CALCULUS II – EXERCISE SET – 8 – SOLUTIONS

1. $\int_0^1 dx \int_0^x (xy + y^2)\, dy$

$= \int_0^1 dx \left(\frac{xy^2}{2} + \frac{y^3}{3} \right) \Big|_{y=0}^{y=x}$

$= \frac{5}{6} \int_0^1 x^3\, dx = \frac{5}{24}.$

2. $\int_0^1 \int_0^y (xy + y^2)\, dx\, dy$

$= \int_0^1 \left(\frac{x^2 y}{2} + xy^2 \right) \Big|_{x=0}^{x=y} dy$

$= \frac{3}{2} \int_0^1 y^3\, dy = \frac{3}{8}.$

3. $\int_0^{\pi} \int_{-x}^{x} \cos y\, dy\, dx$

$= \int_0^{\pi} \sin y \Big|_{y=-x}^{y=x} dx$

$= 2 \int_0^{\pi} \sin x\, dx = -2 \cos x \Big|_0^{\pi} = 4.$

4. $\int_0^2 dy \int_0^y y^2 e^{xy}\, dx$

$$= \int_0^2 y^2\, dy \left(\frac{1}{y} e^{xy} \Big|_{x=0}^{x=y} \right)$$

$$= \int_0^2 y(e^{y^2} - 1)\, dy = \frac{e^{y^2} - y^2}{2} \Big|_0^2 = \frac{e^4 - 5}{2}.$$

5. $\iint_R (x^2 + y^2)\, dA = \int_0^a dx \int_0^b (x^2 + y^2)\, dy$

$$= \int_0^a dx \left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=0}^{y=b}$$

$$= \int_0^a \left(bx^2 + \frac{1}{3} b^3 \right) dx$$

$$= \frac{1}{3} \left(bx^3 + b^3 x \right) \Big|_0^a = \frac{1}{3} (a^3 b + ab^3).$$

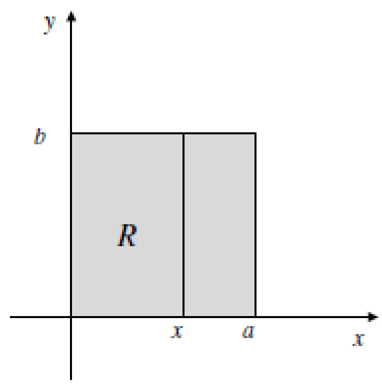

Fig. 14.2.5

6. $\iint_R x^2 y^2\, dA = \int_0^a x^2\, dx \int_0^b y^2\, dy$

$$= \frac{a^3}{3} \frac{b^3}{3} = \frac{a^3 b^3}{9}.$$

7. $$\iint_S (\sin x + \cos y)\, dA$$
$$= \int_0^{\pi/2} dx \int_0^{\pi/2} (\sin x + \cos y)\, dy$$
$$= \int_0^{\pi/2} dx \left(y \sin x + \sin y \right)\Big|_{y=0}^{y=\pi/2}$$
$$= \int_0^{\pi/2} \left(\frac{\pi}{2} \sin x + 1 \right) dx$$
$$= \left(-\frac{\pi}{2} \cos x + x \right)\Big|_0^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

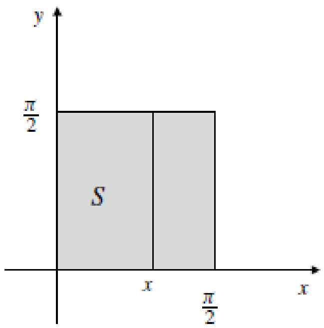

Fig. 14.2.7

8. $$\iint_T (x - 3y)\, dA = \int_0^a dx \int_0^{b(1-(x/a))} (x - 3y)\, dy$$
$$= \int_0^a dx \left(xy - \frac{3}{2} y^2 \right)\Big|_{y=0}^{y=b(1-(x/a))}$$
$$= \int_0^a \left[b\left(x - \frac{x^2}{a} \right) - \frac{3}{2} b^2 \left(1 - \frac{2x}{a} + \frac{x^2}{a^2} \right) \right] dx$$
$$= \left(b\frac{x^2}{2} - \frac{b}{a}\frac{x^3}{3} - \frac{3}{2} b^2 x + \frac{3}{2}\frac{b^2 x^2}{a} - \frac{1}{2}\frac{b^2 x^3}{a^2} \right)\Big|_0^a$$
$$= \frac{a^2 b}{6} - \frac{ab^2}{2}.$$

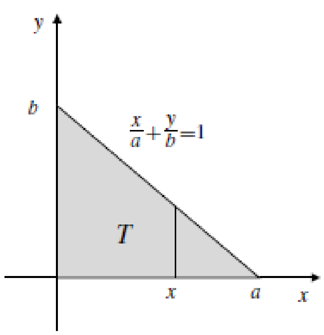

Fig. 14.2.8

9. $$\iint_R xy^2\, dA = \int_0^1 x\, dx \int_{x^2}^{\sqrt{x}} y^2\, dy$$
$$= \int_0^1 x\, dx \left(\frac{1}{3} y^3 \right)\Big|_{y=x^2}^{y=\sqrt{x}}$$
$$= \frac{1}{3} \int_0^1 \left(x^{5/2} - x^7 \right) dx$$
$$= \frac{1}{3} \left(\frac{2}{7} x^{7/2} - \frac{x^8}{8} \right)\Big|_0^1$$
$$= \frac{1}{3} \left(\frac{2}{7} - \frac{1}{8} \right) = \frac{3}{56}.$$

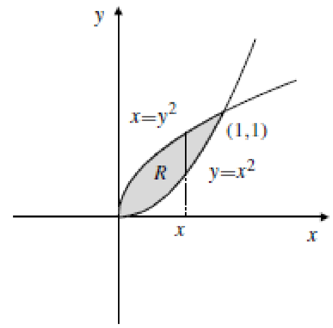

Fig. 14.2.9

10. $$\iint_D x \cos y\, dA$$
$$= \int_0^1 x\, dx \int_0^{1-x^2} \cos y\, dy$$
$$= \int_0^1 x\, dx\, (\sin y)\Big|_{y=0}^{y=1-x^2}$$
$$= \int_0^1 x \sin(1 - x^2)\, dx \quad \text{Let } u = 1 - x^2, \quad du = -2x\, dx$$
$$= -\frac{1}{2} \int_1^0 \sin u\, du = \frac{1}{2} \cos u \Big|_1^0 = \frac{1 - \cos(1)}{2}.$$

