Advanced Calculus

Part I

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These slides[1] are based mainly on the textbooks:

J. W. Brown and R. V. Churchill, Complex Variables and Applications, Sixth Edition, McGrawHill

S. L. Ross, Differential Equations, 3rd Edition, Wiley

Some old exam questions and their solutions are available on DYS.
You may prepare and bring an A4 size formula sheet to the exams.
You may bring a calculator to the exams.

[1] These slides are intended for educational use only; not for sale under any circumstances.

A fact: I always post the solution key on DYS in few minutes after I announce the midterm grades.

FAQ Can I see my graded exam paper?

Ans. Yes, but not before the first lecture following the exam. In that lecture I announce the times when you can see your exam paper.

Definition

A complex number z is an ordered pair $z = (x, y)$ of real numbers x and y with operations of addition and multiplication.

Identify the pairs $(x, 0)$ with real numbers x.
$\therefore$ Complex numbers include the real numbers as a subset.

Complex numbers of the form $(0, y)$ are called imaginary numbers. In $z = (x, y)$, x is known as the real part and y is known as the imaginary part of z.

Related functions:

$$\mathrm{Re}\,(z) = x, \quad \mathrm{Im}\,(z) = y$$

Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Define:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2 \ , \ y_1 + y_2)$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2 \ , \ y_1 x_2 + x_1 y_2)$$

Note that

$$(x, y) = (x, 0) + (0, 1)(y, 0) \tag{1}$$

Let x denote $(x, 0)$ and let i denote the pure imaginary number $(0, 1)$ we can rewrite (1) as

$$(x, y) = x + iy \tag{2}$$

Note that

$$i^2 = (0,1)(0,1) = (-1,0) = -1$$

In view of expression (2) addition and multiplication can be written as

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

Note that

$$(x,y)(a,0) = (ax, ay)$$

$$(a,0)(x,y) = (ax, ay)$$

We therefore define

$$a(x,y) = (ax, ay)$$

$$(x,y)a = (ax, ay)$$

Commutative laws:

$$z_1 + z_2 = z_2 + z_1 \ , \quad z_1 z_2 = z_2 z_1$$

Associative laws:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \ , \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

Distributive law:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

The additive identity $0 = (0, 0)$ and multiplicative identity $1 = (1, 0)$ satisfy

$$z + 0 = z \quad \text{and} \quad z \cdot 1 = z$$

for each complex number z

Additive inverse of z is $(-z)$. That is $z + (-z) = 0$.
Multiplicative inverse z^{-1} of z can be computed as

$$zz^{-1} = 1$$

Let $z = (x, y)$ and $z^{-1} = (u, v)$; then

$$(x, y)(u, v) = 1$$

$$(xu - yv, yu + xv) = (1, 0)$$

$$\rightarrow \quad (xu - yv) = 1 \quad \text{and} \quad (yu + xv) = 0$$

$$\rightarrow \quad u = \frac{x}{x^2 + y^2}\,, \quad v = \frac{-y}{x^2 + y^2}$$

The multiplicative inverse of $z = (x, y)$ is, then,

$$z^{-1} = \left(\frac{x}{x^2 + y^2}\,, \frac{-y}{x^2 + y^2} \right)$$

If a product $z_1 z_2$ is zero, then so is at least one of the factors z_1 and z_2.

For the matrices A and B, the product $AB = \underline{0}$ does not imply $A = \underline{0}$ or $B = \underline{0}$. For instance,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Suppose that $z_1 z_2 = 0$ and $z_1 \neq 0$. We will show that $z_2 = 0$. The inverse z_1^{-1} exists, and according to the definition of multiplication, any complex number times zero is zero. Hence

$$z_2 = 1 \cdot z_2 = (z_1^{-1} z_1) z_2 = z_1^{-1}(z_1 z_2) = z_1^{-1} \cdot 0 = 0$$

Division by a nonzero complex number is defined as:

$$\frac{z_1}{z_2} = z_1 z_2^{-1}$$

If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ then

$$\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right), \ z_2 \neq 0$$

The quotient z_1/z_2 is not defined when $z_2 = 0$

Useful Identities

$$\frac{1}{z_1} = z_1^{-1}$$

$$\frac{1}{z_1 z_2} = \frac{1}{z_1}\frac{1}{z_2}$$

$$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}$$

$$\frac{z_1 z_2}{z_3 z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)$$

Example

$$\left(\frac{1}{2-3i}\right)\left(\frac{1}{1+i}\right)$$

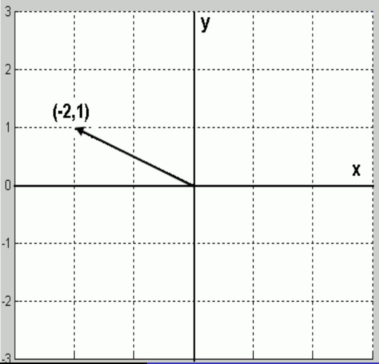
$$=\frac{1}{5-i}=\frac{1}{5-i}\left(\frac{5+i}{5+i}\right)=\frac{5+i}{26}=\frac{5}{26}+i\frac{1}{26}$$

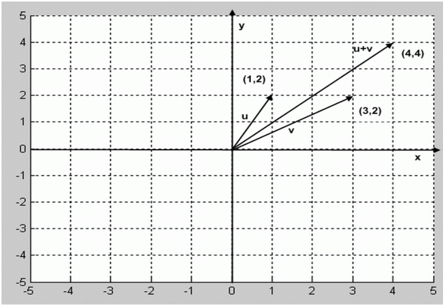
1) Verify that
a) $(\sqrt{2}-i)-i(1-\sqrt{2}i)=-2i$
b) $(2,-3)(-2,1)=(-1,8)$
2) Verify that each of the two numbers $z=1\mp i$ satisfies the equation $z^2-2z+2=0$
3) Solve the equation $z^2+z+1=0$ for $z=(x,y)$ by writing

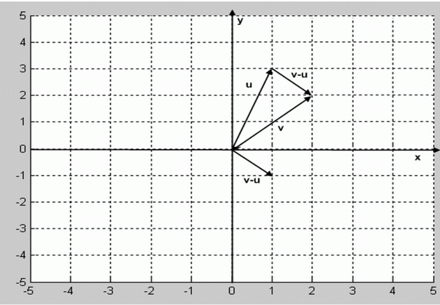
$$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$

and then solving a pair of simultaneous equation in x and y.
Ans. $z=\left(-\frac{1}{2},\mp\frac{\sqrt{3}}{2}\right)$

View $z = x + iy$ as a point whose cartesian coordinates (x, y).
Example: The number $-2 + i$ is represented by the point $(-2, 1)$.
The number z can also be thought of as a vector from the origin to the point (x, y).
The xy plane may be called the complex plane, or the z plane.
The x axis is called the real axis, the y axis is called the imaginary axis.







$u + v = (1 + 2i) + (3 + 2i)$ yields $4 + 4i$; and
$v - u = (2 + 2i) - (1 + 3i)$ yields $1 - i$.

The modulus or absolute value of a complex number $z = x + iy$ is defined as the nonnegative real number $\sqrt{x^2 + y^2}$ and is denoted by $|z|$; that is

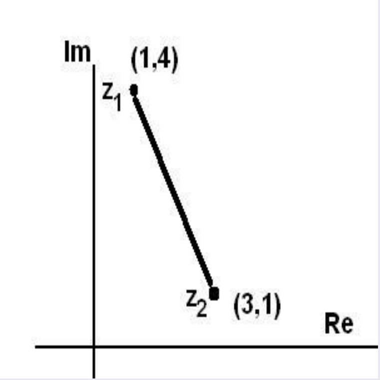
$$|z| = \sqrt{x^2 + y^2}$$

Geometrically the number $|z|$ is the distance between the point

(x, y) and the origin.

$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is the distance between z_1 and z_2.

Example



$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(1-3)^2 + (4-1)^2} = \sqrt{13}$$

Representing Circle in the Complex Plane

The points lying on the circle with center z_0 and radius R satisfy the equation $|z - z_0| = R$.
Note that

$$|(x+iy)-(x_0+iy_0)| = |(x-x_0)+i(y-y_0)| = \sqrt{(x-x_0)^2+(y-y_0)^2} = R$$

Example

The points z satisfying equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius $R = 2$. The equation may be written as $|z - (1 - 3i)| = 2$.

$$|(x+iy)-(1-3i)| = |x-1+i(y+3)| = \sqrt{(x-1)^2+(y+3)^2} = 2$$

Complex Conjugate

The complex conjugate of $z = x + iy$ is the complex number $x - iy$ and is denoted by $\bar{z}$; that is

$$\bar{z} = x - iy$$

Useful Identities

$$\overline{\overline{z}} = z\ , \quad \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}\ , \quad \overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$$

$$\frac{\overline{z_1}}{\overline{z_2}} = \overline{\left(\frac{z_1}{z_2}\right)}\ , \quad z\overline{z} = |z|^2$$

$$|z_1 z_2| = |z_1||z_2|\ , \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\ , \quad \operatorname{Re} z = \frac{z + \overline{z}}{2}$$

$$\operatorname{Im} z = \frac{z - \overline{z}}{2i},\ \operatorname{Re} z \leq |z|$$

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{3}$$

Proof

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z_1} + \overline{z_2})$$

$$= z_1\overline{z_1} + (z_1\overline{z_2} + z_2\overline{z_1}) + z_2\overline{z_2}$$

But $z_1\overline{z_2} + z_2\overline{z_1} = z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2\ \mathrm{Re}(z_1\overline{z_2}) \leq 2|z_1\overline{z_2}| =$
$2|z_1||\overline{z_2}| = 2|z_1||z_2|$

$$\text{and so} \quad |z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$\text{or} \quad |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

Since moduli are nonnegative the inequality (3) follows.

Generalization of the triangle inequality

$$|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$$

Exercises

1. Show that $\overline{(2+i)^2} = 3 - 4i$
2. Show that $\left|(2\overline{z}+5)(\sqrt{2}-i)\right| = \sqrt{3}\,|2z+5|$

Polar Form

Let $z = x + iy$ be a complex number. Its **polar** representation is:

$$z = r\,(\cos\theta + i\sin\theta),$$

where r is the **modulus** of z and θ is the **argument** of z. Modulus is not allowed to be negative. The argument is always in radians!!! We have

$$r = \sqrt{x^2 + y^2} \geq 0$$

and θ is any angle such that

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} \quad \& \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} \tag{4}$$

The argument of z is not defined when $z = 0$; equivalently, when $r = 0$.

