CALCULUS II 1.st MIDTERM EXAM SOLUTIONS

1 (a) $\sum \frac{2^{1/k}\pi^k}{k(k+1)(k+2)}(x-2)^k$

Applying the ratio test $\left|\frac{a_{k+1}}{a_k}\right| = \frac{2^{1/k+1}\pi^{k+1}}{(k+1)(k+2)(k+3)}(x-2)^{k+1}\frac{k(k+1)(k+2)}{2^{1/k}\pi^k(x-2)^k}$

$= (x-2)\pi 2^{\frac{-1}{k(k+1)}}$

$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|(x-2)\pi 2^{\frac{-1}{k(k+1)}}\right| = |(x-2)\pi|$

$|(x-2)\pi| < 1 \implies -1 < (x-2)\pi < 1 \implies \frac{-1}{\pi} < x-2 < \frac{1}{\pi} \implies 2-\frac{1}{\pi} < x < 2+\frac{1}{\pi}$

Interval of convergence $(2-\frac{1}{\pi}, 2+\frac{1}{\pi})$

For $x = 2-\frac{1}{\pi}$ the series $\sum \frac{2^{1/k}(-1)^k}{k(k+1)(k+2)} \implies a_k \geqslant a_{k+1}$

$\implies \lim_{k\to\infty}\frac{2^{1/k}(-1)^k}{k(k+1)(k+2)} = 0$

According to arterning series test this series absolutely convergent.

For $x = 2+\frac{1}{\pi}$ the series $\sum \frac{2^{1/k}}{k(k+1)(k+2)}$ Since $2^{1/k} \leq 2$ if $\sum \frac{2}{k(k+1)(k+2)}$ is convergent then $\sum \frac{2^{1/k}}{k(k+1)(k+2)}$ is also convergent from comparison test.So let us look at the series.$\sum \frac{2}{k(k+1)(k+2)} = \sum(\frac{1}{k} - \frac{2}{(k+1)} + \frac{1}{(k+2)})$ using integral test yields that this series convergent.Because

$\int \frac{1}{k}dk - 2\int \frac{1}{k+1}dk + \int \frac{1}{k+2}dk = \lim_{n\to\infty}(\ln k - 2\ln(k+1) + \ln(k+2))\rfloor_1^n = \lim_{n\to\infty}\frac{k(k+2)}{(k+1)^2}\rfloor_1^n = \ln\frac{4}{3} < \infty$

(b) $\sum(-1)^k \sin(\frac{1}{k})$

The series $\sum a_n$ absolutely convergence if $\sum |a_n|$ converges.

$\lim_{k\to\infty}\left|(-1)^k \sin(\frac{1}{k})\right| = \left|\sin(\frac{1}{k})\right|$ $a_k = \left|\sin(\frac{1}{k})\right| \geq 0 \implies a_{k+1} = \left|\sin(\frac{1}{k+1})\right| \leq a_k \implies \lim_{k\to\infty} a_k = 0$ .So $\sum(-1)^k \sin(\frac{1}{k})$ is absolutely convergent

2 Find an equation for the plane which passes through the point (3,-1,5) and is parallel to the plane 4x+2y-7z+5=0

$P(x_0, y_0, z_0) = P(3, -1, 5)$

$n = Ai + Bj + Ck = 4i + 2j - 7k$

Plane $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$\Longrightarrow 4(x - 3) + 2(y + 1) - 7(z - 5) = 0$

$4x + 2y - 7z + 25 = 0$

3 Find $\dfrac{d^2y}{dx^2}$ at the indicated point without eliminating the parameter

$x(t) = e^t, y(t) = e^{-t}, t = 0$

$$\frac{d^2y}{dx^2} = \frac{1}{\frac{dx}{dt}}\frac{d}{dt}\frac{dy}{dx} \Longrightarrow \frac{dx}{dt} = e^t \ , \ \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-e^{-t}}{e^t} = -e^{-2t}$$

$$\frac{d^2y}{dx^2} = \frac{1}{e^t}\frac{d}{dt}(-e^{-2t}) = \frac{1}{e^t}(2e^{-2t}) = 2e^{-3t}$$

for $t = 0 \Longrightarrow \dfrac{d^2y}{dx^2} = 2\dfrac{1}{e^0} = 2$

4 Calculate the area enclosed by the given curve.Take $a > 0$, $r = a\sqrt{\cos 2\theta}$ from $\theta = -\dfrac{\pi}{4}$ to $\theta = \dfrac{\pi}{4}$

Area $= {}_a^b \dfrac{1}{2} r^2 d\theta$

$$\text{Area} = {}_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(a\sqrt{\cos 2\theta})^2 d\theta = {}_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2} a^2(\cos 2\theta) d\theta = \frac{a^2}{2}{}_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta d\theta$$

$$= \frac{a^2}{2}\frac{1}{2}\sin 2\theta\lfloor_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{a^2}{4}(\sin\frac{\pi}{2} - \sin(-\frac{\pi}{2})) = \frac{a^2}{4}(\sin\frac{\pi}{2} + \sin\frac{\pi}{2})$$

$$= \frac{a^2}{4}(1 + 1) = \frac{a^2}{2}$$

5 Derive the distance from a point to a plane.

Let $S(x, y, z)$ and $P(x_0, y_0, z_0)$, $n = ai + bj + ck$

$PS = (x - x_0, y - y_0, z - z_0)$

$|n| = \sqrt{a^2 + b^2 + c^2}$

The distance from S to the plane is

$$d = \left|PS \ \frac{n}{|n|}\right| = \left|((x - x_0)i + (y - y_0)j + (z - z_0)k)(\frac{a}{\sqrt{a^2 + b^2 + c^2}}i + \frac{b}{\sqrt{a^2 + b^2 + c^2}}j + \frac{c}{\sqrt{a^2 + b^2 +}}\right.$$

$$d = \left|\frac{(x - x_0)a}{\sqrt{a^2 + b^2 + c^2}} + \frac{(y - y_0)b}{\sqrt{a^2 + b^2 + c^2}} + \frac{(z - z_0)c}{\sqrt{a^2 + b^2 + c^2}}\right|$$