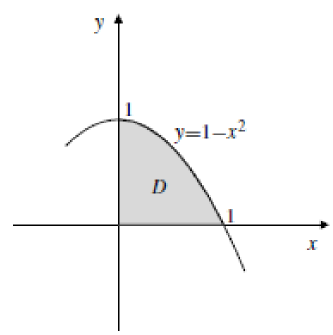

Fig. 14.2.10

11. For intersection: $xy = 1$, $2x + 2y = 5$.
Thus $2x^2 - 5x + 2 = 0$, or $(2x - 1)(x - 2) = 0$. The intersections are at $x = 1/2$ and $x = 2$. We have

$$\iint_D \ln x\, dA = \int_{1/2}^{2} \ln x\, dx \int_{1/x}^{(5/2)-x} dy$$
$$= \int_{1/2}^{2} \ln x \left(\frac{5}{2} - x - \frac{1}{x}\right) dx$$
$$= \int_{1/2}^{2} \ln x \left(\frac{5}{2} - x\right) dx - \frac{1}{2}(\ln x)^2\Big|_{1/2}^{2}$$

$$U = \ln x \qquad dV = \left(\frac{5}{2} - x\right) dx$$
$$dU = \frac{dx}{x} \qquad V = \frac{5}{2}x - \frac{x^2}{2}$$

$$= -\frac{1}{2}\left((\ln 2)^2 - (\ln \tfrac{1}{2})^2\right) + \left(\frac{5}{2}x - \frac{x^2}{2}\right) \ln x\Big|_{1/2}^{2}$$
$$- \int_{1/2}^{2} \left(\frac{5}{2} - \frac{x}{2}\right) dx$$
$$= (5-2)\ln 2 - \left(\frac{5}{4} - \frac{1}{8}\right) \ln\frac{1}{2} - \frac{15}{4} + \frac{15}{16}$$
$$= \frac{33}{8}\ln 2 - \frac{45}{16}.$$

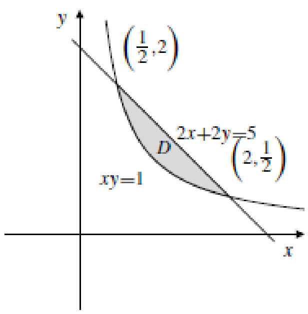

Fig. 14.2.11

12. $$\iint_T \sqrt{a^2 - y^2}\, dA = \int_0^a \sqrt{a^2 - y^2}\, dy \int_y^a dx$$
$$= \int_0^a (a - y)\sqrt{a^2 - y^2}\, dy$$
$$= a \int_0^a \sqrt{a^2 - y^2}\, dy - \int_0^a y\sqrt{a^2 - y^2}\, dy$$

Let $u = a^2 - y^2$
$du = -2y\, dy$

$$= a\frac{\pi a^2}{4} + \frac{1}{2}\int_{a^2}^{0} u^{1/2}\, du$$
$$= \frac{\pi a^3}{4} - \frac{1}{3}u^{3/2}\Big|_0^{a^2} = \left(\frac{\pi}{4} - \frac{1}{3}\right)a^3.$$

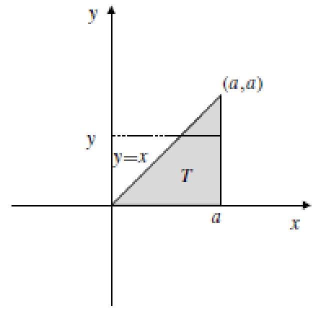

Fig. 14.2.12

13. $$\iint_R \frac{x}{y}e^y\, dA = \int_0^1 \frac{e^y}{y}\, dy \int_y^{\sqrt{y}} x\, dx$$
$$= \frac{1}{2}\int_0^1 (1 - y)e^y\, dy$$
$$U = 1 - y \qquad dV = e^y\, dy$$
$$dU = -dy \qquad V = e^y$$
$$= \frac{1}{2}\left[(1 - y)e^y\Big|_0^1 + \int_0^1 e^y\, dy\right]$$
$$= -\frac{1}{2} + \frac{1}{2}(e - 1) = \frac{e}{2} - 1.$$

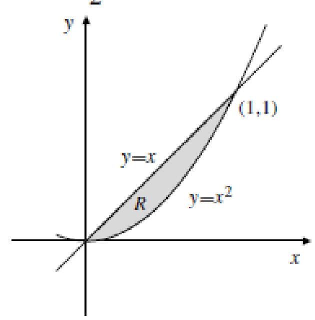

Fig. 14.2.13

14. $$\iint_T \frac{xy}{1 + x^4}\, dA = \int_0^1 \frac{x}{1 + x^4}\, dx \int_0^x y\, dy$$
$$= \frac{1}{2}\int_0^1 \frac{x^3}{1 + x^4}\, dx$$
$$= \frac{1}{8}\ln(1 + x^4)\Big|_0^1 = \frac{\ln 2}{8}.$$

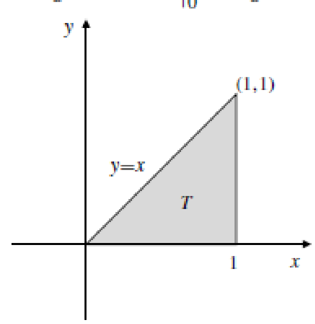

Fig. 14.2.14

15. $\displaystyle\int_0^1 dy \int_y^1 e^{-x^2}\,dx = \int_R e^{-x^2}\,dx \quad (R \text{ as shown})$

$\displaystyle= \int_0^1 e^{-x^2}\,dx \int_0^x dy$

$\displaystyle= \int_0^1 xe^{-x^2}\,dx \qquad \text{Let } u = x^2$

$\qquad\qquad du = 2x\,dx$

$$= \frac{1}{2}\int_0^1 e^{-u}\,du = -\frac{1}{2}e^{-u}\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

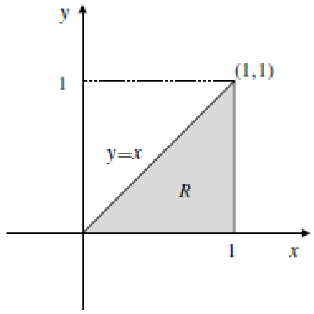

Fig. 14.2.15

16. $\displaystyle\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x}\,dx = \iint_R \frac{\sin x}{x}\,dA \quad (R \text{ as shown})$

$$= \int_0^{\pi/2} \frac{\sin x}{x}\,dx \int_0^x dy = \int_0^{\pi/2} \sin x\,dx = 1.$$

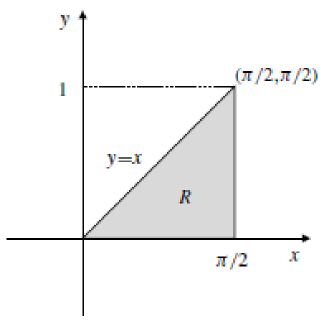

Fig. 14.2.16

17. $\displaystyle\int_0^1 dx \int_x^1 \frac{y^\lambda}{x^2+y^2}\,dy \qquad (\lambda > 0)$

$\displaystyle= \iint_R \frac{y^\lambda}{x^2+y^2}\,dA \quad (R \text{ as shown})$

$\displaystyle= \int_0^1 y^\lambda\,dy \int_0^y \frac{dx}{x^2+y^2}$

$\displaystyle= \int_0^1 y^\lambda\,dy\, \frac{1}{y}\left(\tan^{-1}\frac{x}{y}\right)\Big|_{x=0}^{x=y}$

$$= \frac{\pi}{4}\int_0^1 y^{\lambda-1}\,dy = \frac{\pi y^\lambda}{4\lambda}\Big|_0^1 = \frac{\pi}{4\lambda}.$$