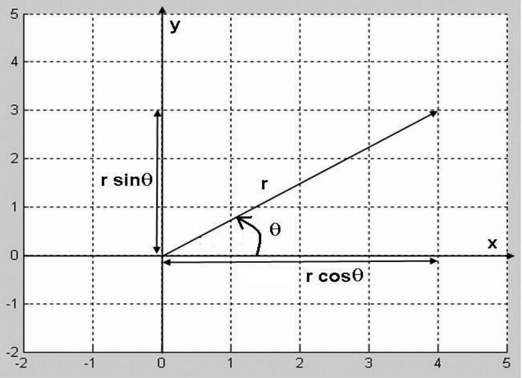


Figure: Polar form illustrations

$r = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$
$\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$; $\rightarrow \theta = 0.643$ radians.
$\therefore r = 5$ and $\arg z = \{0.643 + 2k\pi : \; k = 0, \pm 1, \ldots\}$

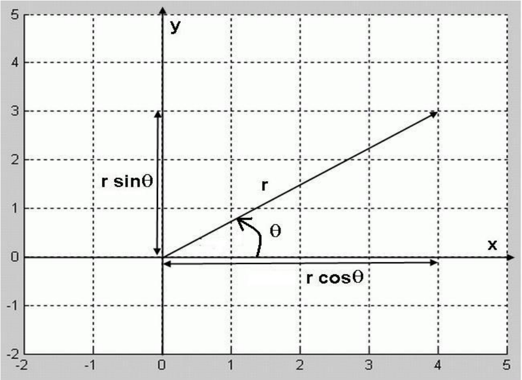


Figure: Polar form illustrations

Notice that arg function generates radians, within the current framework; you cannot say arg $z = 36.86$ degrees!!!

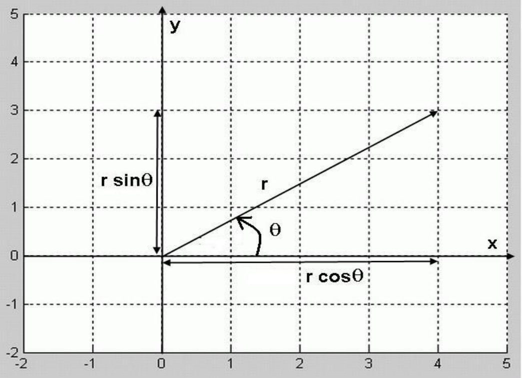


Figure: Polar form illustrations

$$\operatorname{Re} z = r\cos\theta\ , \quad \operatorname{Im} z = r\sin\theta\ , \quad (4)$$

If θ satisfies (4) then so do $\theta + 2k\pi$ $(k = 0, \mp 1, \mp 2, \ldots)$.
$\therefore$ (4) does not determine a unique value of argument z.

$$\text{Note that } r = \sqrt{x^2 + y^2} = |z| = \sqrt{z\bar{z}}$$

Arg z: Principal value of the argument

If θ is restricted to the interval $-\pi < \theta \leq \pi$, then there is a unique value of θ that satisfies (4).
Called the principal value of the argument and denoted by **Arg z**.

If $z = x + iy$, then

$$-\pi < \operatorname{Arg} z \leq \pi \ , \quad \cos(\operatorname{Arg} z) = \frac{x}{|z|} \ , \quad \sin(\operatorname{Arg} z) = \frac{y}{|z|}$$

The set of all values of the argument will be denoted by

$$\arg z = \{\theta + 2k\pi : k = 0, \mp 1, \mp 2, \dots\}$$

where θ is any angle that satisfies (4). In particular we have

$$\arg z = \{\operatorname{Arg} z + 2k\pi : k = 0, \mp 1, \mp 2, \dots\}$$

Unlike Arg z, which is single valued, arg z is multivalued or set valued.

Example

Find the modulus, argument, principal value of the argument, and polar form of the given number.

a) $z_1 = 5$ **b)** $z_2 = -3i$ **c)** $z_3 = \sqrt{3} + i$

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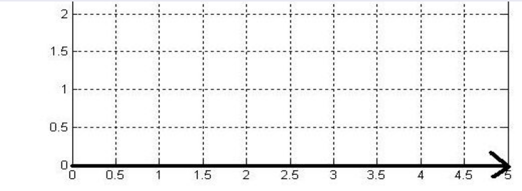
a) $r = |z_1| = 5$, an argument of z_1 is 0. Thus

$$\arg z_1 = \{2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

Since 0 is in interval $(-\pi, \pi]$, Arg $z_1 = 0$. The polar representation is

$$5 = 5(\cos 0 + i \sin 0)$$



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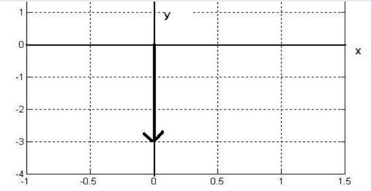
b)

$$r = |z_2| = |-3i| = 3\ ,$$

$$\arg z_2 = \{\frac{3\pi}{2} + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

$\operatorname{Arg} z_2 = \frac{-\pi}{2}$;   it is the element of $\arg z_2$ that lies in $(-\pi\,,\pi]$.

$$-3i = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 3\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right)$$



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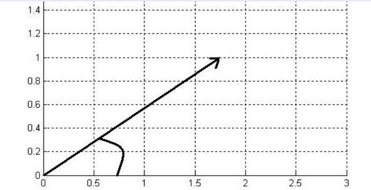
c)

$$r = |z_3| = |\sqrt{3} + i| = \sqrt{3+1} = 2$$

$$\cos\theta = \frac{x}{r} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta = \frac{y}{r} = \frac{1}{2} \rightarrow \theta = \frac{\pi}{6}$$

$$\arg z_3 = \{\frac{\pi}{6} + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

$$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$



Arithmetic in Polar Form

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Thus:

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2 = \{\theta_1 + \theta_2 + 2k\pi : k = 0, \mp 1, \mp 2, \ldots\}$$

$$|z_1 z_2| = |z_1||z_2|$$

When we multiply two complex numbers in polar form, we multiply their moduli and add their arguments.

Inverse of $z = r(\cos\theta + i\sin\theta)$ is

$$z^{-1} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \frac{1}{r}(\cos\theta - i\sin\theta)$$

Because it satisfies $zz^{-1} = 1$.

When $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Example

Let $\quad z_1 = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \quad$ and $\quad z_2 = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$$z_1 z_2 = 2\cdot 3\left[\cos\left(\frac{\pi}{4} + \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)\right]$$

$$= 6\left[\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right]$$

Example

$$z_1 = 5\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right], \quad z_2 = 2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

$$\frac{z_1}{z_2} = \frac{5}{2}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{2}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{2}\right)\right]$$

$$= \frac{5}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{5}{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

Exponential Form

$$\underbrace{e^{i\theta} = \cos\theta + i\sin\theta}_{Euler's\ formula} \quad \rightarrow \quad z = re^{i\theta}$$

Some identities:

$$z_1 z_2 = r_1 r_2\, e^{i(\theta_1+\theta_2)}$$

$$z^{-1} = \frac{1}{r} e^{i(-\theta)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}\ , \quad z_2 \neq 0$$

The circle $|z - z_0| = R$, whose center is z_0 and whose radius is R has the parametric representation

$$z = z_0 + R\, e^{i\theta} \qquad 0 \leq \theta \leq 2\pi$$

Example

$$\left\{z = 3 + 2i + 2e^{i\theta} \ : \ 0 \leq \theta \leq \pi\right\}$$

θ	z
0	$5.0000 + 2.0000i$
0.1	$4.9900 + 2.1997i$
0.2	$4.9601 + 2.3973i$
0.3	$4.9107 + 2.5910i$
0.4	$4.8421 + 2.7788i$
0.5	$4.7552 + 2.9589i$
0.6	$4.6507 + 3.1293i$
0.7	$4.5297 + 3.2884i$
0.8	$4.3934 + 3.4347i$
0.9	$4.2432 + 3.5667i$
1.0	$4.0806 + 3.6829i$
1.1	$3.9072 + 3.7824i$
1.2	$3.7247 + 3.8641i$

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θ	z
0	$5.0000 + 2.0000i$
1.3	$3.5350 + 3.9271i$
1.4	$3.3399 + 3.9709i$
$\vdots$	$\vdots$
3.1	$1.0017 + 2.0832i$
3.2	$1.0034 + 1.8833i$

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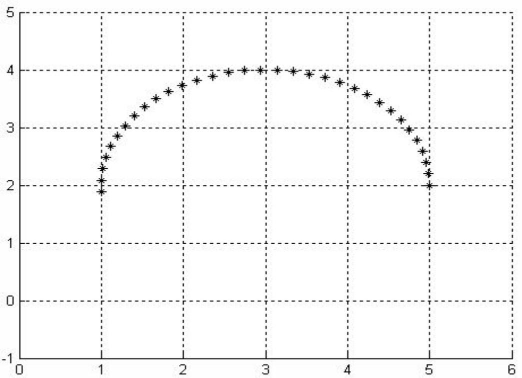

Integral powers of a nonzero complex number $z = re^{i\theta}$ are given by

$$z^n = r^n e^{in\theta} \ , \quad n = 2, 3, \dots$$

De Moivre's Formula

$$(e^{i\theta})^n = e^{in\theta} \ \rightarrow \ (\cos\theta + i\sin\theta)^n = \cos(n\theta) + isin(n\theta) \quad n = 2, 3, \dots$$

Example

Let us solve the equation

$$z^6 = 1$$

Write $z = re^{i\theta}$ and look for values of r and θ such that

$$(re^{i\theta})^6 = 1$$

or

$$r^6 e^{i6\theta} = 1e^{i(0+2k\pi)}$$

$$r^6 = 1 \quad \text{and} \quad 6\theta = 0 + 2k\pi \ , \quad k = 0, \pm 1, \ldots$$

Consequently $r = 1$ and $\theta = 2k\pi/6$ and it follows that the complex numbers

$$z = e^{i\frac{2k\pi}{6}} \ , \quad k = 0, \pm 1, \ldots$$

are 6-th roots of unity.

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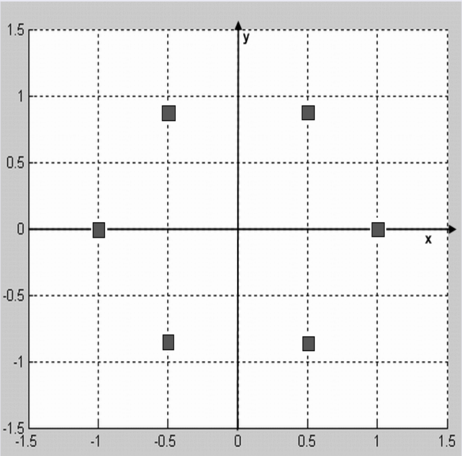


Figure: Roots of $z^6 = 1$

$z^n = 1$ has n distinct roots.

