA} = -3\mathbf{i} + 2\mathbf{j}$

(c) $\overrightarrow{AC} = 2\mathbf{i} - 5\mathbf{j}$ (d) $\overrightarrow{BD} = -2\mathbf{i} + 4\mathbf{j}$

(e) $\overrightarrow{DA} = -\mathbf{i} - 2\mathbf{j}$ (f) $\overrightarrow{AB} - \overrightarrow{BC} = 4\mathbf{i} + \mathbf{j}$

(g) $\overrightarrow{AC} - 2\overrightarrow{AB} + 3\overrightarrow{CD} = -7\mathbf{i} + 20\mathbf{j}$

(h) $\frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right) = 2\mathbf{i} - \frac{5}{3}\mathbf{j}$

2. $\mathbf{u} = \mathbf{i} - \mathbf{j}$
$\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

a) $\mathbf{u} + \mathbf{v} = \mathbf{i} + 2\mathbf{k}$
$\mathbf{u} - \mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$
$2\mathbf{u} - 3\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} - 6\mathbf{k}$

b) $|\mathbf{u}| = \sqrt{1+1} = \sqrt{2}$
$|\mathbf{v}| = \sqrt{1+4} = \sqrt{5}$

c) $\hat{\mathbf{u}} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$
$\hat{\mathbf{v}} = \frac{1}{\sqrt{5}}(\mathbf{j} + 2\mathbf{k})$

d) $\mathbf{u} \bullet \mathbf{v} = 0 - 1 + 0 = -1$

e) The angle between $\mathbf{u}$ and $\mathbf{v}$ is
$\cos^{-1}\frac{-1}{\sqrt{10}} \approx 108.4°$.

f) The scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ is
$\frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} = \frac{-1}{\sqrt{5}}$.

g) The vector projection of $\mathbf{v}$ along $\mathbf{u}$ is
$\frac{(\mathbf{v} \bullet \mathbf{u})\mathbf{u}}{|\mathbf{u}|^2} = -\frac{1}{2}(\mathbf{i} - \mathbf{j})$.

3. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$
$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

a) $\mathbf{u} + \mathbf{v} = 6\mathbf{i} - 10\mathbf{k}$
$\mathbf{u} - \mathbf{v} = 8\mathbf{j}$
$2\mathbf{u} - 3\mathbf{v} = -3\mathbf{i} + 20\mathbf{j} + 5\mathbf{k}$

b) $|\mathbf{u}| = \sqrt{9+16+25} = 5\sqrt{2}$
$|\mathbf{v}| = \sqrt{9+16+25} = 5\sqrt{2}$

c) $\hat{\mathbf{u}} = \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
$\hat{\mathbf{v}} = \frac{1}{5\sqrt{2}}(3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})$

d) $\mathbf{u} \bullet \mathbf{v} = 9 - 16 + 25 = 18$

e) The angle between $\mathbf{u}$ and $\mathbf{v}$ is
$\cos^{-1}\frac{18}{50} \approx 68.9°$.

f) The scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ is
$\frac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} = \frac{18}{5\sqrt{2}}$.

g) The vector projection of $\mathbf{v}$ along $\mathbf{u}$ is
$\frac{(\mathbf{v} \bullet \mathbf{u})\mathbf{u}}{|\mathbf{u}|^2} = \frac{9}{25}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$.

4. If $a = (-1, 1)$, $B = (2, 5)$ and $C = (10, -1)$, then $\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{BC} = 8\mathbf{i} - 6\mathbf{j}$. Since $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$, therefore, $\overrightarrow{AB} \perp \overrightarrow{BC}$. Hence, ΔABC has a right angle at B.

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1. $(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 5\mathbf{i} + 13\mathbf{j} + 7\mathbf{k}$

2. $(\mathbf{j} + 2\mathbf{k}) \times (-\mathbf{i} - \mathbf{j} + \mathbf{k}) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

3. If $A = (1, 2, 0)$, $B = (1, 0, 2)$, and $C = (0, 3, 1)$, then $\overrightarrow{AB} = -2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{AC} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, and the area of triangle ABC is

$$\frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{|-4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}|}{2} = \sqrt{6} \text{ sq. units.}$$

4. A vector perpendicular to the plane containing the three given points is

$$(-a\mathbf{i} + b\mathbf{j}) \times (-a\mathbf{i} + c\mathbf{k}) = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}.$$

A unit vector in this direction is

$$\frac{bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}.$$

The triangle has area $\frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.