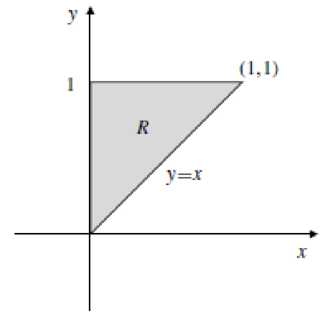

Fig. 14.2.17

18. $\displaystyle\int_0^1 dx \int_x^{x^{1/3}} \sqrt{1-y^4}\,dy$

$\displaystyle= \iint_R \sqrt{1-y^4}\,dA \quad (R \text{ as shown})$

$\displaystyle= \int_0^1 y\sqrt{1-y^4}\,dy - \int_0^1 y^3\sqrt{1-y^4}\,dy$

$\qquad \text{Let } u = y^2 \qquad \text{Let } v = 1 - y^4$

$\qquad du = 2y\,dy \qquad dv = -4y^3\,dy$

$\displaystyle= \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du + \frac{1}{4}\int_1^0 v^{1/2}\,dv$

$$= \frac{1}{2}\left(\frac{\pi}{4} \times 1^2\right) + \frac{1}{6}v^{3/2}\Big|_1^0 = \frac{\pi}{8} - \frac{1}{6}.$$

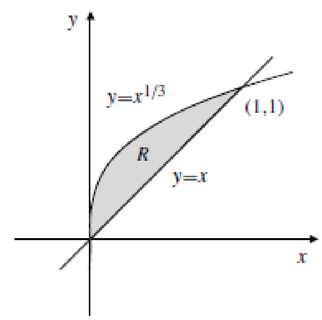

Fig. 14.2.18

19. $\displaystyle V = \int_0^1 dx \int_0^x (1-x^2)\,dy$

$$= \int_0^1 (1-x^2)x\,dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \text{ cu. units.}$$

20. $\displaystyle V = \int_0^1 dy \int_0^y (1-x^2)\,dx$

$$= \int_0^1 \left(y - \frac{y^3}{3}\right) dy = \frac{1}{2} - \frac{1}{12} = \frac{5}{12} \text{ cu. units.}$$

21. $$V = \int_0^1 dx \int_0^{1-x} (1 - x^2 - y^2)\,dy$$
$$= \int_0^1 \left((1 - x^2)y - \frac{y^3}{3} \right)\Bigg|_{y=0}^{y=1-x} dx$$
$$= \int_0^1 \left((1 - x^2)(1 - x) - \frac{(1 - x)^3}{3} \right) dx$$
$$= \int_0^1 \left(\frac{2}{3} - 2x^2 + \frac{4x^3}{3} \right) dx = \frac{2}{3} - \frac{2}{3} + \frac{1}{3} = \frac{1}{3} \text{ cu. units.}$$

22. $z = 1 - y^2$ and $z = x^2$ intersect on the cylinder $x^2 + y^2 = 1$. The volume lying below $z = 1 - y^2$ and above $z = x^2$ is

$$V = \iint_{x^2+y^2 \le 1} (1 - y^2 - x^2)\,dA$$
$$= 4 \int_0^1 dx \int_0^{\sqrt{1-x^2}} (1 - x^2 - y^2)\,dy$$
$$= 4 \int_0^1 dx \left((1 - x^2)y - \frac{y^3}{3} \right)\Bigg|_{y=0}^{y=\sqrt{1-x^2}}$$
$$= \frac{8}{3} \int_0^1 (1 - x^2)^{3/2}\,dx \quad \text{Let } x = \sin u, \quad dx = \cos u\,du$$
$$= \frac{8}{3} \int_0^{\pi/2} \cos^4 u\,du = \frac{2}{3} \int_0^{\pi/2} (1 + \cos 2u)^2\,du$$
$$= \frac{2}{3} \int_0^{\pi/2} \left(1 + 2\cos 2u + \frac{1 + \cos 4u}{2} \right) du$$
$$= \frac{2}{3}\frac{3}{2}\frac{\pi}{2} = \frac{\pi}{2} \text{ cu. units.}$$

23. $$V = \int_1^2 dx \int_0^x \frac{1}{x + y}\,dy$$
$$= \int_1^2 dx \left(\ln(x + y)\Big|_{y=0}^{y=x} \right)$$
$$= \int_1^2 (\ln 2x - \ln x)\,dx = \ln 2 \int_1^2 dx = \ln 2 \text{ cu. units.}$$

24. $$V = \int_0^{\pi^{1/4}} dy \int_0^y x^2 \sin(y^4)\,dx$$
$$= \frac{1}{3} \int_0^{\pi^{1/4}} y^3 \sin(y^4)\,dy \quad \text{Let } u = y^4, \quad du = 4y^3\,dy$$
$$= \frac{1}{12} \int_0^{\pi} \sin u\,du = \frac{1}{6} \text{ cu. units.}$$

25. $$\text{Vol} = \iint_E (1 - x^2 - 2y^2)\,dA$$
$$= 4 \int_0^1 dx \int_0^{\sqrt{(1-x^2)/2}} (1 - x^2 - 2y^2)\,dy$$
$$= 4 \int_0^1 \left(\frac{1}{\sqrt{2}}(1 - x^2)^{3/2} - \frac{2}{3}\frac{(1 - x^2)^{3/2}}{2\sqrt{2}} \right) dx$$
$$= \frac{4\sqrt{2}}{3} \int_0^1 (1 - x^2)^{3/2}\,dx \quad \text{Let } x = \sin\theta, \quad dx = \cos\theta\,d\theta$$
$$= \frac{4\sqrt{2}}{3} \int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{4\sqrt{2}}{3} \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta$$
$$= \frac{\sqrt{2}}{3} \int_0^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2} \right) d\theta$$
$$= \frac{\sqrt{2}}{3} \left[\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta \right]\Bigg|_0^{\pi/2} = \frac{\pi}{2\sqrt{2}} \text{ cu. units.}$$

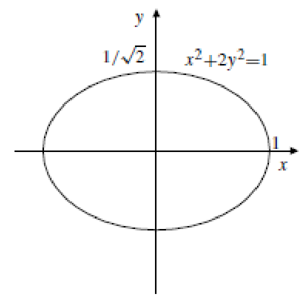

Fig. 14.2.25

26. $$\text{Vol} = \iint_T \left(2 - \frac{x}{a} - \frac{y}{b} \right) dA$$
$$= \int_0^a dx \int_0^{b(1-(x/a))} \left(2 - \frac{x}{a} - \frac{y}{b} \right) dy$$
$$= \int_0^a \left[\left(2 - \frac{x}{a} \right) b \left(1 - \frac{x}{a} \right) - \frac{1}{2b} b^2 \left(1 - \frac{x}{a} \right)^2 \right] dx$$
$$= \frac{b}{2} \int_0^a \left(3 - \frac{4x}{a} + \frac{x^2}{a^2} \right) dx$$
$$= \frac{b}{2} \left(3x - \frac{2x^2}{a} + \frac{x^3}{3a^2} \right)\Bigg|_0^a = \frac{2}{3}ab \text{ cu. units.}$$

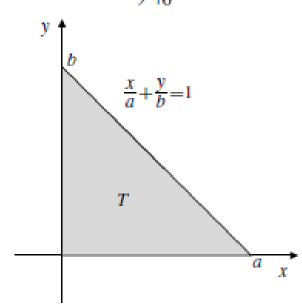

Fig. 14.2.26

27. Vol = 8 × part in the first octant

$$= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy$$
$$= 8\int_0^a (a^2-x^2)\,dx$$
$$= 8\left(a^2x - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{16}{3}a^3 \text{ cu. units.}$$