Example

Find all values of $(-8i)^{\frac{1}{3}}$. Let $z = (-8i)^{\frac{1}{3}}$ It is equivalent to solving $z^3 = -8i$. Or $(re^{i\theta})^3 = 8e^{i(\frac{-\pi}{2}+2k\pi)}$, $\quad k = 0, \pm 1, \ldots$. That is,

$$r^3 e^{i3\theta} = 8e^{i(\frac{-\pi}{2}+2k\pi)}, \qquad k = 0, \pm 1, \ldots$$

$$r^3 = 8 \text{ and } 3\theta = \frac{-\pi}{2} + 2k\pi, \quad k = 0, \pm 1, \ldots$$

The roots are $z_k = 2e^{i(\frac{-\pi}{6}+\frac{2k\pi}{3})}$, $\quad k = 0, \pm 1, \ldots$

1) Find one value of arg z when

a) $z = \dfrac{-2}{1+\sqrt{3}i}$ **b)** $\dfrac{i}{-2-2i}$ **c)** $(\sqrt{3}-i)^6$

2) By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to cartesian coordinates, show that

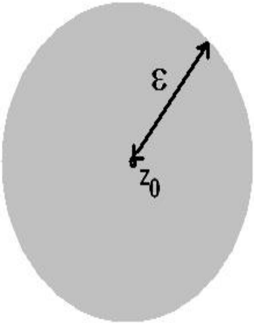
a) $i(1-\sqrt{3}i)(\sqrt{3}+i) = 2(1+\sqrt{3}i)$
b) $5i/(2+i) = 1+2i$
c) $(-1+i)^7 = -8(1+i)$
d) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$

3) In each case find all the roots in cartesian form, exhibit them geometrically

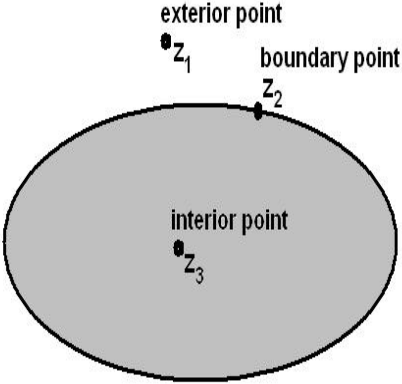
a) $(2i)^{1/2}$ b) $(-1-\sqrt{3}i)^{1/2}$ **c)** $(-16)^{1/4}$

4) Find the four roots of the equation $z^4+4=0$ and use them to factor z^4+4 into quadratic factors with real coefficients.
Ans. $(z^2+2z+2)(z^2-2z+2)$

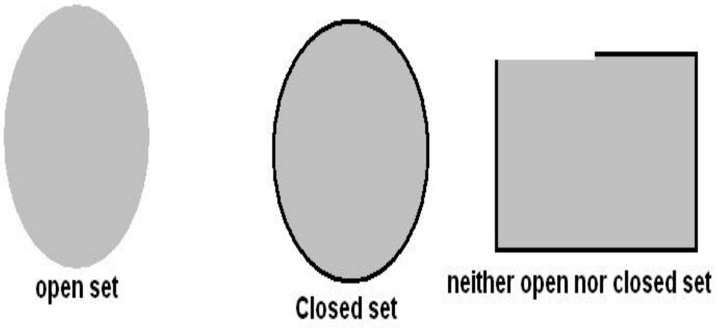
ε **neighborhood** of a given point z_0 is defined as the set of points satisfying $|z - z_0| < \varepsilon$. It consists of all points z lying inside but not on a circle centered at z_0 and with a specified radius ε.



A point z_0 is said to be **an interior point** of a set S whenever there is some neighborhood of z_0 that contains only points of S; it is called **an exterior point** of S when there exists a neighborhood of it containing no points of S. If z_0 is neither of these, it is **a boundary point** of S. A boundary point is therefore a point all of whose neighborhoods contain points in S and points not in S. The totality of all boundary points is called the **boundary** of S.



A set is **open** if it contains none of its boundary points. Consequently a set is open iff each of its points is an interior point. A set is **closed** if it contains all its boundary points; and the **closure** $\overline{S}$ of S is the closed set consisting of all points in S together with the boundary of S.

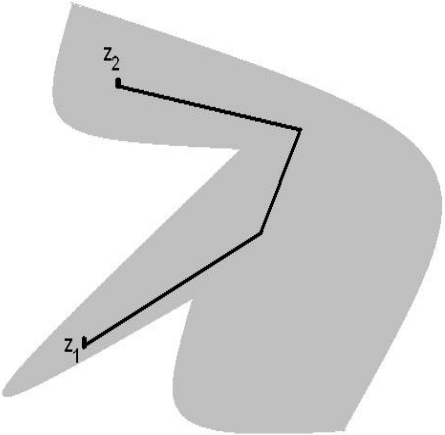


Example

$|z| < 1$	open set
$|z| \leq 1$	closed set
$0 < |z| \leq 1$	neither open nor closed set

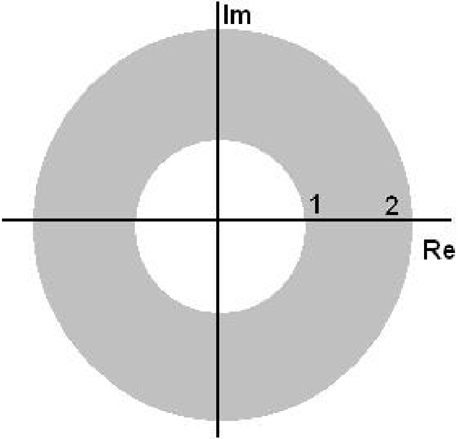
Connected set

An open set S is **connected** if each pair of points z_1 and z_2 in it can be joined by a polygonal path, consisting of a finite number of line segments joined end to end, that lies entirely in S.



Example

The annulus $1 < |z| < 2$ is open and connected.



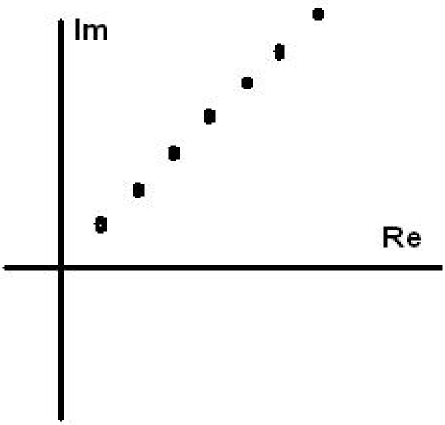
An open set that is connected is called **a domain**. A domain together with some, none, or all of its boundary points is referred to as a region.
A set S is **bounded** if every point of S lies inside some circle
$|z| = R$,
otherwise it is **unbounded**.

Example

$$\{x + iy : \ 1 \leq x \leq 2, 0 \leq y \leq 3\} \text{ is bounded}$$

Example

$$\{1+i, 2+2i, 3+3i, \ldots\} \text{ is unbounded}$$



Unbounded; because no circle can contain the set above.

A point z_0 is said to be **an accumulation point** of a set S if each neighborhood of z_0 contains at least one point of S distinct from z_0. It follows that if a set S is closed, then it contains each of its accumulation points. Converse is also true.
Informally, **accumulation points** of a set (or sequence) are points where there are infinitely many other points of the set (or sequence) "nearby."

Example

If you take any interval (a,b) of real (or rational) numbers, EVERY point in [a,b] is an accumulation point. If you take the integers as a subset of the real (or rational) numbers, NO point is an accumulation point.

If you take the set $S = \{J - \frac{1}{n} : \;\; J \in Z, \; n \in N\}$, then the accumulation points are exactly the integers.

The sequence $\{\frac{1}{n} : \;\; n \in N\}$ has one accumulation point, namely 0.

Exercises

1) Sketch the following sets and determine which are both open and connected (and therefore domains):

a) $|z - 2 + i| \leq 1$ **b)** $|2z + 3| > 4$ **c)** $\operatorname{Im} z > 1$

d) $\operatorname{Im} z = 1$ **e)** $0 \leq \arg z \leq \pi/4 \quad (z \neq 0)$ **f)** $|z - 4| \geq |z|$

g) $0 < |z - z_0| < \delta$ where z_0 is a fixed point and δ is a positive number.

Ans. b) c) & g) are domains.

2) Which set in Exercise 1 is neither open nor closed?
Ans. e)

3) Which sets in Exercise 1 are bounded? Ans. a) and g)

4) Determine the accumulation points of each of the following sets.

a) $z_n = i^n$, $n = 1, 2, \ldots$

b) $z_n = i^n/n$, $n = 1, 2, \ldots$

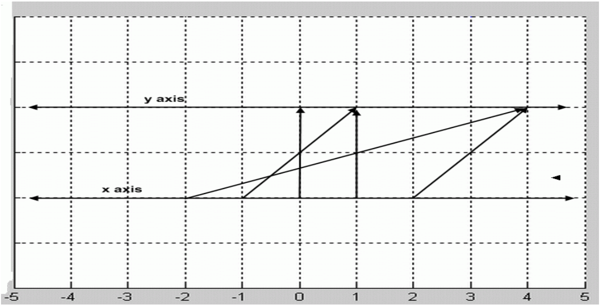
c)$0 \leq \arg z \leq \pi/2$, $z \neq 0$

d) $z_n = (-1)^n(1+i)(n-1)/n$, $n = 1, 2, \ldots$

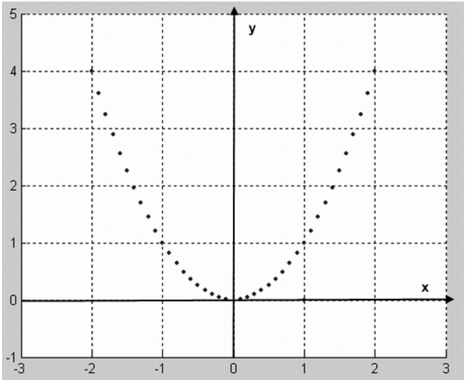
Ans. a) none b) 0 d)$\pm(1+i)$

5) Sketch a) $|z| > 0$, b) $|z| < \infty$, c) $0 < |z| < \infty$ d)$1 < |z| < 2$

Lets plot the function $y = x^2$.



We never do this way because in the past, Descartes thought about placing the y axis vertically, and plotting as we already know:

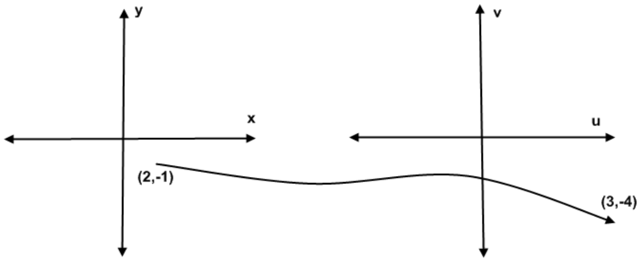


There is no method comparable to Descartes's procedure for plotting complex functions. Instead, plotting in this case is done analogously to the way we plotted $y = x^2$ using two horizontal axes and pieces of string to show the correspondence.

