CALCULUS II – EXERCISE SET – 6 – SOLUTIONS

1. $f(x, y) = x^2 + 2y^2 - 4x + 4y$
$f_1(x, y) = 2x - 4 = 0 \quad \text{if } x = 2$
$f_2(x, y) = 4y + 4 = 0 \quad \text{if } y = -1.$
Critical point is $(2, -1)$. Since $f(x, y) \to \infty$ as $x^2 + y^2 \to \infty$, f has a local (and absolute) minimum value at that critical point.

2. $f(x, y) = xy - x + y, \quad f_1 = y - 1, \quad f_2 = x + 1$
$A = f_{11} = 0, \quad B = f_{12} = 1, \quad C = f_{22} = 0.$
Critical point $(-1, 1)$ is a saddle point since $B^2 - AC > 0$.

3. $f(x, y) = x^3 + y^3 - 3xy$
$f_1(x, y) = 3(x^2 - y), \qquad f_2(x, y) = 3(y^2 - x).$
For critical points: $x^2 = y$ and $y^2 = x$. Thus $x^4 - x = 0$, that is, $x(x - 1)(x^2 + x + 1) = 0$. Thus $x = 0$ or $x = 1$. The critical points are $(0, 0)$ and $(1, 1)$. We have

$$A = f_{11}(x, y) = 6x, \qquad B = f_{12}(x, y) = -3,$$
$$C = f_{22}(x, y) = 6y.$$

At $(0, 0)$: $A = C = 0$, $B = -3$. Thus $AC < B^2$, and $(0, 0)$ is a saddle point of f.
At $(1, 1)$: $A = C = 6$, $B = -3$, so $AC > B^2$. Thus f has a local minimum value at $(1, 1)$.

4. $f(x, y) = x^4 + y^4 - 4xy, \quad f_1 = 4(x^3 - y), \quad f_2 = 4(y^3 - x)$

$A = f_{11} = 12x^2, \quad B = f_{12} = -4, \quad C = f_{22} = 12y^2.$
For critical points: $x^3 = y$ and $y^3 = x$. Thus $x^9 = x$, or $x(x^8 - 1) = 0$, and $x = 0$, 1, or -1. The critical points are $(0, 0)$, $(1, 1)$ and $(-1, -1)$.
At $(0, 0)$, $B^2 - AC = 16 - 0 > 0$, so $(0, 0)$ is a saddle point.
At $(1, 1)$ and $(-1, -1)$, $B^2 - AC = 16 - 144 < 0$, $A > 0$, so f has local minima at these points.

5. $f(x, y) = \dfrac{x}{y} + \dfrac{8}{x} - y$
$f_1(x, y) = \dfrac{1}{y} - \dfrac{8}{x^2} = 0 \quad \text{if } 8y = x^2$
$f_2(x, y) = -\dfrac{x}{y^2} - 1 = 0 \quad \text{if } x = -y^2.$
For critical points: $8y = x^2 = y^4$, so $y = 0$ or $y = 2$. $f(x, y)$ is not defined when $y = 0$, so the only critical point is $(-4, 2)$. At $(-4, 2)$ we have

$$A = f_{11} = \frac{16}{x^3} = -\frac{1}{4}, \qquad B = f_{12} = -\frac{1}{y^2} = -\frac{1}{4},$$
$$C = f_{22} = \frac{2x}{y^3} = -1.$$

Thus $B^2 - AC = \dfrac{1}{16} - \dfrac{1}{4} < 0$, and $(-4, 2)$ is a local maximum.

6. $f(x, y) = \cos(x + y), \quad f_1 = -\sin(x + y) = f_2.$
All points on the lines $x + y = n\pi$ (n is an integer) are critical points. If n is even, $f = 1$ at such points; if n is odd, $f = -1$ there. Since $-1 \le f(x, y) \le 1$ at all points in $\mathbb{R}^2$, f must have local and absolute maximum values at points $x + y = n\pi$ with n even, and local and absolute minimum values at such points with n odd.

7. $f(x, y) = x \sin y$. For critical points we have

$$f_1 = \sin y = 0, \qquad f_2 = x \cos y = 0.$$

Since $\sin y$ and $\cos y$ cannot vanish at the same point, the only critical points correspond to $x = 0$ and $\sin y = 0$. They are $(0, n\pi)$, for all integers n. All are saddle points.

8. $f(x, y) = \cos x + \cos y, \quad f_1 = -\sin x, \quad f_2 = -\sin y$
$A = f_{11} = -\cos x, \quad B = f_{12} = 0, \quad C = f_{22} = -\cos y.$
The critical points are points $(m\pi, n\pi)$, where m and n are integers.
Here $B^2 - AC = -\cos(m\pi)\cos(n\pi) = (-1)^{m+n+1}$ which is negative if $m + n$ is even, and positive if $m + n$ is odd.
If $m + n$ is odd then f has a saddle point at $(m\pi, n\pi)$.
If $m + n$ is even and m is odd then f has a local (and absolute) minimum value, -2, at $(m\pi, n\pi)$. If $m + n$ is even and m is even then f has a local (and absolute) maximum value, 2, at $(m\pi, n\pi)$.

23. Let the length, width, and height of the box be x, y, and z, respectively. Then $V = xyz$. The total surface area of the bottom and sides is

$$S = xy + 2xz + 2yz = xy + 2(x+y)\frac{V}{xy} = xy + \frac{2V}{x} + \frac{2V}{y},$$

where $x > 0$ and $y > 0$. Since $S \to \infty$ as $x \to 0+$ or $y \to 0+$ or $x^2 + y^2 \to \infty$, S must have a minimum value at a critical point in the first quadrant. For CP:

$$0 = \frac{\partial S}{\partial x} = y - \frac{2V}{x^2}$$
$$0 = \frac{\partial S}{\partial y} = x - \frac{2V}{y^2}.$$

Thus $x^2 y = 2V = xy^2$, so that $x = y = (2V)^{1/3}$ and $z = V/(2V)^{2/3} = 2^{-2/3}V^{1/3}$.

24. Let the length, width, and height of the box be x, y, and z, respectively. Then $V = xyz$. If the top and side walls cost $\$k$ per unit area, then the total cost of materials for the box is

$$C = 2kxy + kxy + 2kxz + 2kyz = k\left[3xy + 2(x+y)\frac{V}{xy}\right] = k\left[3xy + \frac{2V}{x} + \frac{2V}{y}\right],$$

where $x > 0$ and $y > 0$. Since $C \to \infty$ as $x \to 0+$ or $y \to 0+$ or $x^2 + y^2 \to \infty$, C must have a minimum value at a critical point in the first quadrant. For CP:

$$0 = \frac{\partial C}{\partial x} = k\left(3y - \frac{2V}{x^2}\right)$$
$$0 = \frac{\partial C}{\partial y} = k\left(3x - \frac{2V}{y^2}\right).$$

Thus $3x^2 y = 2V = 3xy^2$, so that $x = y = (2V/3)^{1/3}$ and $z = V/(2V/3)^{2/3} = (9V/4)^{1/3}$.