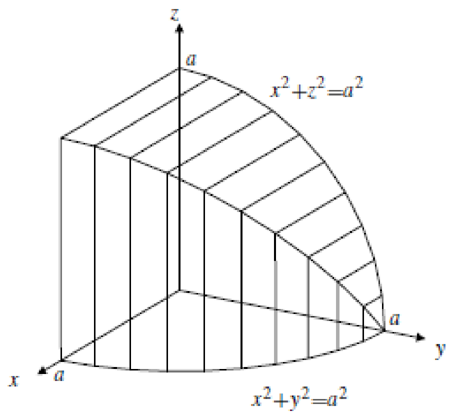

Fig. 14.2.27

28. The part of the plane $z = 8 - x$ lying inside the elliptic cylinder $x^2 = 2y^2 = 8$ lies above $z = 0$. The part of the plane $z = y - 4$ inside the cylinder lies below $z = 0$. Thus the required volume is

$$\text{Vol} = \iint_{x^2+2y^2\le 8}\Big(8 - x - (y-4)\Big)\,dA$$
$$= \iint_{x^2+2y^2\le 8} 12\,dA \qquad \text{(by symmetry)}$$
$$= 12 \times \text{area of ellipse } \frac{x^2}{8} + \frac{y^2}{4} = 1$$
$$= 12 \times \pi(2\sqrt{2})(2) = 48\sqrt{2}\pi \text{ cu. units.}$$

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1. $$\iint_Q e^{-x-y}\,dA = \int_0^\infty e^{-x}\,dx \int_0^\infty e^{-y}\,dy$$
$$= \left(\lim_{R\to\infty}(-e^{-x})\Bigg|_0^R\right)^2 = 1 \text{ (converges)}$$

2. $$\iint_Q \frac{dA}{(1+x^2)(1+y^2)} = \int_0^\infty \frac{dx}{1+x^2}\int_0^\infty \frac{dy}{1+y^2}$$
$$= \left(\lim_{R\to\infty}(\tan^{-1}x)\Bigg|_0^R\right)^2 = \frac{\pi^2}{4}$$
(converges)

3. $\displaystyle \iint_S \frac{y}{1+x^2}\,dA = \int_0^1 y\,dy \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$
$$= \frac{1}{2}\left(\lim_{\substack{S\to-\infty \\ R\to\infty}} \tan^{-1}x)\bigg|_S^R\right) = \frac{\pi}{2} \text{ (converges)}$$

4. $\displaystyle \iint_T \frac{1}{x\sqrt{y}}\,dA = \int_0^1 \frac{dx}{x}\int_x^{2x}\frac{dy}{\sqrt{y}}$
$$= \int_0^1 \frac{2(\sqrt{2x}-\sqrt{x})}{x}\,dx$$
$$= 2(\sqrt{2}-1)\int_0^1 \frac{dx}{\sqrt{x}} = 4(\sqrt{2}-1) \text{ (converges)}$$

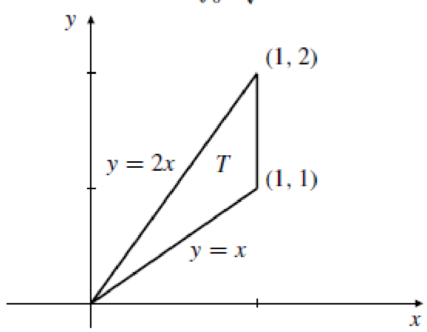

Fig. 14.3.4

5. $\displaystyle \iint_Q \frac{x^2+y^2}{(1+x^2)(1+y^2)}\,dA$
$$= 2\iint_Q \frac{x^2\,dA}{(1+x^2)(1+y^2)} \quad \text{(by symmetry)}$$
$$= 2\int_0^\infty \frac{x^2\,dx}{1+x^2}\int_0^\infty \frac{dy}{1+y^2} = \pi\int_0^\infty \frac{x^2\,dx}{1+x^2},$$
which diverges to infinity, since $x^2/(1+x^2) \ge 1/2$ on $[1,\infty)$.

6. $\displaystyle \iint_H \frac{dA}{1+x+y} = \int_0^\infty dx \int_0^1 \frac{1}{1+x+y}\,dy$
$$= \int_0^\infty \left(\ln(1+x+y)\bigg|_{y=0}^{y=1}\right)dx$$
$$= \int_0^\infty \ln\left(\frac{2+x}{1+x}\right)dx = \int_0^\infty \ln\left(1+\frac{1}{1+x}\right)dx.$$

Since $\displaystyle\lim_{u\to0+}\frac{\ln(1+u)}{u} = 1$, we have $\ln(1+u) \ge u/2$ on some interval $(0, u_0)$. Therefore
$$\ln\left(1+\frac{1}{1+x}\right) \ge \frac{1}{2(1+x)}$$
on some interval (x_0,∞), and
$$\int_0^\infty \ln\left(1+\frac{1}{1+x}\right)dx \ge \int_{x_0}^\infty \frac{1}{2(1+x)}\,dx,$$
which diverges to infinity. Thus the given double integral diverges to infinity by comparison.

7. $\displaystyle \iint_{\mathbb{R}^2} e^{-(|x|+|y|)}\,dA = 4\iint_{\substack{x\ge0\\y\ge0}} e^{-(x+y)}\,dA$
$$= 4\int_0^\infty e^{-x}\,dx\int_0^\infty e^{-y}\,dy$$
$$= 4\left(\lim_{R\to\infty} -e^{-x}\bigg|_0^R\right)^2 = 4$$
(The integral converges.)

8. On the strip S between the parallel lines $x+y=0$ and $x+y=1$ we have $e^{-|x+y|} = e^{-(x+y)} \ge 1/e$. Since S has infinite area,
$$\iint_S e^{-|x+y|}\,dA = \infty.$$
Since $e^{-|x+y|} > 0$ for all (x,y) in $\mathbb{R}^2$, we have
$$\iint_{\mathbb{R}^2} e^{-|x+y|}\,dA > \iint_S e^{-|x+y|}\,dA,$$
and the given integral diverges to infinity.

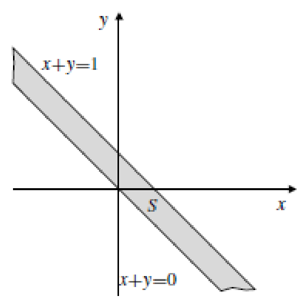

Fig. 14.3.8

9. $\displaystyle \iint_T \frac{1}{x^3}e^{-y/x}\,dA = \int_1^\infty \frac{dx}{x^3}\int_0^x e^{-y/x}\,dy$
$$= \int_1^\infty \frac{dx}{x^3}\left(-xe^{-y/x}\bigg|_{y=0}^{y=x}\right)$$
$$= \left(1-\frac{1}{e}\right)\int_1^\infty \frac{dx}{x^2}$$
$$= \left(1-\frac{1}{e}\right)\lim_{R\to\infty}\left(-\frac{1}{x}\bigg|_1^R\right) = 1-\frac{1}{e}$$
(The integral converges.)