Let $w = z^2$, where $z = x + iy$ and $w = u + iv$.

$$\rightarrow u + iv = (x + iy)^2 = x^2 - y^2 + i2xy$$

The component functions are: $u = x^2 - y^2$, $v = 2xy$.
Example: $z = 2 - i \rightarrow w = 3 - 4i$



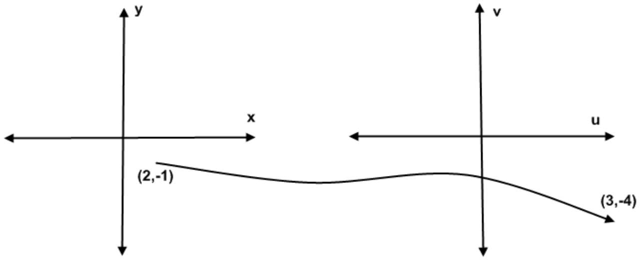
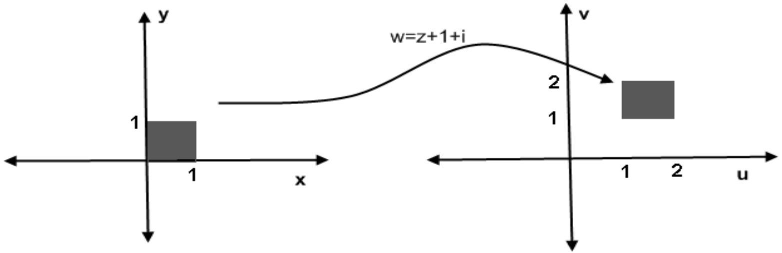


Image of $(2, -1)$ under the mapping/transformation/function $w = z^2$ is $(3, -4)$.
$w = z^2$ transforms/maps $(2, -1)$ to $(3, -4)$.
For the input $(2, -1)$ the function $w = z^2$ outputs $(3, -4)$.

Terminology

Function, mapping, transformation, image, input, output
$f(x) = x^2$: real-valued function of a real variable
$f(x) = x + i6x$: complex-valued function of a real variable
$f(z) = x + ix$: complex-valued function of a complex variable
$f(z) = x^2 + y^2$: real-valued function of a complex variable

Example

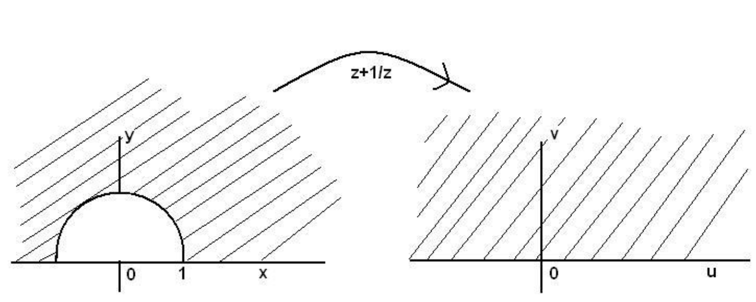


Example

Let us show that, under the transformation

$$w = z + \frac{1}{z} \tag{5}$$

the image of the shaded region on the left is the shaded upper half plane shown on the right.



Boundary in the z plane maps to boundary in the w plane

Image of $x + i0$ when $x > 1$: $w = x + \frac{1}{x}$, it is positive and increasing, starts from 2 and goes to ∞.

when $x < -1$: $w = x + \frac{1}{x}$, negative and decreases as x decreases, starts from -2 and goes to $-\infty$.

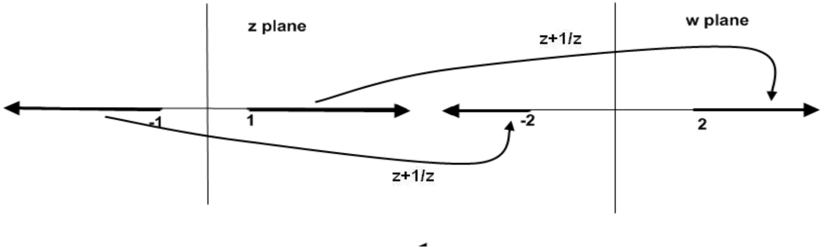
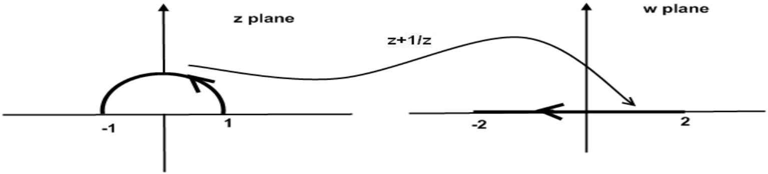


Image of the upper half of the circle $|z| = 1$ (or $z = e^{i\theta}\ , \quad 0 \leq \theta < \pi$)

$$w = e^{i\theta} + \frac{1}{e^{i\theta}} = e^{i\theta} + e^{-i\theta} = 2\cos\theta\ , \text{ it is real}$$

Recall that
$e^{i\theta} = \cos\theta + i\sin\theta,\ e^{-i\theta} = \cos\theta - i\sin\theta \rightarrow e^{i\theta} + e^{-i\theta} = 2\cos\theta$
As θ varies from 0 to π, the image varies from 2 to -2.



Let's check images of larger semicircles (i.e. , $z = re^{i\theta}$ with $r > 1$ and $0 \leq \theta \leq \pi$)

$$w = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + \frac{1}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) = a\cos\theta + ib\sin\theta$$

$$\text{where } a = r + \frac{1}{r}\,; \quad b = r - \frac{1}{r}$$

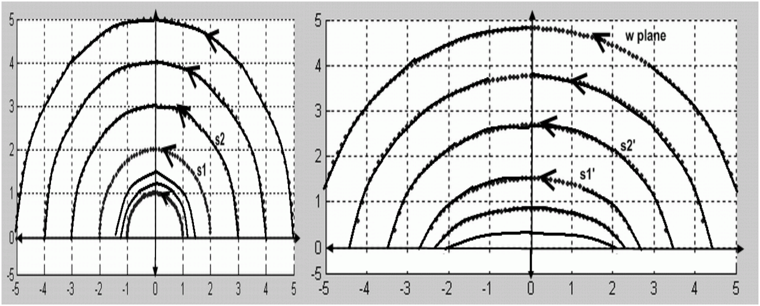
Real and imaginary components of w are
$u = a\cos\theta\,; \quad v = b\sin\theta.$

$$\left(\frac{u}{a}\right)^2 + \left(\frac{v}{b}\right)^2 = 1$$

This represents an ellipse with foci at the points

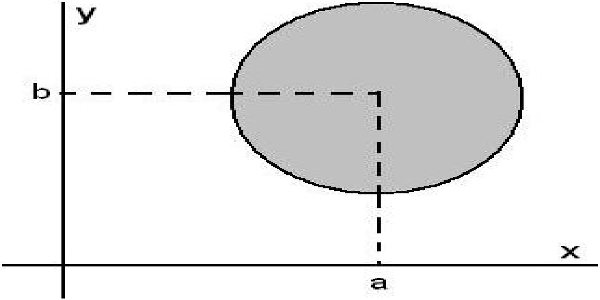
$$\pm\sqrt{a^2 - b^2} = \pm\sqrt{(r + \frac{1}{r})^2 - (r - \frac{1}{r})^2} = \pm 2.$$

Definition: Ellipse is the set of all points for which the sum of the distance to two fixed points (called foci) is constant.



We can repeat this for all possible upper semicircles with radius > 1. Their images will cover the upper w-plane.

Preliminaries (Two variable real valued function's limit) Let f be defined on the interior of a circle centered at the point (a, b) except possibly at (a, b) itself.

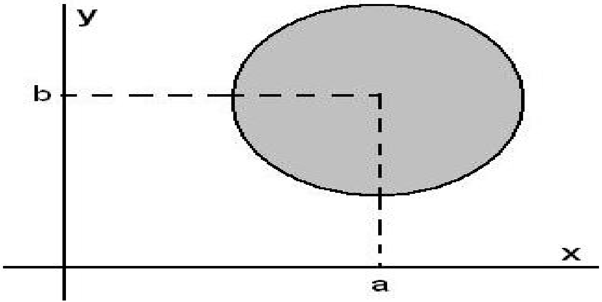


Function f is defined in the shaded region (neighborhood of (a, b)); its values are real.

If (x, y) gets closer to (a, b), what happens to the values of $f(x, y)$?

Does it approach some fixed L value.

If the answer is yes then we say that "limit of f as (x, y) approaches (a, b) is L.

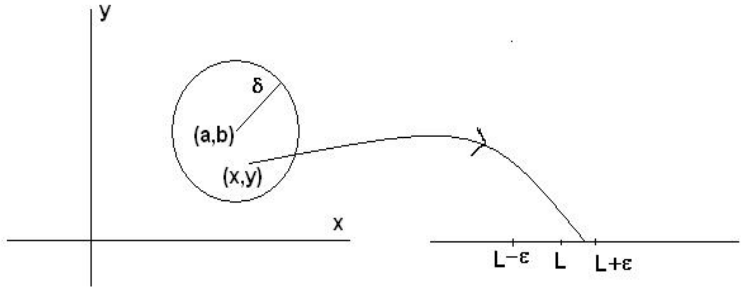


Formal definition is as follows:

We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$



An Interpretation

Recall the right limit and left limit of the real calculus.

If approaching t_0 from right and left gives the same limit value, then we say that the real function has a limit at t_0.

In two variable functions we have more than two directions to approach. Approaching from all possible directions must give the same limit value.

Particularly, approaching from two specific directions, from the right and from the top, must give the same limit value.

This can be implemented by first setting $y = b$ and taking the limit wrt x, then setting $x = a$ and taking the limit wrt y.

Example

$$\lim_{(x,y)\to(1,0)} \frac{y}{x+y-1}$$

First consider the vertical line path along the line $x = 1$ we have

$$\lim_{(1,y)\to(1,0)} \frac{y}{1+y-1} = \lim_{y\to 0} 1 = 1$$

Consider the horizontal line $y = 0$ and compute the limit as x approaches 1.

$$\lim_{(x,0)\to(1,0)} \frac{0}{x+0-1} = \lim_{x\to 1} 0 = 0$$

Since approaching from two different directions results in two different values; there is no limit.

End of the Preliminaries

Complex functions' limit

$$\lim_{z \to z_0} f(z) = L \quad \text{or} \quad f(z) \to L \text{ as } z \to z_0$$

if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - L| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta \tag{6}$$

If the limit of a function exists at a point, then it is unique.

Note that, $|f(z) - L|$ is the distance between $f(z)$ and L.

If the distance between $f(z)$ and L tends to zero as z tends to z_0, then we say the function f has limit L as $z \to z_0$.

Thus

$$\lim_{z \to z_0} f(z) = L \quad \text{if and only if} \quad \lim_{z \to z_0} |f(z) - L| = 0$$

Note that the value of f at z_0 is immaterial, and need not even be defined at z_0.

Recall the deleted neighborhood concept in the real calculus. We check the right limit and the left limit, but we don't check functions value at t_0.

Numerical verification of a limit value

If all the $f(z)$ values in a sufficiently small deleted neighborhood are not close to each other, then the function does not have limit at z_0.