5. A vector perpendicular to $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$ is

$$\pm(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = \pm(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}),$$

which has length 3. A unit vector in that direction is

$$\pm\left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right).$$

6. A vector perpendicular to $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and to $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is the cross product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -7\mathbf{i} - 6\mathbf{j} - 4\mathbf{k},$$

which has length $\sqrt{101}$. A unit vector with positive $\mathbf{k}$ component that is perpenducular to $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{-1}{\sqrt{101}}\mathbf{u} \times \mathbf{v} = \frac{1}{\sqrt{101}}(7\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}).$$

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2. The plane through $(0, 2, -3)$ normal to $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ has equation

$$4(x - 0) - (y - 2) - 2(z + 3) = 0,$$

or $4x - y - 2z = 4$.

3. The plane through the origin having normal $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ has equation $x - y + 2z = 0$.

4. The plane passing through $(1, 2, 3)$, parallel to the plane $3x + y - 2z = 15$, has equation $3z + y - 2z = 3 + 2 - 6$, or $3x + y - 2z = -1$.

5. The plane through $(1, 1, 0)$, $(2, 0, 2)$, and $(0, 3, 3)$ has normal

$$(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 7\mathbf{i} + 5\mathbf{j} - \mathbf{k}.$$

It therefore has equation

$$7(x - 1) + 5(y - 1) - (z - 0) = 0,$$

or $7x + 5y - z = 12$.

6. The plane passing through $(-2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$ has equation

$$\frac{x}{-2} + \frac{y}{3} + \frac{z}{4} = 1,$$

or $6x - 4y - 3z = -12$.

7. The normal $\mathbf{n}$ to a plane through $(1, 1, 1)$ and $(2, 0, 3)$ must be perpendicular to the vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ joining these points. If the plane is perpendicular to the plane $x + 2y - 3z = 0$, then $\mathbf{n}$ must also be perpendicular to $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, the normal to this latter plane. Hence we can use

$$\mathbf{n} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = -\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

The plane has equation

$$-(x - 1) + 5(y - 1) + 3(z - 1) = 0,$$

or $x - 5y - 3z = -7$.

8. Since $(-2, 0, -1)$ does not lie on $x - 4y + 2z = -5$, the required plane will have an equation of the form

$$2x + 3y - z + \lambda(x - 4y + 2z + 5) = 0$$

for some λ. Thus

$$-4 + 1 + \lambda(-2 - 2 + 5) = 0,$$

so $\lambda = 3$. The required plane is $5x - 9y + 5z = -15$.

9. A plane through the line $x + y = 2$, $y - z = 3$ has equation of the form

$$x + y - 2 + \lambda(y - z - 3) = 0.$$

This plane will be perpendicular to $2x + 3y + 4z = 5$ if

$$(2)(1) + (1 + \lambda)(3) - (\lambda)(4) = 0,$$

that is, if $\lambda = 5$. The equation of the required plane is

$$x + 6y - 5z = 17.$$

15. The line through $(1, 2, 3)$ parallel to $2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ has equations given in vector parametric form by

$$\mathbf{r} = (1 + 2t)\mathbf{i} + (2 - 3t)\mathbf{j} + (3 - 4t)\mathbf{k},$$

or in scalar parametric form by

$$x = 1 + 2t, \quad y = 2 - 3t, \quad z = 3 - 4t,$$

or in standard form by

$$\frac{x-1}{2} = \frac{y-2}{-3} - \frac{z-3}{-4}.$$

16. The line through $(-1, 0, 1)$ perpendicular to the plane $2x - y + 7z = 12$ is parallel to the normal vector $2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ to that plane. The equations of the line are, vector parametric form,

$$\mathbf{r} = (-1 + 2t)\mathbf{i} - t\mathbf{j} + (1 + 7t)\mathbf{k},$$

or in scalar parametric form,

$$x = -1 + 2t, \quad y = -t, \quad z = 1 + 7t,$$

or in standard form

$$\frac{x+1}{2} = \frac{y}{-1} = \frac{z-1}{7}.$$

17. A line parallel to the line with equations

$$x + 2y - z = 2, \qquad 2x - y + 4z = 5$$

is parallel to the vector

$$(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

Since the line passes through the origin, it has equations

$$\mathbf{r} = 7t\mathbf{i} - 6t\mathbf{j} - 5t\mathbf{k} \qquad \text{(vector parametric)}$$
$$x = 7t, \quad y = -6t, \quad z = -5t \qquad \text{(scalar parametric)}$$
$$\frac{x}{7} = \frac{y}{-6} = \frac{z}{-5} \qquad \text{(standard form).}$$