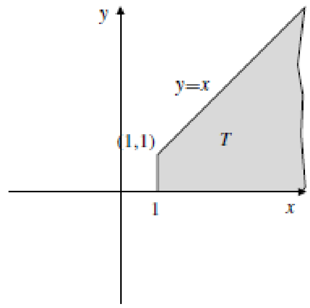

Fig. 14.3.9

10. $$\iint_T \frac{dA}{x^2+y^2} = \int_1^\infty dx \int_0^x \frac{dy}{x^2+y^2}$$
$$= \int_1^\infty dx \left(\frac{1}{x} \tan^{-1} \frac{y}{x} \bigg|_{y=0}^{y=x} \right)$$
$$= \frac{\pi}{4} \int_1^\infty \frac{dx}{x} = \infty$$
(The integral diverges to infinity.)

11. Since $e^{-xy} > 0$ on Q we have

$$\iint_Q e^{-xy}\,dA > \iint_R e^{-xy}\,dA,$$

where R satisfies $1 \le x < \infty$, $0 \le y \le 1/x$. Thus

$$\iint_Q e^{-xy}\,dA > \int_1^\infty dx \int_0^{1/x} e^{-xy}\,dy > \frac{1}{e} \int_1^\infty \frac{dx}{x} = \infty.$$

The given integral diverges to infinity.

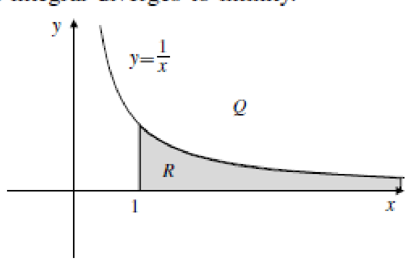

Fig. 14.3.11

12. $$\iint_R \frac{1}{x} \sin\frac{1}{x}\,dA = \int_{2/\pi}^\infty \frac{1}{x} \sin\frac{1}{x}\,dx \int_0^{1/x} dy$$
$$= \int_{2/\pi}^\infty \frac{1}{x^2} \sin\frac{1}{x}\,dx \quad \text{Let } u = 1/x, \quad du = -1/x^2\,dx$$
$$= -\int_{\pi/2}^0 \sin u\,du = \cos u \bigg|_{\pi/2}^0 = 1$$
(The integral converges.)

22. The average value of x^2 over the rectangle R is

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\iint_R x^2\,dA \\
&= \frac{1}{(b-a)(d-c)}\int_a^b x^2\,dx \int_c^d dy \\
&= \frac{1}{b-a}\,\frac{b^3-a^3}{3} = \frac{a^2+ab+b^2}{3}.
\end{aligned}$$

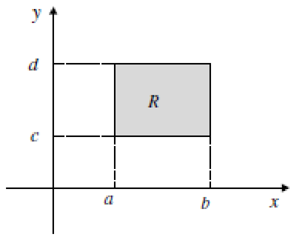

Fig. 14.3.22

23. The average value of x^2+y^2 over the triangle T is

$$\begin{aligned}
&\frac{2}{a^2}\iint_T (x^2+y^2)\,dA \\
&= \frac{2}{a^2}\int_0^a dx \int_0^{a-x}(x^2+y^2)\,dy \\
&= \frac{2}{a^2}\int_0^a dx \left(x^2y+\frac{y^3}{3}\right)\bigg|_{y=0}^{y=a-x} \\
&= \frac{2}{3a^2}\int_0^a \left[3x^2(a-x)+(a-x)^3\right]dx \\
&= \frac{2}{3a^2}\int_0^a \left[a^3-3a^2x+6ax^2-4x^3\right]dx = \frac{a^2}{3}.
\end{aligned}$$

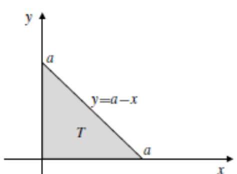

Fig. 14.3.23

24. The area of region R is

$$\int_0^1 (\sqrt{x}-x^2)\,dx = \frac{1}{3} \text{ sq. units.}$$

The average value of $1/x$ over R is

$$\begin{aligned}
3\iint_R \frac{dA}{x} &= 3\int_0^1 \frac{dx}{x}\int_{x^2}^{\sqrt{x}} dy \\
&= 3\int_0^1 \left(x^{-1/2}-x\right)dx = \frac{9}{2}.
\end{aligned}$$

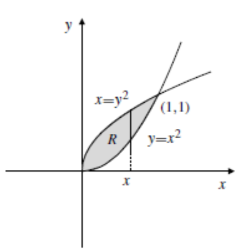

Fig. 14.3.24

1. $$\iint_D (x^2+y^2)\,dA = \int_0^{2\pi} d\theta \int_0^a r^2 r\,dr$$
$$= 2\pi \frac{a^4}{4} = \frac{\pi a^4}{2}$$

2. $$\iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi} d\theta \int_0^a r\,r\,dr = \frac{2\pi a^3}{3}$$

3. $$\iint_D \frac{dA}{\sqrt{x^2+y^2}} = \int_0^{2\pi} d\theta \int_0^a \frac{r\,dr}{r} = 2\pi a$$

4. $$\iint_D |x|\,dA = 4\int_0^{\pi/2} d\theta \int_0^a r\cos\theta\, r\,dr$$
$$= 4\sin\theta\Big|_0^{\pi/2} \frac{a^3}{3} = \frac{4a^3}{3}$$

5. $\iint_D x^2\,dA = \frac{\pi a^4}{4}$; by symmetry the value of this integral is half of that in Exercise 1.

6. $$\iint_D x^2y^2\,dA = 4\int_0^{\pi/2} d\theta \int_0^a r^4\cos^2\theta\sin^2\theta r\,dr$$
$$= \frac{a^6}{6}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$
$$= \frac{a^6}{12}\int_0^{\pi/2} \Big(1-\cos(4\theta)\Big)\,d\theta = \frac{\pi a^6}{24}$$

7. $$\iint_Q y\,dA = \int_0^{\pi/2} d\theta \int_0^a r\sin\theta\, r\,dr$$
$$= (-\cos\theta)\Big|_0^{\pi/2} \frac{a^3}{3} = \frac{a^3}{3}$$

8. $\iint_Q (x+y)\,dA = \frac{2a^3}{3}$; by symmetry, the value is twice that obtained in the previous exercise.