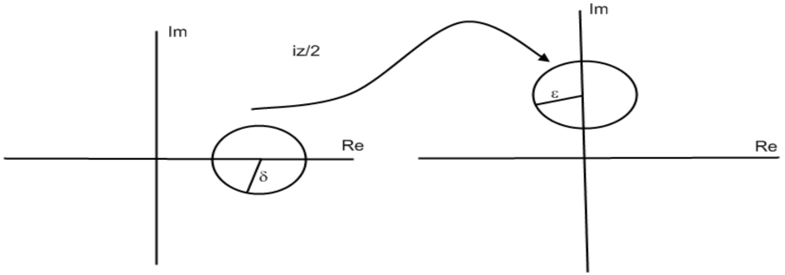
If the limit exists at z_0 and if we take two points very close to z_0 then at these points the function's values must be very close to each other.

Example

$$\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$$

$$|f(z) - L| = \left| \frac{iz}{2} - \frac{i}{2} \right| = \left| \frac{i(z-1)}{2} \right| = |i| \frac{|z-1|}{2} = \frac{|z-1|}{2}$$

Hence $\left| f(z) - \frac{i}{2} \right| < \varepsilon$ whenever $0 < |z-1| < 2\varepsilon$



Theorem

Given a complex-valued function $f(z) = u(x, y) + i\, v(x, y)$ *and complex numbers* $L = a + ib\,,\ z_0 = x_0 + iy_0$, *then*

$$\lim_{z \to z_0} f(z) = L \iff$$

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = a \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = b$$

Theorem

Suppose $\lim_{z\to z_0} f(z)$ *and* $\lim_{z\to z_0} g(z)$ *both exist and* c_1, c_2 *are complex constants. Then:*

$$\lim_{z\to z_0}[c_1\, f(z)+c_2\, g(z)] = c_1 \lim_{z\to z_0} f(z) + c_2 \lim_{z\to z_0} g(z)$$

$$\lim_{z\to z_0}[f(z)g(z)] = \lim_{z\to z_0} f(z)\cdot \lim_{z\to z_0} g(z)$$

$$\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}\,, \quad \textit{provided} \lim_{z\to z_0} g(z)\neq 0$$

Theorem

(i) Suppose that $f(z) \to 0$ *as* $z \to z_0$ *and* $|g(z)| \leq |f(z)|$ *in a deleted neighborhood of* z_0. *Then* $g(z) \to 0$ *as* $z \to z_0$.
(ii) Suppose that $f(z) \to 0$ *as* $z \to z_0$ *and* $g(z)$ *is bounded in a deleted neighborhood of* z_0. *Then* $f(z)g(z) \to 0$ *as* $z \to z_0$.

Example

Evaluate $\lim_{z\to 0} ye^{\frac{i}{|z|}}$. Let $f(z) = y$ and $g(z) = e^{\frac{i}{|z|}}$. As $z \to 0$, $f(z) \to 0$. Also, for $z \neq 0$, since $\frac{1}{|z|}$ is a purely real number, $|e^{\frac{i}{|z|}}| = 1$. Thus we can apply Theorem 3-ii and conclude that

$$\lim_{z\to 0} ye^{\frac{i}{|z|}} = 0$$

Recall that for a real x:

$$|e^{ix}| = |\cos x + i \sin x| = \sqrt{\cos^2 x + \sin^2 x} = 1$$

Example

$$lim_{z\to 0} z = 0;\ \ lim_{z\to 0}\overline{z} = 0;\ \ lim_{z\to 0} x = 0;\ \ lim_{z\to 0} y = 0$$

Example

Suppose that $\lim_{z\to i} f(z) = 2 + i$ and $\lim_{z\to i} g(z) = 3 - i$. Find

$$L = \lim_{z\to i}\left[(f(z))^2 + \frac{(3+i)\,g(z)}{z}\right]$$

Solution:

$$\begin{aligned} L &= \lim_{z\to i}(f(z))^2 + \lim_{z\to i}(3+i)\frac{g(z)}{z} \\ &= \left(\lim_{z\to i} f(z)\right)^2 + (3+i)\frac{\lim_{z\to i} g(z)}{\lim_{z\to i} z} \\ &= (2+i)^2 + (3+i)\frac{3-i}{i} \\ &= 3 - 6i \end{aligned}$$

Suppose f is defined in a neighborhood of z_0. We say that f is continuous at the point z_0 if $\lim_{z\to z_0} f(z)$ exists and equals $f(z_0)$. We say f is continuous on a set S if it is continuous at every point in S

Properties of Continuous Functions

If f and g are continuous at z_0, and c_1, c_2 are complex constants, then the following functions are continuous at z_0.

$$c_1 f + c_2 g \ , \quad f \cdot g \ , \quad \frac{f}{g} \quad \text{provided} \quad g(z_0) \neq 0$$

If g is continuous at z_0 and f is continuous at $g(z_0)$, then the composition $h = f(g)$ is continuous at z_0.
The function $f = u + iv$ is continuous at $z_0 = x_0 + iy_0$ if and only if u and v are continuous at (x_0, y_0).

Removable discontinuities

Example

$$\lim_{z\to i}\frac{z-i}{z^2+1}=\lim_{z\to i}\frac{z-i}{(z-i)(z+i)}=\lim_{z\to i}\frac{1}{z+i}=\frac{1}{2i}=-\frac{1}{2}i$$

We can make the function continuous by redefining the function $f(z)=\frac{z-i}{z^2+1}$ at i. $f(i)=-i/2$ makes the function continuous at i. Originally the function f was discontinuous at i, however we made it continuous by redefining the function at this point. The newly defined function is:

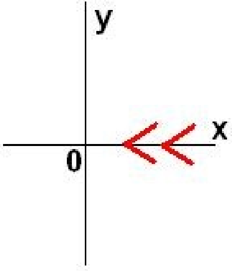
$$f(z)=\begin{cases}\frac{z-i}{z^2+1} & \text{for } z\neq i\\ -i/2 & \text{for } z=i\end{cases}$$

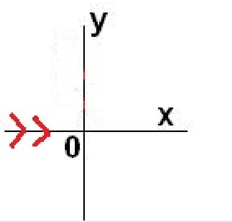
The function above is continuous. In the formation of this, we removed the discontinuity at $z=i$. This discontinuity is called *a removable discontinuity*.

The Nonremovable Discontinuities of Arg z

Example

The principal branch of the argument Arg z takes the value of the argument z that is in the interval $-\pi < \operatorname{Arg} z \leq \pi$. It is not defined at $z = 0$ and hence Arg z is not continuous at $z = 0$. We will show that $z = 0$ is not a removable discontinuity by showing that $\lim_{z\to 0} \operatorname{Arg} z$ does not exist. Indeed if $z = x > 0$ then Arg $z = 0$ so $\lim_{z=x\downarrow 0} \operatorname{Arg} z = 0$, where the down-arrow denotes the limit from the right. However, if $z = x < 0$, then Arg $z = \pi$ and so $\lim_{z=x\uparrow 0} \operatorname{Arg} z = \pi$. By the uniqueness of the limit, we conclude that $\lim_{z\to 0} \operatorname{Arg} z$ does not exist.





Let f be defined on a neighborhood of z_0. If

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, then f is said to be differentiable at the point z_0, and the number

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

is called derivative of f at z_0.
We can also define the derivative as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Example

Suppose that $f(z) = z^2$. At any point z_0

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - {z_0}^2}{\Delta z} = \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0$$

A numerical verification

Notice that when $z_0 = 2 + i$ we have:

$$\left.\frac{(z_0+\Delta z)^2 - {z_0}^2}{\Delta z}\right|_{\Delta z = 0.0001} = 4.0001 + 2.0000i$$

$$\left.\frac{(z_0+\Delta z)^2 - {z_0}^2}{\Delta z}\right|_{\Delta z = 0.0001i} = 4.0000 + 2.0001i$$

These results are very close to $2z_0 = 2(2 + i) = 4 + 2i$.

Example

$$f(z) = |z|^2 \to f'(z_0) = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z_0} + \overline{\Delta z}) - z_0 \overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{z_0} + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z} \qquad (7)$$

Continued from the previous page

At $z_0 = 2 + i$ we have

$$\overline{z_0} + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z}\bigg|_{0.0001} = 4.0001$$

$$\overline{z_0} + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z}\bigg|_{0.0001i} = -2.0001i$$

In the neighborhood of $z_0 = 2 + i$ we obtained two nonmatching results. This shows that there is no derivative at $z_0 = 2 + i$. Below, we will show analytically that there is no derivative when $z_0 \neq 0$.

Continued from the previous page

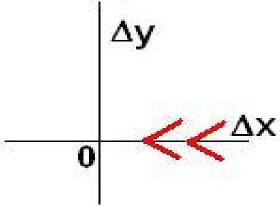
$$f'(z_0) = \lim_{\Delta z \to 0} \overline{z_0} + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z}$$

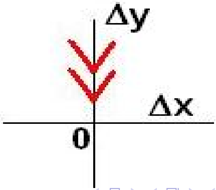
At $z_0 = 0$, (7) reduces to $f'(0) = \lim_{\Delta z \to 0} \overline{\Delta z} = 0$. So, the derivative exists at $z_0 = 0$, and it is equal to 0.

Continued from the previous page

$$f'(z_0) = \lim_{\Delta z \to 0} \overline{z_0} + \overline{\Delta z} + z_0 \frac{\overline{\Delta z}}{\Delta z}$$

If the limit exists when $z_0 \neq 0$ that limit may be found by letting the variable $\Delta z = \Delta x + i\Delta y$ approach 0 in any manner. In particular, when Δz approaches 0 through real values $\Delta z = \Delta x + i0$ we may write $\overline{\Delta z} = \Delta z$. Hence if the limit exists its value must be $\overline{z_0} + z_0$. However when Δz approaches 0 through the pure imaginary values $\Delta z = 0 + i\Delta y$ so that $\overline{\Delta z} = -\Delta z$, the limit is found to be $\overline{z_0} - z_0$. Since a limit is unique, it follows that $\overline{z_0} + z_0 = \overline{z_0} - z_0$ or $z_0 = 0$. So the derivative exists only at $z_0 = 0$.





This example shows that a function can be differentiable at a certain point but nowhere else.

Note that $f(z) = |z|^2 = x^2 + y^2 \rightarrow u(x,y) = x^2 + y^2 \,; \quad v(x,y) = 0$

The function $f(z) = |z|^2$ is continuous at each point in the plane since its components are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there. It is however true that the existence of the derivative of a function at a point implies the continuity of the function at that point.