18. A line parallel to $x + y = 0$ and to $x - y + 2z = 0$ is parallel to the cross product of the normal vectors to these two planes, that is, to the vector

$$(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 2(\mathbf{i} - \mathbf{j} - \mathbf{k}).$$

Since the line passes through $(2, -1, -1)$, its equations are, in vector parametric form

$$\mathbf{r} = (2 + t)\mathbf{i} - (1 + t)\mathbf{j} - (1 + t)\mathbf{k},$$

or in scalar parametric form

$$x = 2 + t, \quad y = -(1 + t), \quad z = -(1 + t),$$

or in standard form

$$x - 2 = -(y + 1) = -(z + 1).$$

19. A line making equal angles with the positive directions of the coordinate axes is parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$. If the line passes through the point $(1, 2, -1)$, then it has equations

$$\mathbf{r} = (1 + t)\mathbf{i} + (2 + t)\mathbf{j} + (-1 + t)\mathbf{k} \qquad \text{(vector parametric)}$$
$$x = 1 + t, \quad y = 2 + t, \quad z = -1 + t \qquad \text{(scalar parametric)}$$
$$x - 1 = y - 2 = z + 1 \qquad \text{(standard form).}$$

13. $\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad (0 \le t \le 1)$

$$v = \sqrt{(2t)^2 + (2t)^2 + (3t^2)^2} = t\sqrt{8 + 9t^2}$$

$$\text{Length} = \int_0^1 t\sqrt{8+9t^2}\,dt \quad \text{Let } u = 8 + 9t^2, \quad du = 18t\,dt$$

$$= \frac{1}{18}\frac{2}{3}u^{3/2}\Big|_8^{17} = \frac{17\sqrt{17} - 16\sqrt{2}}{27} \text{ units.}$$

14. $\mathbf{r} = t\mathbf{i} + \lambda t^2\mathbf{j} + t^3\mathbf{k}, \quad (0 \le t \le T)$

$v = \sqrt{1 + (2\lambda t)^2 + 9t^4} = \sqrt{(1+3t^2)^2}$

if $4\lambda^2 = 6$, that is, if $\lambda = \pm\sqrt{3/2}$. In this case, the length of the curve is

$$s(T) = \int_0^T (1 + 3t^2)\,dt = T + T^3.$$

15. $$\text{Length} = \int_1^T \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_1^T \sqrt{4a^2t^2 + b^2 + \frac{c^2}{t^2}}\,dt \text{ units.}$$

If $b^2 = 4ac$ then

$$\text{Length} = \int_1^T \sqrt{\left(2at + \frac{c}{t}\right)^2}\,dt = \int_1^T \left(2at + \frac{c}{t}\right) dt = a(T^2 - 1) + c\ln T \text{ units.}$$

16. $x = a\cos t\sin t = \frac{a}{2}\sin 2t,$

$y = a\sin^2 t = \frac{a}{2}(1 - \cos 2t),$

$z = bt.$

The curve is a circular helix lying on the cylinder

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}.$$

Its length, from $t = 0$ to $t = T$, is

$$L = \int_0^T \sqrt{a^2\cos^2 2t + a^2\sin^2 2t + b^2}\,dt = T\sqrt{a^2 + b^2} \text{ units.}$$

17. $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 2\pi$

$\mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \mathbf{k}$

$v = |\mathbf{v}| = \sqrt{(1+t^2) + 1} = \sqrt{2 + t^2}.$

The length of the curve is

$$L = \int_0^{2\pi} \sqrt{2 + t^2}\,dt \quad \text{Let } t = \sqrt{2}\tan\theta, \quad dt = \sqrt{2}\sec^2\theta\,d\theta$$

$$= 2\int_{t=0}^{t=2\pi} \sec^3\theta\,d\theta$$

$$= \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{t=0}^{t=2\pi}$$

$$= \frac{t\sqrt{2+t^2}}{2} + \ln\left(\frac{\sqrt{2+t^2}}{\sqrt{2}} + \frac{t}{\sqrt{2}}\right)\Big|_0^{2\pi}$$

$$= \pi\sqrt{2 + 4\pi^2} + \ln\left(\sqrt{1 + 2\pi^2} + \sqrt{2}\pi\right) \text{ units.}$$

The curve is called a conical helix because it is a spiral lying on the cone $x^2 + y^2 = z^2$.