9. $$\iint_Q e^{x^2+y^2}\,dA = \int_0^{\pi/2} d\theta \int_0^a e^{r^2}\, r\,dr$$
$$= \frac{\pi}{2}\left(\frac{1}{2}e^{r^2}\right)\Big|_0^a = \frac{\pi(e^{a^2}-1)}{4}$$

10. $\displaystyle \iint_Q \frac{2xy}{x^2+y^2}\,dA = \int_0^{\pi/2} d\theta \int_0^a \frac{2r^2\sin\theta\cos\theta}{r^2}\,r\,dr$

$$= \frac{a^2}{2}\int_0^{\pi/2} \sin(2\theta)\,d\theta = -\frac{a^2\cos(2\theta)}{4}\bigg|_0^{\pi/2} = \frac{a^2}{2}$$

11. $\displaystyle \iint_S (x+y)\,dA = \int_0^{\pi/3} d\theta \int_0^a (r\cos\theta + r\sin\theta)r\,dr$

$$= \int_0^{\pi/3} (\cos\theta+\sin\theta)\,d\theta \int_0^a r^2\,dr$$

$$= \frac{a^3}{3}(\sin\theta - \cos\theta)\bigg|_0^{\pi/3}$$

$$= \left[\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)-(-1)\right]\frac{a^3}{3} = \frac{(\sqrt{3}+1)a^3}{6}$$

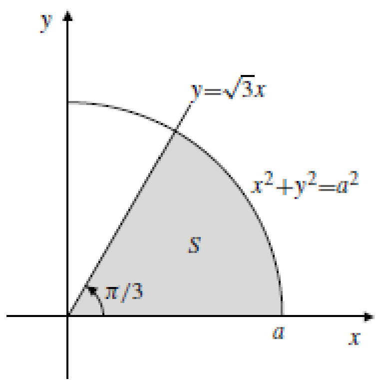

Fig. 14.4.11

12. $\displaystyle \iint_S x\,dA = 2\int_0^{\pi/4} d\theta \int_{\sec\theta}^{\sqrt{2}} r\cos\theta\,r\,dr$

$$= \frac{2}{3}\int_0^{\pi/4} \cos\theta\left(2\sqrt{2}-\sec^3\theta\right)d\theta$$

$$= \frac{4\sqrt{2}}{3}\sin\theta\bigg|_0^{\pi/4} - \frac{2}{3}\tan\theta\bigg|_0^{\pi/4}$$

$$= \frac{4}{3}-\frac{2}{3}=\frac{2}{3}$$

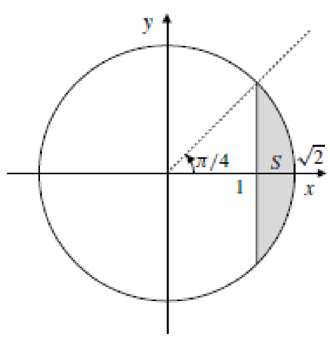

Fig. 14.4.12

13. $\iint_T (x^2+y^2)\,dA = \int_0^{\pi/4} d\theta \int_0^{\sec\theta} r^3\,dr$

$= \frac{1}{4}\int_0^{\pi/4} \sec^4\theta\,d\theta$

$= \frac{1}{4}\int_0^{\pi/4} (1+\tan^2\theta)\sec^2\theta\,d\theta$ Let $u = \tan\theta$
$du = \sec^2\theta\,d\theta$

$= \frac{1}{4}\int_0^1 (1+u^2)\,du$

$= \frac{1}{4}\left(u + \frac{u^3}{3}\right)\Big|_0^1 = \frac{1}{3}$

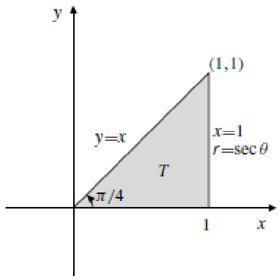

Fig. 14.4.13

14. $\iint_{x^2+y^2\le 1} \ln(x^2+y^2)\,dA = \int_0^{2\pi} d\theta \int_0^1 (\ln r^2) r\,dr$

$= 4\pi \int_0^1 r\ln r\,dr$

$U = \ln r \quad dV = r\,dr$

$dU = \frac{dr}{r} \quad V = \frac{r^2}{2}$

$= 4\pi\left[\frac{r^2}{2}\ln r\Big|_0^1 - \frac{1}{2}\int_0^1 r\,dr\right]$

$= 4\pi\left[0 - 0 - \frac{1}{4}\right] = -\pi$

(Note that the integral is improper, but converges since $\lim_{r\to 0+} r^2\ln r = 0$.)

15. The average distance from the origin to points in the disk D: $x^2+y^2 \le a^2$ is

$$\frac{1}{\pi a^2}\iint_D \sqrt{x^2+y^2}\,dA = \frac{1}{\pi a^2}\int_0^{2\pi} d\theta \int_0^a r^2\,dr = \frac{2a}{3}.$$

16. The annular region R: $0 < a \le \sqrt{x^2+y^2} \le b$ has area $\pi(b^2-a^2)$. The average value of $e^{-(x^2+y^2)}$ over the region is

$$\frac{1}{\pi(b^2-a^2)}\iint_R e^{-(x^2+y^2)}\,dA$$

$= \frac{1}{\pi(b^2-a^2)}\int_0^{2\pi} d\theta \int_a^b e^{-r^2} r\,dr$ Let $u = r^2$
$du = 2r\,dr$

$= \frac{1}{\pi(b^2-a^2)}(2\pi)\frac{1}{2}\int_{a^2}^{b^2} e^{-u}\,du$

$= \frac{1}{b^2-a^2}\left(e^{-a^2} - e^{-b^2}\right).$

1. R is symmetric about the coordinate planes and has volume $8abc$. Thus

$$\iiint_R (1+2x-3y)\,dV = \text{volume of } R + 0 - 0 = 8abc.$$

2. $$\iiint_B xyz\,dV = \int_0^1 x\,dx \int_{-2}^0 y\,dy \int_1^4 z\,dz = \frac{1}{2}\left(-\frac{4}{2}\right)\left(\frac{16-1}{2}\right) = -\frac{15}{2}.$$

3. The hemispherical dome $x^2+y^2+z^2 \le 4$, $z \ge 0$, is symmetric about the planes $x=0$ and $y=0$. Therefore

$$\iiint_D (3+2xy)\,dV = 3\iiint_D dV + 2\iiint_D xy\,dV = 3\times\frac{2}{3}\pi(2^3) + 0 = 16\pi.$$

4. $$\begin{aligned}\iiint_R x\,dV &= \int_0^a x\,dx \int_0^{b\left(1-\frac{x}{a}\right)} dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\\ &= c\int_0^a x\,dx \int_0^{b\left(1-\frac{x}{a}\right)} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy\\ &= c\int_0^a x\left[b\left(1-\frac{x}{a}\right)^2 - \frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right] dx\\ &= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 x\,dx \quad \text{Let } u = 1-(x/a),\ du = -(1/a)\,dx\\ &= \frac{a^2bc}{2}\int_0^1 u^2(1-u)\,du = \frac{a^2bc}{24}.\end{aligned}$$

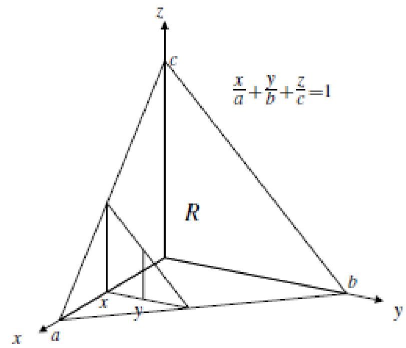

Fig. 14.5.4

5. R is the cube $0 \le x, y, z \le 1$. By symmetry,

$$\iiint_R (x^2+y^2)\,dV = 2\iiint_R x^2\,dV = 2\int_0^1 x^2\,dx \int_0^1 dy \int_0^1 dz = \frac{2}{3}.$$