Let c be a complex constant, and let f be a function whose derivative exists at a point z.

$$\frac{d}{dz}c = 0\,, \quad \frac{d}{dz}z = 1\,, \quad \frac{d}{dz}[cf] = cf'$$

If the derivatives of two functions f and F exist at a point z_0, then

$$\frac{d}{dz}(f + F) = f' + F'$$

$$\frac{d}{dz}(f \cdot F) = f' \cdot F + F' \cdot f$$

and when $F(z_0) \neq 0$

$$\frac{d}{dz}\left(\frac{f}{F}\right) = \frac{F \cdot f' - f \cdot F'}{F^2}$$

If n is a positive integer $\frac{d}{dz}z^n = n\,z^{n-1}$
This formula remains valid when n is a negative integer, provided $z \neq 0$.
There is also a chain rule for differentiating composite functions. Suppose that f has a derivative at z_0 and g has a derivative at point $f(z_0)$. Then the function $F = g(f)$ has a derivative at z_0, and

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

Use the results of the section above to find $f'(z)$ when

a) $f(z) = 3z^2 - 2z + 4$

b) $f(z) = (1 - 4z^2)^3$

c) $f(z) = \frac{z-1}{2z+1}\ , \quad z \neq -\frac{1}{2}$

d) $f(z) = \frac{(1+z^2)^4}{z^2}\ , \quad z \neq 0$

Suppose that the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists.
Writing $z_0 = x_0 + iy_0$ and $\Delta z = \Delta x + i\Delta y$, by the first theorem on limits we have

$$\operatorname{Re}[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} \operatorname{Re}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \tag{8}$$

$$\operatorname{Im}[f'(z_0)] = \lim_{(\Delta x, \Delta y) \to (0,0)} \operatorname{Im}\left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] \tag{9}$$

where

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y} \tag{10}$$

It is important to keep in mind that expressions (8) and (9) are valid as $(\Delta x, \Delta y)$ tends to $(0,0)$ in any manner that we may choose. In particular, let $(\Delta x, \Delta y)$ tend to $(0,0)$ horizontally through the points $(\Delta x, 0)$ as indicated in the figure below.

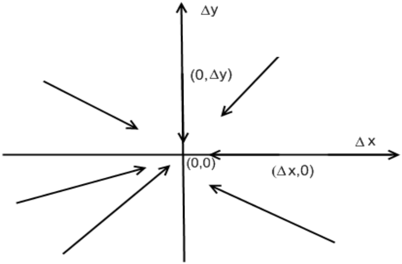
$$\mathrm{Re}[f'(z_0)] = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} =: u_x(x_0, y_0)$$

$$\mathrm{Im}[f'(z_0)] = \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} =: v_x(x_0, y_0)$$

That is,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \tag{11}$$

where $u_x(x_0, y_0)$ and $v_x(x_0, y_0)$ denote the 1st order partial derivatives with respect to x of the functions u and v at (x_0, y_0).



We might have let $(\Delta x, \Delta y)$ tend to zero vertically through the points $(0, \Delta y)$. In that case $\Delta x = 0$ in equation (10); and we obtain the expression

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0) \tag{12}$$

which can also be written

$$f'(z_0) = -i(u_y(x_0, y_0) + iv_y(x_0, y_0))$$

Recall

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \tag{11}$$

Equate (11) and (12) to obtain

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \tag{13}$$

Theorem

Suppose that

$$f(z) = u(x,y) + iv(x,y)$$

and f' exist at a point $z_0 = x_0 + iy_0$. Then the 1st order partial derivatives of u and v with respect to x and y must exist at (x_0, y_0) and they must satisfy the Cauchy-Riemann equations (13) at that point. Also $f'(z_0)$ is given in terms of these partial derivatives by either equation (11) or (12).

Recall the Caucy-Riemann Equations:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \tag{13}$$

Also recall:

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \tag{11}$$

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0) \tag{12}$$

Example

Consider the function

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

Note here that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Thus

$$u_x(x, y) = 2x = v_y(x, y) \; ; \quad u_y(x, y) = -2y = -v_x(x, y)$$

$$\rightarrow f'(z) = 2x + i2y = 2(x + iy) = 2z$$

The same result as before.

Example

Let $f(z) = |z|^2$

$$\to u(x, y) = x^2 + y^2 \quad \text{and} \quad v(x, y) = 0$$

$$\to u_x(x, y) = 2x \quad \text{and} \quad v_y(x, y) = 0$$

$$v_x(x, y) = 0 \quad \text{and} \quad u_y(x, y) = 2y$$

Cauchy-Riemann equations are satisfied only at $(x, y) = (0, 0)$

Caucy-Riemann Equations:

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0) \tag{13}$$

Derivative of f at z_0 exists $\longrightarrow$ Cauchy-Riemann equations hold at z_0. The converse does not necessarily hold.

LOGIC's RULE: $A \to B$ and $B' \to A'$ are the same.

DEFINE: A:Derivative of f at z_0 exists.
B:Cauchy-Riemann equations hold at z_0.

A CONSEQUENCE: $B' \to A'$ means "If Cauchy-Riemann equations do not hold at z_0 then derivative of f at z_0 does not exist.

Theorem

Let the function

$$f(z) = u(x, y) + iv(x, y)$$

be defined throughout some ε *neighborhood of a point* $z_0 = x_0 + iy_0$. *Suppose that the 1st order partial derivatives of the functions* u *and* v *with respect to* x *and* y *exist everywhere in that neighborhood and that they are continuous at* (x_0, y_0). *Then if those partial derivatives satisfy the Cauchy-Riemann equations*

$$u_x = v_y \; ; \quad u_y = -v_x$$

at (x_0, y_0), *then the derivative* $f'(z_0)$ *exists.* $\square$

This is called a sufficiency theorem for existence of the derivative.

Cauchy-Riemann equations hold at z_0, and u_x, u_y, v_x, and v_y are continuous at z_0 $\longrightarrow$ Derivative of f at z_0 exists.

Example

Suppose that $f(z) = e^x(\cos y + i \sin y)$ where y is to be taken in radians when $\cos y$ and $\sin y$ are evaluated. Then

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere and since those derivatives are everywhere continuous, the conditions in the theorem are satisfied at all points in the complex plane.

$$\to f'(z) = u_x(x, y) + iv_x(x, y) = e^x(\cos y + i \sin y)$$

Theorem

Let the function

$$f(z) = u(r,\theta) + iv(r,\theta)$$

be defined throughout some ε neighborhood of a nonzero point

$$z_0 = r_0(\cos\theta_0 + i\sin\theta_0)$$

Suppose that the 1st order partial derivatives of the functions u and v with respect to r and θ exist everywhere in that neighborhood and that they are continuous at z_0. Then if those partial derivatives satisfy the polar form

$$u_r = \frac{1}{r}v_\theta \ , \quad \frac{1}{r}u_\theta = -v_r \tag{14}$$

of the Cauchy-Riemann equations at (r_0, θ_0), then the derivative $f'(z_0)$ exists and equals $e^{-i\theta}[u_r + iv_r]$ $\square$

This is called a sufficiency theorem for existence of the derivative in polar coordinates

The relationship between polar and cartesian coordinates:

$$(x, y) \leftrightarrow (r\cos\theta, r\sin\theta)$$

$$u(x, y) \leftrightarrow u(r\cos\theta, r\sin\theta)\,,\;\; v(x, y) \leftrightarrow v(r\cos\theta, r\sin\theta)$$

The following partial derivatives will be useful later on:

$$\left.\begin{aligned} u_r &= \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \\ u_\theta &= \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y} \end{aligned}\right\} \quad (i)$$

$$\left.\begin{aligned} v_r &= \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y} \\ v_\theta &= \frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial v}{\partial x} + r\cos\theta\frac{\partial v}{\partial y} \end{aligned}\right\} \quad (ii)$$

$$(i) \to \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \to$$

$$\left.\begin{aligned} u_x &= \cos\theta u_r - \frac{1}{r}\sin\theta u_\theta \\ u_y &= \sin\theta u_r + \frac{1}{r}\cos\theta u_\theta \end{aligned}\right\}(iii)$$

$$(ii) \to \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_r \\ v_\theta \end{bmatrix} \to$$

$$\left.\begin{aligned} v_x &= \cos\theta v_r - \frac{1}{r}\sin\theta v_\theta \\ v_y &= \sin\theta v_r + \frac{1}{r}\cos\theta v_\theta \end{aligned}\right\}(iv)$$

C-R in cartesian coordinates

$$u_x = v_y$$
$$u_y = -v_x$$

$$(iii)\&(iv) \to \left.\begin{array}{l} \cos\theta u_r - \frac{1}{r}\sin\theta u_\theta = \sin\theta v_r + \frac{1}{r}\cos\theta v_\theta \\ \\ \sin\theta u_r + \frac{1}{r}\cos\theta u_\theta = -\cos\theta v_r + \frac{1}{r}\sin\theta v_\theta \end{array}\right\}$$

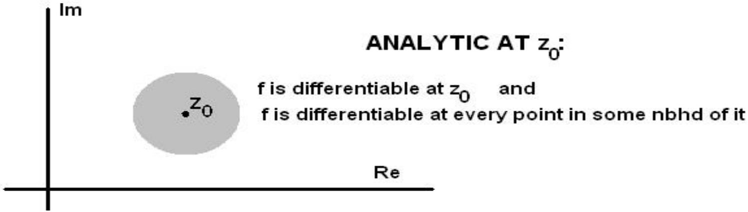
$$\to \boxed{-\frac{1}{r}u_\theta = v_r \; ; \quad u_r = \frac{1}{r}v_\theta}$$ C-R in polar coordinates.

$$
\begin{aligned}
f'(z) = u_x + iv_x &= \cos\theta u_r - \frac{1}{r}\sin\theta u_\theta + i\cos\theta v_r - i\frac{1}{r}\sin\theta v_\theta \\
&= \cos\theta u_r + \sin\theta v_r + i\cos\theta v_r - i\sin\theta u_r \\
&= (\cos\theta - i\sin\theta)u_r + (i\cos\theta + \sin\theta)v_r \\
&= e^{-i\theta}[u_r + iv_r]
\end{aligned}
$$

Exercises

1) Use Theorem 4 on C-R to show that f' does not exist at any point if f is
a) $\overline{z}$ **b)** $z - \overline{z}$ **c)** $2x + ixy^2$
2) Use the sufficiency theorem to show that f' and its derivative f'' exist everywhere and find f' and f'' when
a $f(z) = iz + 2$ **b** $f(z) = e^{-x}e^{-iy}$ **c** $f(z) = z^3$
d) $f(z) = \cos x \cosh y - i \sin x \sinh y$
Ans. **b)** $f'(z) = -f(z)$, $f''(z) = f(z)$ **d)** $f''(z) = -f(z)$
3) Find f' when
a) $f(z) = 1/z$ **b)** $f(z) = x^2 + iy^2$ **c)** $f(z) = z \operatorname{Im} z$
Ans. **a)** $f'(z) = 1/z^2$, $(z \neq 0)$ **b)** $f'(x + ix) = 2x$
c) $f'(0) = 0$

A function f of the complex variable z is *analytic* at a point z_0 if its derivative exists at z_0 and exists also at each point z in some neighborhood of z_0.



Example

$f(z) = z^2$ is analytic everywhere. But the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at $z = 0$ and not throughout any neighborhood.

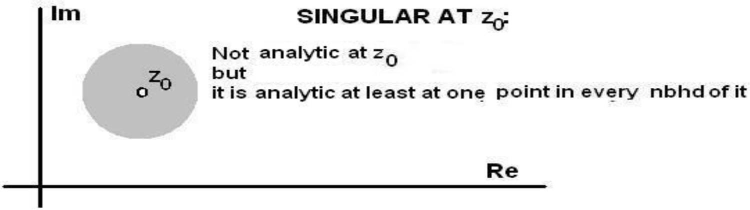
An **entire function** is a function that is analytic at each point in the entire xy plane.

