CALCULUS II – EXERCISE SET – 7 – SOLUTIONS

1. First we observe that $f(x, y) = x^3y^5$ must have a maximum value on the line $x + y = 8$ because if $x \to -\infty$ then $y \to \infty$ and if $x \to \infty$ then $y \to -\infty$. In either case $f(x, y) \to -\infty$.
Let $L = x^3y^5 + \lambda(x + y - 8)$. For CPs of L:

$$0 = \frac{\partial L}{\partial x} = 3x^2y^5 + \lambda$$
$$0 = \frac{\partial L}{\partial y} = 5x^3y^4 + \lambda$$
$$0 = \frac{\partial L}{\partial \lambda} = x + y - 8.$$

The first two equations give $3x^2y^5 = 5x^3y^4$, so that either $x = 0$ or $y = 0$ or $3y = 5x$. If $x = 0$ or $y = 0$ then $f(x, y) = 0$. If $3y = 5x$, then $x + \frac{5}{3}x = 8$, so $8x = 24$ and $x = 3$. Then $y = 5$, and $f(x, y) = 3^35^5 = 84,375$. This is the maximum value of f on the line.

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4. Let $f(x, y, z) = x + y - z$, and define the Lagrangian

$$L = x + y - z + \lambda(x^2 + y^2 + z^2 - 1).$$

Solutions to the constrained problem will be found among the critical points of L. To find these we have

$$0 = \frac{\partial L}{\partial x} = 1 + 2\lambda x,$$
$$0 = \frac{\partial L}{\partial y} = 1 + 2\lambda y,$$
$$0 = \frac{\partial L}{\partial z} = -1 + 2\lambda z,$$
$$0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1.$$

Therefore $2\lambda x = 2\lambda y = -2\lambda z$. Either $\lambda = 0$ or $x = y = -z$. $\lambda = 0$ is not possible. (It implies $0 = 1$ from the first equation.) From $x = y = -z$ we obtain $1 = x^2 + y^2 + z^2 = 3x^2$, so $x = \pm\frac{1}{\sqrt{3}}$. L has critical points at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{-\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
At the first $f = \sqrt{3}$, which is the maximum value of f on the sphere; at the second $f = -\sqrt{3}$, which is the minimum value.

5. The distance D from $(2, 1, -2)$ to (x, y, z) is given by

$$D^2 = (x - 2)^2 + (y - 1)^2 + (z + 2)^2.$$

We can extremize D by extremizing D^2. If (x, y, z) lies on the sphere $x^2+y^2+z^2=1$, we should look for critical points of the Lagrangian

$$L = (x-2)^2 + (y-1)^2 + (z+2)^2 + \lambda(x^2+y^2+z^2-1).$$

Thus

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2(x-2) + 2\lambda x \quad \Leftrightarrow \quad x = \frac{2}{1+\lambda} \\
0 &= \frac{\partial L}{\partial y} = 2(y-1) + 2\lambda y \quad \Leftrightarrow \quad y = \frac{1}{1+\lambda} \\
0 &= \frac{\partial L}{\partial z} = 2(z+2) + 2\lambda z \quad \Leftrightarrow \quad z = \frac{-2}{1+\lambda} \\
0 &= \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1.
\end{aligned}$$

Substituting the solutions of the first three equations into the fourth, we obtain

$$\begin{aligned}
&\frac{1}{(1+\lambda)^2}(4+1+4) = 1 \\
&(1+\lambda)^2 = 9 \\
&1+\lambda = \pm 3.
\end{aligned}$$

Thus we must consider the two points $P = (\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$, and $Q = (-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$ for giving extreme values for D. At P, $D = 2$. At Q, $D = 4$. Thus the greatest and least distances from $(2, 1, -2)$ to the sphere $x^2+y^2+z^2=1$ are 4 units and 2 units respectively.

6. Let $L = x^2+y^2+z^2+\lambda(xyz^2-2)$. For critical points:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2x + \lambda yz^2 \quad \Leftrightarrow \quad -\lambda xyz^2 = 2x^2 \\
0 &= \frac{\partial L}{\partial y} = 2y + \lambda xz^2 \quad \Leftrightarrow \quad -\lambda xyz^2 = 2y^2 \\
0 &= \frac{\partial L}{\partial z} = 2z + 2\lambda xyz \quad \Leftrightarrow \quad -\lambda xyz^2 = z^2 \\
0 &= \frac{\partial L}{\partial \lambda} = xyz^2 - 2.
\end{aligned}$$

From the first three equations, $x^2 = y^2$ and $z^2 = 2x^2$. The fourth equation then gives $x^2y^24z^4 = 4$, or $x^8 = 1$. Thus $x^2 = y^2 = 1$ and $z^2 = 2$.
The shortest distance from the origin to the surface $xyz^2 = 2$ is

$$\sqrt{1+1+2} = 2 \text{ units}.$$

7. We want to minimize $V = \dfrac{4\pi abc}{3}$ subject to the constraint $\dfrac{1}{a^2} + \dfrac{4}{b^2} + \dfrac{1}{c^2} = 1$. Note that abc cannot be zero. Let

$$L = \frac{4\pi abc}{3} + \lambda\left(\frac{1}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} - 1\right).$$