6. As in Exercise 5,

$$\iiint_R (x^2+y^2+z^2)\,dV = 3\iiint_R x^2\,dV = \frac{3}{3} = 1.$$

7. The set R: $0 \le z \le 1-|x|-|y|$ is a pyramid, one quarter of which lies in the first octant and is bounded by the coordinate planes and the plane $x+y+z=1$. (See the figure.) By symmetry, the integral of xy over R is 0. Therefore,

$$\begin{aligned}\iiint_R (xy+z^2)\,dV &= \iiint_R z^2\,dV\\ &= 4\int_0^1 z^2\,dz \int_0^{1-z} dy \int_0^{1-z-y} dx\\ &= 4\int_0^1 z^2\,dz \int_0^{1-z} (1-z-y)\,dy\\ &= 4\int_0^1 z^2\left[(1-z)^2 - \frac{1}{2}(1-z)^2\right] dz\\ &= 2\int_0^1 (z^2-2z^3+z^4)\,dz = \frac{1}{15}.\end{aligned}$$

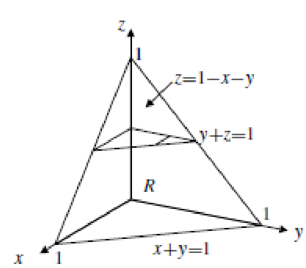

Fig. 14.5.7

8. R is the cube $0 \le x, y, z \le 1$. We have

$$\begin{aligned}
&\iiint_R yz^2 e^{-xyz}\, dV \\
&= \int_0^1 z\,dz \int_0^1 dy \left(-e^{-xyz}\right)\Big|_{x=0}^{x=1} \\
&= \int_0^1 z\,dz \int_0^1 (1 - e^{-yz})\, dy \\
&= \int_0^1 z\left(1 + \frac{1}{z} e^{-yz}\Big|_{y=0}^{y=1}\right) dz \\
&= \frac{1}{2} + \int_0^1 (e^{-z} - 1)\, dz \\
&= \frac{1}{2} - 1 - e^{-z}\Big|_0^1 = \frac{1}{2} - \frac{1}{e}.
\end{aligned}$$

9.
$$\begin{aligned}
\iiint_R \sin(\pi y^3)\, dV &= \int_0^1 \sin(\pi y^3)\, dy \int_0^y dz \int_0^y dx \\
&= \int_0^1 y^2 \sin(\pi y^3)\, dy = -\frac{\cos(\pi y^3)}{3\pi}\Big|_0^1 \\
&= \frac{2}{3\pi}.
\end{aligned}$$

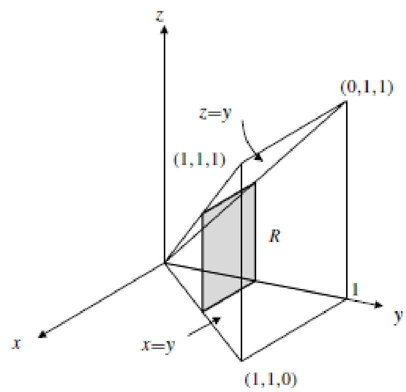

Fig. 14.5.9

10.
$$\begin{aligned}
\iiint_R y\, dV &= \int_0^1 y\,dy \int_{1-y}^1 dz \int_0^{2-y-z} dx \\
&= \int_0^1 y\,dy \int_{1-y}^1 (2 - y - z)\, dz \\
&= \int_0^1 y\,dy \left((2-y)z - \frac{z^2}{2}\right)\Big|_{z=1-y}^{z=1} \\
&= \int_0^1 y\left((2-y)y - \frac{1}{2}\left(1 - (1-y)^2\right)\right) dy \\
&= \int_0^1 \frac{1}{2}\left(2y^2 - y^3\right) dy = \frac{5}{24}.
\end{aligned}$$

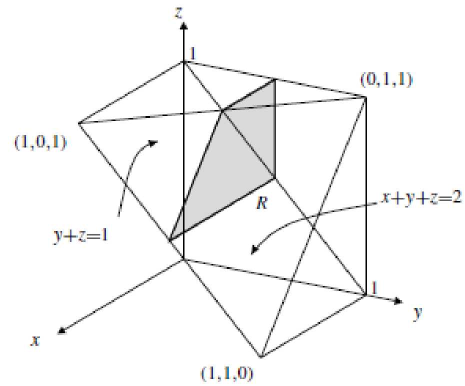

Fig. 14.5.10

11. R is bounded by $z = 1$, $z = 2$, $y = 0$, $y = z$, $x = 0$, and $x = y + z$. These bounds provide an iteration of the triple integral without our having to draw a diagram.

$$\begin{aligned}
&\iiint_R \frac{dV}{(x+y+z)^3} \\
&= \int_1^2 dz \int_0^z dy \int_0^{y+z} \frac{dx}{(x+y+z)^3} \\
&= \int_1^2 dz \int_0^z dy \left(\frac{-1}{2(x+y+z)^2}\right)\Big|_{x=0}^{x=y+z} \\
&= \frac{3}{8}\int_1^2 dz \int_0^z \frac{dy}{(y+z)^2} \\
&= \frac{3}{8}\int_1^2 \left(\frac{-1}{y+z}\right)\Big|_{y=0}^{y=z} dz \\
&= \frac{3}{16}\int_1^2 \frac{dz}{z} = \frac{3}{16}\ln 2.
\end{aligned}$$

12. We have

$$\begin{aligned}
&\iiint_R \cos x \cos y \cos z\, dV \\
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \cos y\, dy \int_0^{\pi - x - y} \cos z\, dz \\
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \cos y\, dy\, (\sin z)\Big|_{z=0}^{z=\pi - x - y} \\
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \cos y \sin(x+y)\, dy
\end{aligned}$$

recall that $\sin a \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$

$$\begin{aligned}
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \frac{1}{2}\left[\sin(x + 2y) + \sin x\right] dy \\
&= \frac{1}{2}\int_0^\pi \cos x\, dx \left[-\frac{\cos(x+2y)}{2} + y \sin x\right]\Big|_{y=0}^{y=\pi - x} \\
&= \frac{1}{2}\int_0^\pi \left(-\frac{\cos x \cos(2\pi - x)}{2} + \frac{\cos^2 x}{2}\right.
\end{aligned}$$

$$+ (\pi - x)\cos x \sin x\Big)\,dx$$

$$= \frac{1}{2}\int_0^\pi \frac{\pi - x}{2}\sin 2x\,dx$$

$$U = \pi - x \quad dV = \sin 2x\,dx$$
$$dU = -dx \quad V = -\frac{\cos 2x}{2}$$

$$= \frac{1}{4}\left[-\frac{\pi - x}{2}\cos 2x\Big|_0^\pi - \frac{1}{2}\int_0^\pi \cos 2x\,dx\right]$$

$$= \frac{1}{8}\left[\pi - \frac{\sin 2x}{2}\Big|_0^\pi\right] = \frac{\pi}{8}.$$