Example

Since the derivative of a polynomial exists everywhere, it follows that every polynomial is an entire function.

Definition

If a function f is not analytic at a point z_0 and if it is analytic at some point in every neighborhood of z_0, then z_0 is called a **singular point** of f.



Example

The function $f(z) = 1/z(z \neq 0)$ whose derivative is $f'(z) = -1/z^2$. It is analytic at every point except for $z = 0$, where it is not even defined. The point $z = 0$ is therefore a singular point.

If two functions are analytic in a domain D, their sum and their product are both analytic in D. Similarly, their quotient is analytic in D provided the function in the denominator does not vanish at any point in D.

Composition of two analytic function is analytic.

The Exponential Function

$e^z = e^x(\cos y + i \sin y)$

Some Properties

$\frac{d}{dz} e^z = e^z$

$e^{i\theta} = \cos\theta + i \sin\theta$

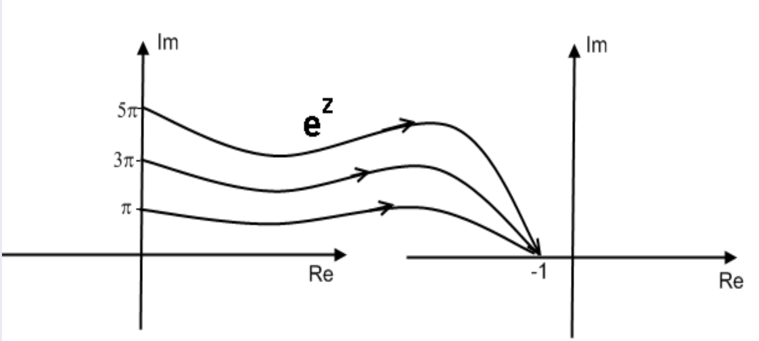
$e^{z_1} e^{z_2} = e^{z_1+z_2}$

$e^{z_1} / e^{z_2} = e^{z_1-z_2}$

$(e^z)^n = e^{nz}$

Example

$e^z = -1$ can be written as
$e^x e^{iy} = 1e^{i(\pi(2n+1))}$, $n = 0, \pm 1, \pm 2, \ldots \rightarrow x = \ln 1$
$\rightarrow x = 0$; $y = (2n+1)\pi \rightarrow z = i(2n+1)\pi$, $w = e^z$ is a many to one mapping.



From the equations

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

it follows that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

for every real number x. Sine and cosine functions may be defined for complex variables:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

The sine and cosine functions are entire since they are linear combinations of the entire functions e^{iz} and e^{-iz}. Knowing the derivatives of those exponential functions, we obtain

$$\frac{d}{dz}\sin z = \cos z \; ; \quad \frac{d}{dz}\cos z = -\sin z$$

Note that,

$$\frac{d}{dz}\sin z = \frac{d}{dz}\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \frac{1}{2i}\left(\frac{d}{dz}e^{iz} - \frac{d}{dz}e^{-iz}\right)$$

$$= \frac{1}{2i}\left(ie^{iz} - (-ie^{-iz})\right) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

Other Properties

$\sin(-z) = -\sin z\,; \quad \cos(-z) = \cos z$
$\sin^2 z + \cos^2 z = 1$
$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
$\sin 2z = 2 \sin z \cos z\,; \quad \cos 2z = \cos^2 z - \sin^2 z$
$\sin(z + \pi/2) = \cos z$
Note that,

$$\sin(-z) = \frac{e^{i(-z)} - e^{-i(-z)}}{2i} = \frac{e^{-iz} - e^{iz}}{2i} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

Trigonometric functions may be expressed in terms of the components of z. For instance

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= (\cos x + i \sin x)\frac{e^{-y}}{2i} - (\cos x - i \sin x)\frac{e^{y}}{2i}$$

$$= \sin x \left(\frac{e^{y} + e^{-y}}{2}\right) + i \cos x \left(\frac{e^{y} - e^{-y}}{2}\right)$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y \tag{15}$$

$$\sin z = \sin x \cosh y + i \cos x \sinh y \tag{15}$$

Likewise

$$\cos z = \cos x \cosh y - i \sin x \sinh y \tag{16}$$

Equations (15) and (16) imply

$$\sin(iy) = i \sinh y \ , \quad \cos(iy) = \cosh y$$

Example

$$\sin(3+2i) = \underbrace{\sin 3 \cosh 2}_{0.1411 \times 3.7622} + i \underbrace{\cos 3 \sinh 2}_{-0.9900 \times 3.6269} = 0.5309 - 3.5906i$$

We have evaluated sin and cos in radians!!!

$\sin \bar{z}$ and $\cos \bar{z}$ are complex conjugates of $\sin z$ and $\cos z$.

Example

$$\sin \overline{3+2i} = \overline{\sin(3+2i)}$$

Note that $\quad \sin(z+2\pi) = \sin z \qquad \sin(z+\pi) = -\sin z$

$$\cos(z+2\pi) = \cos z \qquad \cos(z+\pi) = -\cos z$$

One may use (15) and (16) to show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Show that,

$$|\sin z|^2 = (\sin x \cosh y + i \cos x \sinh y)(\sin x \cosh y - i \cos x \sinh y)$$

$$= \sin^2 x + \sinh^2 y$$

Some definitions:

$$\tan z = \frac{\sin z}{\cos z} \qquad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z} \qquad \csc z = \frac{1}{\sin z}$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z$$

$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

Exercises

1) Show that
$\exp(2 \pm 3\pi i) = -e^2$; $\qquad e^{(2+\pi i)/4} = \sqrt{e/2}\,(1+i)$
2) State why $2z^2 - 3 - ze^z + e^{-z}$ is entire.
3) Prove that the function $\exp(\overline{z})$ is not analytic anywhere.
4) Prove that $1 + \tan z = \sec z$
5) Find all the roots of $\cos z = 2$.
Ans. $2n\pi - i\cosh^{-1} 2$; that is $2n\pi \pm i\ln(2+\sqrt{3})$, $n \in \mathbb{Z}$

Hyperbolic Functions-Definition

$$\sinh z = \frac{e^z - e^{-z}}{2} \; , \quad \cosh z = \frac{e^z + e^{-z}}{2}$$

$$\rightarrow \quad \frac{d}{dz} \sinh z = \cosh z \; , \quad \frac{d}{dz} \cosh z = \sinh z$$

Some Identities

$$-i\sinh(iz) = \sin z\ , \quad -i\sin(iz) = \sinh z$$

$$\cosh(iz) = \cos z\ , \quad \cos(iz) = \cosh z$$

$$\sinh(-z) = -\sinh z\ , \quad \cosh(-z) = \cosh(z)$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\sinh z = \sinh x \cos y + i \cosh x \sin y$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

For nonzero point $z = re^{i\Theta}$, $(-\pi < \Theta \leq \pi)$, the logarithmic function is defined by

$$\log z = \ln r + i(\Theta + 2n\pi)\ , \quad n = 0, \mp 1, \mp 2, \ldots$$

or

$$\log z = \ln |z| + i \arg z$$

$\log z$ is a multivalued function.
Its principal value $\operatorname{Log} z$ is a single valued function and it is defined as

$$\operatorname{Log} z = \ln r + i\Theta \ , \quad r > 0 \ , \quad -\pi < \Theta \leq \pi$$

Example

$$\log 1 = \log 1e^{i0} = \ln 1 + i(0 + 2n\pi) = 2n\pi i \ , \quad n = 0, \mp 1, \mp 2, \ldots$$

$$\log(-1) = \log 1e^{i\pi} = \ln 1 + i(\pi + 2n\pi) = (2n+1)\pi i \ n = 0, \mp 1, \ldots$$

$$\log(i) = \log 1e^{i\frac{\pi}{2}} = \ln 1 + i(\tfrac{\pi}{2} + 2n\pi) = (2n + \tfrac{1}{2})\pi i \ n = 0, \mp 1, \ldots$$

$$\log(2+3i) = \ln\sqrt{13} + i(0.98 + 2n\pi) = 1.28 + i(0.98 + 2n\pi), \quad n = 0, \mp 1, \mp 2, \ldots$$

$$\operatorname{Log} 1 = 0$$

$$\operatorname{Log}(-1) = \pi i$$

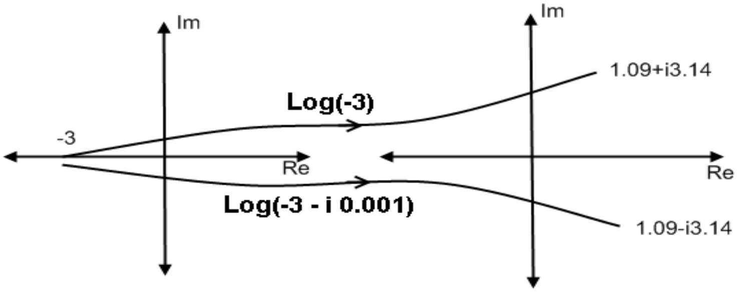
$$\operatorname{Log}(i) = \operatorname{Log} 1e^{i\frac{\pi}{2}} = \ln 1 + i\frac{\pi}{2} = i\frac{1}{2}\pi$$

$$\operatorname{Log}(2+3i) = \ln\sqrt{13} + i0.98 = 1.28 + i0.98$$

$$\operatorname{Log} z = \ln r + i\Theta \ , \quad r > 0 \ , \quad -\pi < \Theta \le \pi$$

The single valued function $\operatorname{Log} z$, whose component functions are $u(r, \Theta) = \ln r$ and $v(r, \Theta) = \Theta$, is not continuous, and therefore not analytic throughout its domain of definition
$r > 0 \ , \quad -\pi < \Theta \le \pi$.
Because when z is on the negative real axis, we can find a neighbouring point z^*, so that the points $\operatorname{Log} z$ and $\operatorname{Log} z^*$ are distant. The following example illustrates this.

Example



Example

- Close points must have close images if the function is continuous there.

Image of z_1 under $\mathrm{Log}(\cdot)$.

$$\mathrm{Log}(-3) = 1.098612289 + i3.141592654$$

Image of z_2 under $\mathrm{Log}(\cdot)$.

$$\mathrm{Log}(-3 - 0.001i) = 1.098612344 - i3.14125932$$

These two close points have distant images.

The function $\operatorname{Log} z$ is not continuous at z_1. Indeed it is not continuous on the negative real axis.
However, for the domain of definition $r > 0\ , \quad -\pi < \Theta < \pi$, the partial derivatives

$$u_r = \frac{1}{r}\ , \quad u_\Theta = 0\ , \quad v_r = 0\ , \quad v_\Theta = 1$$

are continuous in the domain and satisfy the polar form of the C-R equations:

$$u_r = \frac{1}{r} v_\Theta\ , \quad \frac{1}{r} u_\Theta = -v_r$$

In this domain $\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}$.