19. If $\mathcal{C}$ is the curve

$$x = e^t \cos t, \quad y = e^t \sin t, \quad z = t, \qquad (0 \le t \le 2\pi),$$

then the length of $\mathcal{C}$ is

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + 1}\, dt \\
&= \int_0^{2\pi} \sqrt{2e^{2t} + 1}\, dt \quad \text{Let } 2e^{2t} + 1 = v^2 \\
&\qquad\qquad\qquad\qquad\quad 2e^{2t}\, dt = v\, dv \\
&= \int_{t=0}^{t=2\pi} \frac{v^2\, dv}{v^2 - 1} = \int_{t=0}^{t=2\pi} \left(1 + \frac{1}{v^2 - 1}\right) dv \\
&= \left(v + \frac{1}{2} \ln \left|\frac{v-1}{v+1}\right|\right)\Bigg|_{t=0}^{t=2\pi}
\end{aligned}$$

$$\begin{aligned}
&= \sqrt{2e^{4\pi} + 1} - \sqrt{3} + \frac{1}{2} \ln \frac{\sqrt{2e^{2t} + 1} - 1}{\sqrt{2e^{2t} + 1} + 1}\Bigg|_0^{2\pi} \\
&= \sqrt{2e^{4\pi} + 1} - \sqrt{3} + \ln \frac{\sqrt{2e^{2t} + 1} - 1}{\sqrt{2}e^t}\Bigg|_0^{2\pi} \\
&= \sqrt{2e^{4\pi} + 1} - \sqrt{3} + \ln\left(\sqrt{2e^{4\pi} + 1} - 1\right) \\
&\qquad - 2\pi - \ln(\sqrt{3} - 1) \text{ units.}
\end{aligned}$$

20. $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$

$\mathbf{v} = 3t^2\mathbf{i} + 2t\mathbf{j}$

$v = |\mathbf{v}| = \sqrt{9t^4 + 4t^2} = |t|\sqrt{9t^2 + 4}$

The length L between $t = -1$ and $t = 2$ is

$$L = \int_{-1}^{0} (-t)\sqrt{9t^2 + 4}\, dt + \int_0^2 t\sqrt{9t^2 + 4}\, dt.$$

Making the substitution $u = 9t^2 + 4$ in each integral, we obtain

$$\begin{aligned}
L &= \frac{1}{18}\left[\int_4^{13} u^{1/2}\, du + \int_4^{40} u^{1/2}\, du\right] \\
&= \frac{1}{27}\left(13^{3/2} + 40^{3/2} - 16\right) \quad \text{units.}
\end{aligned}$$

1.

$$\mathbf{r} = t\mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}$$
$$\mathbf{v} = \mathbf{i} - 4t\mathbf{j} + 9t^2\mathbf{k}$$
$$v = \sqrt{1 + 16t^2 + 81t^4}$$
$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{v} = \frac{\mathbf{i} - 4t\mathbf{j} + 9t^2\mathbf{k}}{\sqrt{1 + 16t^2 + 81t^4}}.$$

2.

$$\mathbf{r} = a \sin \omega t\mathbf{i} + a \cos \omega t\mathbf{k}$$
$$\mathbf{v} = a\omega \cos \omega t\mathbf{i} - a\omega \sin \omega t\mathbf{k}, \qquad v = |a\omega|$$
$$\hat{\mathbf{T}} = \operatorname{sgn}(a\omega)\Big[\cos \omega t\mathbf{i} - \sin \omega t\mathbf{k}\Big].$$

3.

$$\mathbf{r} = \cos t \sin t\mathbf{i} + \sin^2 t + \cos t\mathbf{k}$$
$$= \frac{1}{2}\sin 2t\mathbf{i} + \frac{1}{2}(1 - \cos 2t)\mathbf{j} + \cos t\mathbf{k}$$
$$\mathbf{v} = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k}$$
$$v = |\mathbf{v}| = \sqrt{1 + \sin^2 t}$$
$$\hat{\mathbf{T}} = \frac{1}{\sqrt{1 + \sin^2 t}}\Big(\cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k}\Big).$$

4.

$$\mathbf{r} = a \cos t\mathbf{i} + b \sin t\mathbf{j} + t\mathbf{k}$$
$$\mathbf{v} = -a \sin t\mathbf{i} + b \cos t\mathbf{j} + \mathbf{k}$$
$$v = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t + 1}$$
$$\hat{\mathbf{T}} = \frac{\mathbf{v}}{v} = \frac{-a \sin t\mathbf{i} + b \cos t\mathbf{j} + \mathbf{k}}{\sqrt{a^2 \sin^2 t + b^2 \cos^2 t + 1}}.$$