Calculus II--------------Midterm # 1--------------------April 5, 2005

Name of the Student:------------------------------No:------------------------------

20 p 1. Draw the graph of the polar curve $r^2 = \cos(2\theta)$ for $0 \le \theta \le \pi$. Then calculate the area bounded by the curve.

Calculate the two areas bounded by the two curves separately.

$$Area_{parametric} = 4 \int_{t=0}^{t=\pi/2} y\,dx = 4 \int_{t=0}^{t=\pi/2} \frac{1}{2}\sin(2t)\cos t\,dt = 4 \int_{t=0}^{t=\pi/2} \sin t \cos^2 t\,dt$$

Let $u = \cos t \Rightarrow du = -\sin t\,dt$

$$\text{Area} = 4\int_0^1 (-u^2)\,du = \frac{4}{3} \quad \text{sq.units}$$

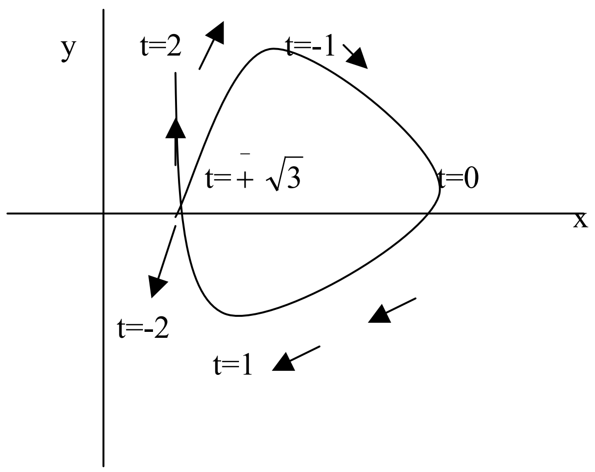
$$Area_{polar} = 4\int_{\theta=0}^{\pi/4} \frac{1}{2} r^2\,d\theta = 2\int_{\theta=0}^{\pi/4} \cos(2\theta)\,d\theta = \sin(2\theta)\,\Big|_0^{\pi/4} = 1 \text{ sq. units}$$

20p 2. Determine the points where the parametric curve has horizontal and vertical tangents, sketch the curve for $-2 \le t \le 2$.

$$x(t) = \frac{4}{1+t^2}, \quad y(t) = t^3 - 3t.$$

$$\frac{dx}{dt} = \frac{-8t}{(1+t^2)^2} = 0 \Rightarrow \text{Vertical tangent } t = 0$$

$$\frac{dy}{dt} = 3t^2 - 3 = 0 \Rightarrow \text{Horizontal tangent } t = 1 \text{ and } t = -1$$



20p 3. Determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$

Ratio test :

$$\rho = \lim_{n\to\infty} \frac{(2n+2)!}{((n+1)!)^3} \frac{(n!)^3}{(2n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^3} = 0 \Rightarrow \text{CONVERGENT}$$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{3^n - n^3}$

Ratio test :

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1} - (n+1)^3} \frac{3^n - n^3}{2^n} = \frac{2}{3} \lim_{n\to\infty} \frac{3^n - n^3}{3^n - \frac{(n+1)^3}{3}} = \frac{2}{3} \lim_{n\to\infty} \frac{1 - \frac{n^3}{3^n}}{1 - \frac{(n+1)^3}{3^{n+1}}}$$

$$\rho = \frac{2}{3} < 1 \Rightarrow \text{CONVERGENT}$$

20p 4. Find the interval of convergence for the power series: $\sum_{n=1}^{\infty} \frac{e^n}{n^3} (4-x)^n$.

The radius of convergence :

$$R = \lim_{n\to\infty} \frac{e^n}{n^3} \frac{(n+1)^3}{e^{n+1}} = \frac{1}{e}$$

Ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{e^{n+1}}{(n+1)^3} (4-x)^{n+1} \frac{n^3}{e^n (4-x)^n} \right| = \lim_{n\to\infty} \left| \frac{en^3}{(n+1)^3} (4-x) \right| < 1$$

for convergence

$$|4-x| < \frac{1}{e} \Rightarrow \frac{-1}{e} < 4-x < \frac{1}{e}$$

$$\Rightarrow \frac{1}{e} > -4 + x > \frac{-1}{e}$$

$$\Rightarrow \frac{1}{e} + 4 > x > 4 - \frac{1}{e}$$

At $x = 4 + \frac{1}{e}$ the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges.

At $x = 4 - \frac{1}{e}$ the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. Interval of convergence

$4 - \frac{1}{e} \le x \le 4 + \frac{1}{e}$

20p 5. Find the first four nonzero terms for the Maclaurin series of

$f(x) = \sin^2 x = \frac{1 - \cos 2x}{2}$.

$$f'(x) = \sin(2x) \;\Rightarrow f'(0) = 0$$
$$f''(x) = 2\cos(2x) \;\Rightarrow f''(0) = 2$$
$$f'''(x) = -4\sin(2x) \;\Rightarrow f'''(0) = 0$$
$$f^{(4)}(x) = -8\cos(2x) \;\Rightarrow f^{(4)}(0) = -8$$
$$f^{(5)}(x) = 16\sin(2x) \;\Rightarrow f^{(5)}(0) = 0$$
$$f^{(6)}(x) = 32\cos(2x) \;\Rightarrow f^{(6)}(0) = 32$$
$$f^{(7)}(x) = -64\sin(2x) \;\Rightarrow f^{(7)}(0) = 0$$
$$f^{(8)}(x) = -128\cos(2x) \;\Rightarrow f^{(8)}(0) = -128$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \ldots$$
$$f(x) = 0 + 0x + \frac{2}{2!}x^2 + \frac{1}{3!}0x^3 + \frac{-8}{4!}x^4 + \frac{1}{5!}0x^5 + \frac{32}{6!}x^6 + \frac{1}{7!}0x^7 + \frac{-128}{8!}x^8 + \ldots$$
$$f(x) = \frac{2}{2!}x^2 + \frac{-2^3}{4!}x^4 + \frac{2^5}{6!}x^6 + \frac{-2^7}{8!}x^8 + \ldots$$