CALCULUS I 2.nd MIDTERM
ANSWER KEY

1. Find the following integrals

a)

$$\int \sin^3 x \cos^2 x dx = \int (1-\cos^2 x)\sin x \cos^2 x dx = -\int (u^2 - u^4) du$$
$$\sin^2 x = 1 - \cos^2 x \quad u = \cos x \Longrightarrow du = -\sin x dx$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + c = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

1. b)

$$\int \frac{1}{e^x\sqrt{e^{2x}-9}} dx = ?$$
$$e^x = 3\sec u \Longrightarrow e^x dx = 3\sec u \tan u du$$

$$\int \frac{1}{e^x\sqrt{e^{2x}-9}} dx = \int \frac{\tan u du}{3\sec u(3\tan u)} = \frac{1}{9}\int \cos u du$$

$$= \frac{1}{9}\sin u + C = \frac{1}{9}e^{-x}\sqrt{e^{2x}-9} + C$$

1. c)

$$\int x\ln(x+1)dx = \frac{x^2}{2}\ln(x+1) - \int \frac{x^2}{2}\frac{1}{x+1}dx$$

$$u = \ln(x+1) \Longrightarrow du = \frac{1}{x+1}dx$$
$$dv = xdx \Longrightarrow v = \frac{x^2}{2}$$

$$\int x\ln(x+1)dx = \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{1}{2}\int (x - \frac{x}{x+1})dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{1}{2}\int \frac{x+1-1}{x+1}\ dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{1}{2}\int (\frac{x+1}{x+1} - \frac{1}{x+1})dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{1}{2}\int (1 - \frac{1}{x+1})dx$$

$$= \frac{x^2}{2}\ln(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\ln(x+1) + c$$

1. d)

$$\int \frac{1}{x^3+x}dx = \int \frac{1}{x(x^2+1)}dx \Longrightarrow \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{BX+C}{x^2+1}$$
$$\Longrightarrow A=1, B=-1, C=0 \Longrightarrow \frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$
$$\int \frac{1}{x(x^2+1)}dx = \int \frac{1}{x}dx - \int \frac{x}{x^2+1}dx = \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + c$$

2. Find the indicated limit.

$$\lim_{x\longrightarrow 0}(\frac{1+2^x}{2})^{1/x} =?$$

$$y = (\frac{1+2^x}{2})^{1/x} \Longrightarrow \ln y = \frac{1}{x}\ln(\frac{1+2^x}{2})$$

$$\lim_{x\longrightarrow 0}\frac{\ln(\frac{1+2^x}{2})}{x} = \lim_{x\longrightarrow 0}\frac{\frac{2^{x-1}\ln 2}{(\frac{1+2^x}{2})}}{1} = \frac{1}{2}\ln 2 = \ln 2^{1/2}$$

$$\lim_{x\longrightarrow 0}(\frac{1+2^x}{2})^{1/x} = e^{\ln 2^{1/2}} = \sqrt{2}$$

3. Find the Taylor polynomial of $\sqrt{1+x}$ of the first five terms withbase-point 0. Then use to find approximate value for $\sqrt{1,001}$.

$$f(x) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + ... + \frac{f^{n}(a)}{n!}(x-a)^n$$
$$f(x) = \sqrt{1+x}, \quad a=0,$$

$$
\begin{aligned}
f(x) &= (1+x)^{1/2} \\
f^{'}(x) &= \frac{1}{2}(1+x)^{-1/2} \\
f^{''}(x) &= -\frac{1}{4}(1+x)^{-3/2} \\
f^{'''}(x) &= \frac{3}{8}(1+x)^{-5/2} \\
f^{(4)}(x) &= -\frac{15}{16}(1+x)^{-7/2}
\end{aligned}
$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2!} + \frac{3}{8}\frac{x^3}{3!} - \frac{15}{16}\frac{x^4}{4!}$$

$$= 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}$$

$$\sqrt{1,001} = 1 + \frac{1}{2}(0,001) - \frac{(0,001)^2}{8} + \frac{(0,001)^3}{16} - \frac{5(0,001)^4}{128}$$

$$= 1 + 5.10^{-4} - \frac{1}{8}.10^{-6} + \frac{1}{16}.10^{-9} - \frac{5}{128}.10^{-12}$$

4. Find the points on the parabola $y = \frac{1}{8}x^2$ closest to the point (3,2).

$$(x_0, y_0) = (3,2) \Longrightarrow d = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$d = \sqrt{(x-3)^2 + (\frac{1}{8}x^2 - 2)^2} = [(x-3)^2 + (\frac{1}{8}x^2 - 2)^2]^{1/2}$$

$$d^2 = [(x-3)^2 + (\frac{1}{8}x^2 - 2)^2] \Longrightarrow 2dd_x = 2(x-3) + 2(\frac{1}{8}x^2 - 2)(\frac{1}{4}x)$$

$$d_x = \frac{(x-3) + (\frac{1}{8}x^2 - 2)(\frac{1}{4}x)}{[(x-3)^2 + (\frac{1}{8}x^2 - 2)^2]^{1/2}} \Longrightarrow d_x = 0$$

$$(x-3) + (\frac{1}{8}x^2 - 2)(\frac{1}{4}x) = 0 \Longrightarrow \frac{1}{32}x^3 + \frac{1}{2}x - 3 = 0$$

$$x^3 + 16x - 96 = 0 \Longrightarrow x = 3.5 \Longrightarrow y = \frac{12.25}{8} \approx 1.5 \Longrightarrow (x,y) = (3.5, 1.5)$$