For critical points of L:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial a} = \frac{4\pi bc}{3} - \frac{2\lambda}{a^3} \quad \Leftrightarrow \quad \frac{2\pi abc}{3} = \frac{\lambda}{a^2} \\
0 &= \frac{\partial L}{\partial b} = \frac{4\pi ac}{3} - \frac{8\lambda}{b^3} \quad \Leftrightarrow \quad \frac{2\pi abc}{3} = \frac{4\lambda}{b^2} \\
0 &= \frac{\partial L}{\partial c} = \frac{4\pi ab}{3} - \frac{2\lambda}{c^3} \quad \Leftrightarrow \quad \frac{2\pi abc}{3} = \frac{\lambda}{c^2} \\
0 &= \frac{\partial L}{\partial \lambda} = \frac{1}{a^2} + \frac{4}{b^2} + \frac{1}{c^2} - 1.
\end{aligned}$$

$abc \neq 0$ implies $\lambda \neq 0$, and so we must have

$$\frac{1}{a^2} = \frac{4}{b^2} = \frac{1}{c^2} = \frac{1}{3},$$

so $a = \pm\sqrt{3}$, $b = \pm 2\sqrt{3}$, and $c = \pm\sqrt{3}$.

8. Let $L = x^2+y^2+\lambda(3x^2+2xy+3y^2-16)$. We have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2x + 6\lambda x + 2\lambda y \qquad (A) \\
0 &= \frac{\partial L}{\partial y} = 2y + 6\lambda y + 2\lambda x. \qquad (B)
\end{aligned}$$

Multiplying (A) by y and (B) by x and subtracting we get

$$2\lambda(y^2 - x^2) = 0.$$

Thus, either $\lambda = 0$, or $y = x$, or $y = -x$.
$\lambda = 0$ is not possible, since it implies $x = 0$ and $y = 0$, and the point $(0, 0)$ does not lie on the given ellipse.
If $y = x$, then $8x^2 = 16$, so $x = y = \pm\sqrt{2}$.
If $y = -x$, then $4x^2 = 16$, so $x = -y = \pm 2$.
The points on the ellipse nearest the origin are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. The points farthest from the origin are $(2, -2)$ and $(-2, 2)$. The major axis of the ellipse lies along $y = -x$ and has length $4\sqrt{2}$. The minor axis lies along $y = x$ and has length 4.

9. Let $L = xyz + \lambda(x^2+y^2+z^2-12)$. For CPs of L:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = yz + 2\lambda x \qquad (A) \\
0 &= \frac{\partial L}{\partial y} = xz + 2\lambda y \qquad (B) \\
0 &= \frac{\partial L}{\partial z} = xy + 2\lambda z \qquad (C) \\
0 &= \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-12. \qquad (D)
\end{aligned}$$

Multiplying equations (A), (B), and (C) by x, y, and z, respectively, and subtracting in pairs, we conclude that $\lambda x^2 = \lambda y^2 = \lambda z^2$, so that either $\lambda = 0$ or $x^2 = y^2 = z^2$. If $\lambda = 0$, then (A) implies that $yz = 0$, so $xyz = 0$. If $x^2 = y^2 = z^2$, then (D) gives $3x^2 = 12$, so $x^2 = 4$. We obtain eight points (x, y, z) where each coordinate is either 2 or -2. At four of these points xyz =8, which is the maximum value of xyz on the sphere. At the other four $xyz = -8$, which is the minimum value.

10. Let $L = x + 2y - 3z + \lambda(x^2 + 4y^2 + 9z^2 - 108)$. For CPs of L:

$$0 = \frac{\partial L}{\partial x} = 1 + 2\lambda x \qquad (A)$$
$$0 = \frac{\partial L}{\partial y} = 2 + 8\lambda y \qquad (B)$$
$$0 = \frac{\partial L}{\partial z} = -3 + 18\lambda z \qquad (C)$$
$$0 = \frac{\partial L}{\partial \lambda} = x^2 + 4y^2 + 9z^2 - 108. \qquad (D)$$

From (A), (B), and (C),

$$\lambda = -\frac{1}{2x} = -\frac{2}{8y} = \frac{3}{18z},$$

so $x = 2y = -3z$. From (D):

$$x^2 + 4\left(\frac{x^2}{4}\right) + 9\left(\frac{x^2}{9}\right) = 108,$$

so $x^2 = 36$, and $x = \pm 6$. There are two CPs: $(6, 3, -2)$ and $(-6, -3, 2)$. At the first, $x + 2y - 3z = 18$, the maximum value, and at the second, $x + 2y - 3z = -18$, the minimum value.