---

27. $$\int_0^1 dz \int_z^1 dx \int_0^x e^{x^3}\,dy$$

$$= \iiint_R e^{x^3}\,dV \qquad (R \text{ is the pyramid in the figure})$$

$$= \int_0^1 e^{x^3} dx \int_0^x dy \int_0^x dz$$

$$= \int_0^1 x^2 e^{x^3}\,dx = \frac{e-1}{3}.$$

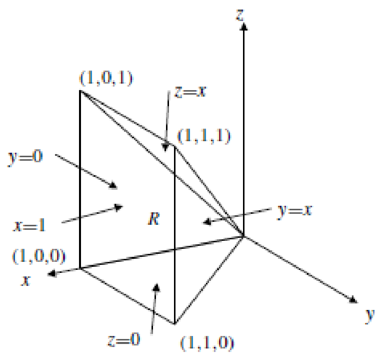

Fig. 14.5.27

28. $$\int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\,dz$$

$$= \iiint_R \frac{\sin(\pi z)}{z(2-z)}\,dV \qquad (R \text{ is the pyramid in the figure})$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\,dz \int_0^z dy \int_0^{1-y} dx$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\,dz \int_0^z (1-y)\,dy$$

$$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\left(z - \frac{z^2}{2}\right)dz$$

$$= \frac{1}{2}\int_0^1 \sin(\pi z)\,dz = \frac{1}{\pi}.$$

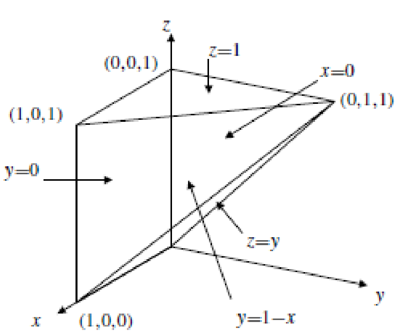

Fig. 14.5.28

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15. $V = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \, d\phi \int_0^a R^2 \, dR$

$= \frac{2\pi a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$ cu. units.

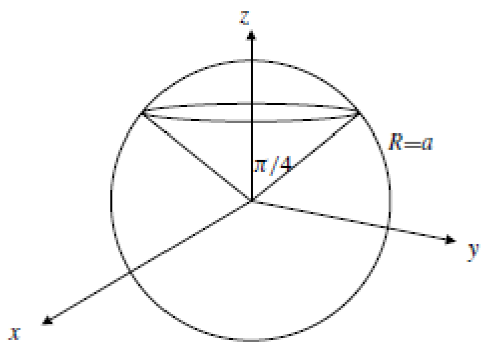

Fig. 14.6.15

16. The surface $z = \sqrt{r}$ intersects the sphere $r^2 + z^2 = 2$ where $r^2 + r - 2 = 0$. This equation has positive root $r = 1$. The required volume is

$$\begin{aligned}
V &= \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{\sqrt{r}}^{\sqrt{2-r^2}} dz \\
&= \int_0^{2\pi} d\theta \int_0^1 \left(\sqrt{2-r^2} - \sqrt{r}\right) r\,dr \\
&= 2\pi \left(\int_0^1 r\sqrt{2-r^2}\,dr - \frac{2}{5}\right) \quad \text{Let } u = 2 - r^2, \ du = -2r\,dr \\
&= \pi \int_1^2 u^{1/2}\,du - \frac{4\pi}{5} \\
&= \frac{2\pi}{3}\left(2\sqrt{2} - 1\right) - \frac{4\pi}{5} = \frac{4\sqrt{2}\pi}{3} - \frac{22\pi}{15} \text{ cu. units.}
\end{aligned}$$

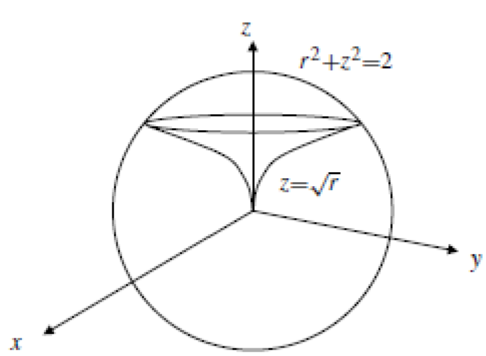

Fig. 14.6.16

17. The paraboloids $z = 10 - r^2$ and $z = 2(r^2 - 1)$ intersect where $r^2 = 4$, that is, where $r = 2$. The volume lying between these surfaces is

$$V = \int_0^{2\pi} d\theta \int_0^2 [10 - r^2 - 2(r^2 - 1)]r\,dr$$
$$= 2\pi \int_0^2 (12r - 3r^3)\,dr = 24\pi \text{ cu. units.}$$

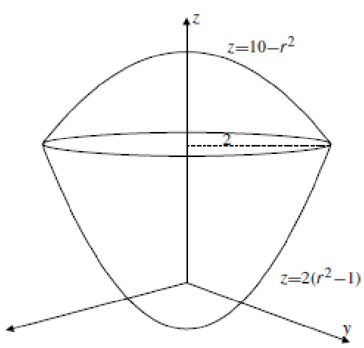

Fig. 14.6.17

18. The paraboloid $z = r^2$ intersects the sphere $r^2 + z^2 = 12$ where $r^4 + r^2 - 12 = 0$, that is, where $r = \sqrt{3}$. The required volume is

$$V = \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left(\sqrt{12 - r^2} - r^2\right) r\,dr$$
$$= 2\pi \int_0^{\sqrt{3}} r\sqrt{12 - r^2}\,dr - \frac{9\pi}{2} \qquad \text{Let } u = 12 - r^2, \; du = -2r\,dr$$
$$= \pi \int_9^{12} u^{1/2}\,du - \frac{9\pi}{2}$$
$$= \frac{2\pi}{3}\left(12^{3/2} - 27\right) - \frac{9\pi}{2} = 16\sqrt{3}\pi - \frac{45\pi}{2} \text{ cu. units.}$$

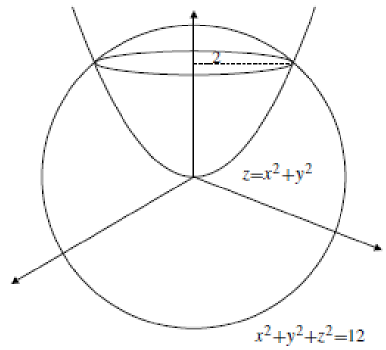

Fig. 14.6.18

19. One half of the required volume V lies in the first octant, inside the cylinder with polar equation $r = 2a\sin\theta$. Thus

$$V = 2\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} (2a - r)r\,dr$$
$$= 2a \int_0^{\pi/2} 4a^2 \sin^2\theta\,d\theta - \frac{2}{3}\int_0^{\pi/2} 8a^3 \sin^3\theta\,d\theta$$
$$= 4a^3 \int_0^{\pi/2} (1 - \cos 2\theta)\,d\theta - \frac{16a^3}{3}\int_0^{\pi/2} \sin^3\theta\,d\theta$$
$$= 2\pi a^3 - \frac{32a^3}{9} \text{ cu. units.}$$

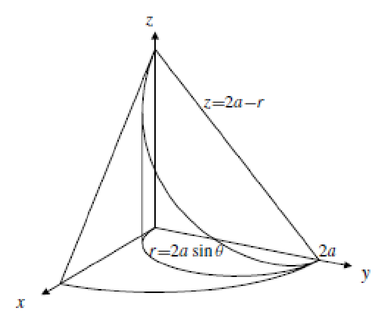

Fig. 14.6.19

20. The required volume V lies above $z = 0$, below $z = 1 - r^2$, and between $\theta = -\pi/4$ and $\theta = \pi/3$. Thus

$$V = \int_{-\pi/4}^{\pi/3} d\theta \int_0^1 (1 - r^2)r\,dr$$
$$= \frac{7\pi}{12}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{7\pi}{48} \text{ cu. units.}$$