Example

A branch of a multivalued function f is any single valued function F, which takes one of the values of f. Branch is required to be analytic in its domain.

$$\log z, \quad (|z| > 0\ , \quad \frac{\pi}{6} < \arg z < \frac{\pi}{6} + 2\pi)$$

is a branch of the log function. Notice that $z = 0$ and arg $z = \frac{\pi}{6}$ line are not included in the branch.

$$\frac{d}{dz}\log z = \frac{1}{z}\ , \quad \underbrace{(|z| > 0\ , \quad \frac{\pi}{6} < \arg z < \frac{\pi}{6} + 2\pi)}_{\textit{domain of the branch}}$$

Why?

Why did we define $\log z$ by

$$\log z = \ln r + i(\Theta + 2n\pi)\ , \quad n = 0, \mp 1, \mp 2, \ldots$$

Because, this definition of $\log z$ solves

$$e^{\log z} = z$$

We verify this in the next slide.

$$e^{\log z} = z \tag{17}$$

Reasoning

$$e^{\log z} = e^{\ln r + i(\Theta + 2k\pi)} = e^{\ln r} e^{i(\Theta + 2k\pi)}$$

$$= r(\cos(\Theta + 2k\pi) + i \sin(\Theta + 2k\pi))$$

$$= r(\cos \Theta + i \sin \Theta) = z$$

However

$$\log(e^z) \neq z$$

because:

$$\log(e^z) = \ln|e^z| + i \arg e^z = \ln|e^{x+iy}| + i \arg e^{x+iy}$$

$$= \ln|e^x||e^{iy}| + i(y + 2n\pi)$$

$$= \ln|e^x| + i(y+2n\pi) = x + i(y+2n\pi) = z + i2n\pi \ , \quad n = 0, \mp 1, \mp 2, \ldots$$

For the principal value of the logarithm, we have:

$$\operatorname{Log} e^z = z$$

More Identities

$$\log(z_1 z_2) = \log z_1 + \log z_2 \tag{18}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 \tag{19}$$

Example

Calculate $log - 6i$ directly and by Identity (18):

$$\log(-2) = \log(2e^{i\pi}) = \ln 2 + i(\pi + 2n\pi), \quad n = 0, \mp 1, \mp 2, \ldots$$

$$\log(3i) = \log(3e^{i\frac{\pi}{2}}) = \ln 3 + i(\frac{\pi}{2} + 2m\pi), \quad m = 0, \mp 1, \mp 2, \ldots$$

Compute $\log(-6i)$:

Direct computation:

$$\log(-6i) = \log(6e^{-i\frac{\pi}{2}}) = \ln 6 + i(-\frac{\pi}{2} + 2k\pi), \quad k = 0, \mp 1, \mp 2, \ldots$$

Continued from the previous page

By (18):

$$\log(-6i) = \log(-2 \times 3i) = \log(-2) + \log(3i)$$

$$= \ln 2 + \ln 3 + i(\pi + 2n\pi) + i(\frac{\pi}{2} + 2m\pi)$$

$$= \ln(2 \cdot 3) + i(\frac{3\pi}{2} + 2n\pi + 2m\pi) = \ln(6) + i(-\frac{\pi}{2} + 2k\pi)$$

Example

Compute $\log(\frac{-3i}{2})$:

Direct computation:

$$\log(-\frac{3i}{2}) = \ln(\frac{3}{2}) + i(-\frac{\pi}{2} + 2k\pi), \quad k = 0, \mp 1, \mp 2, \ldots$$

By (19):

$$\log(-\frac{3i}{2}) = \log(3i) - \log(-2) = \ln 3 + i(\frac{\pi}{2} + 2m\pi) - \ln 2 - i(\pi + 2n\pi)$$

$$= \ln(\frac{3}{2}) + i(-\frac{\pi}{2} + 2m\pi - 2n\pi) = \ln(\frac{3}{2}) + i(-\frac{\pi}{2} + 2k\pi)$$

Generalization of $z = e^{\log z}$:

$z^n = e^{\log z^n}$ or $z^n = e^{n \log z}, \quad n = 0, \mp 1, \mp 2, \ldots$

When $z \neq 0$, $\quad z^{1/n} = e^{\frac{1}{n} \log z}$, $\quad n = \pm 1, \pm 2, \ldots$

Exercises

1) Show that when $n \in \mathbb{Z}$
$\log(-1 + \sqrt{3}i) = \ln 2 + 2(n + \frac{1}{3})\pi i$

2) Find all the roots of the equation $\log z = i\frac{\pi}{2}$.

When $z \neq 0$, and the exponent c is any complex number, the identity $z = e^{\log z}$ can be generalized as:

$$z^c = e^{\log z^c} \text{ or } z^c = e^{c \log z} \tag{20}$$

Example

$i^{-2i} = \exp(-2i \log i) = \exp(-2i(2n + \frac{1}{2})\pi i) = \exp((4n+1)\pi)$,
$n = 0, \mp 1, \mp 2, \ldots$

Given z, we want to evaluate $\sin^{-1} z$. Let us denote the result by w.

$$\sin^{-1} z = w \tag{21}$$

$$\rightarrow z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$\rightarrow e^{iw} - e^{-iw} = 2iz$$

$$\rightarrow (e^{iw})^2 - 2ize^{iw} - 1 = 0$$

This is a quadratic polynomial in e^{iw}. Its solution is:

$$e^{iw} = iz \mp \sqrt{1 - z^2}$$

$$\log(e^{iw}) = \log(iz \mp \sqrt{1 - z^2})$$

$$i(w + 2n\pi) = \log(iz \mp \sqrt{1 - z^2})$$

$$w = -i \log(iz \mp \sqrt{1 - z^2}) - 2n\pi$$

By (21) $w = \sin^{-1} z$, hence

$$\sin^{-1} z = -i \log(iz \mp \sqrt{1 - z^2}) - 2n\pi$$

The righthand side term $-2n\pi$ does not have contribution to the overall righthand side, because, the same term is generated by the preceding term.

$$\sin^{-1} z = -i \log(iz \mp \sqrt{1 - z^2})$$

$\sin^{-1} z$ is a multivalued function.

$$\sin^{-1} z = -i \log(iz \mp \sqrt{1-z^2})$$

Example

$$\sin^{-1}(-i) = -i \log(1 \mp \sqrt{2})$$

$$= -i \log(1+\sqrt{2}) \cup -i \log(1-\sqrt{2})$$

$$-i[\ln(1+\sqrt{2})+2n\pi i] \cup -i[\ln(\sqrt{2}-1)+(2m+1)\pi i], \quad n, m = 0, \pm 1, \pm 2, \ldots$$

Likewise $\cos^{-1} z = -i \log[z \mp i\sqrt{1-z^2}]$, $\quad \tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$

Derivatives of $\sin^{-1} z$ & $\cos^{-1} z$ are defined for specific (appropriate) branches.

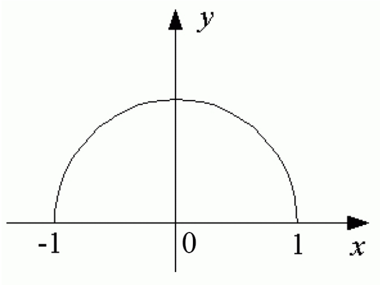
$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}\,, \quad \frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}$$

Inverse hyperbolic functions may be treated likewise

$$\sinh^{-1} z = \log[z + (z^2+1)^{1/2}]$$

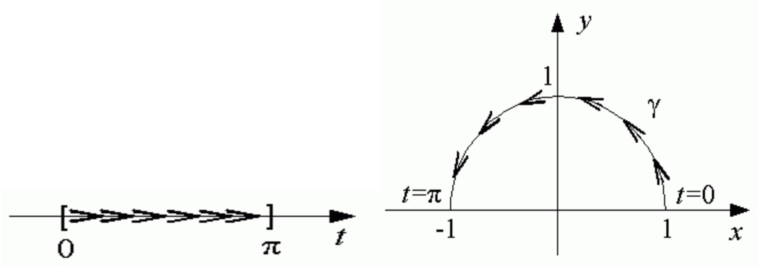
$$\cosh^{-1} z = \log[z + (z^2-1)^{1/2}]$$

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$$



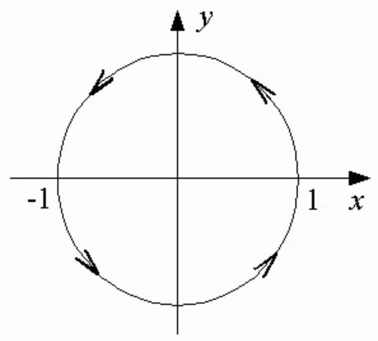
$$y = \sqrt{1-x^2}\,, \quad -1 \le x \le 1$$

Alternatively, this curve can be expressed as a function of one parameter:
$\gamma(t) = \cos(t) + i\sin(t), \quad 0 \leq t \leq \pi$
Each value of t determines a point $\gamma(t$ which traces a curve as t varies.



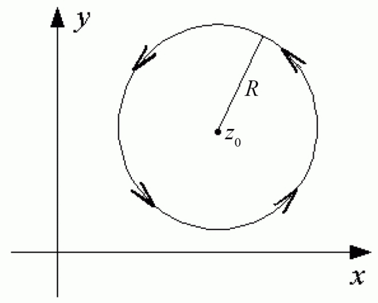
Example

$$\gamma(t) = e^{it} = \cos t + i \sin t \ , \quad 0 \leq t \leq 2\pi$$



Example

$$\gamma(t) = z_0 + Re^{it} \ , \quad 0 \leq t \leq 2\pi$$



Example

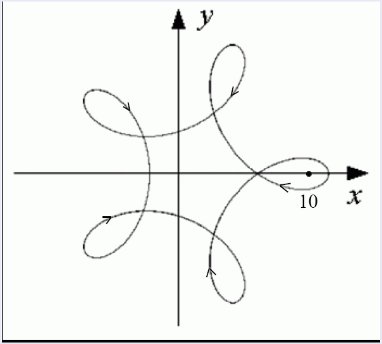
Hypotrochoid

$x(t) = a\cos t + b\cos\frac{at}{2}$

$y(t) = a\sin t - b\sin\frac{at}{2}$

$0 \le t \le 2\pi$

$a = 8\,,\;\; b = 5$ case



Complex Functions of a Real Variable

Given a complex-valued function of a real variable $f = u + iv$, we define the derivative of f in the usual way by

$$f'(t) = \frac{d}{dt} f(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

$$= \lim_{h \to 0} \frac{u(t+h) + iv(t+h) - (u(t) + iv(t))}{h}$$

$$= \lim_{h \to 0} \frac{u(t+h) - u(t)}{h} + i \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}$$

$$\to f'(t) = u'(t) + iv'(t)$$

Some properties

If f and g are complex-valued differentiable functions, α and β are complex numbers, then

$$[\alpha f(t) + \beta g(t)]' = \alpha f'(t) + \beta g'(t)$$

$$[f(t)g(t)]' = f'(t)g(t) + g'(t)f(t)$$

$$\left[\frac{f(t)}{g(t)}\right]' = \frac{f'(t)g(t) - g'(t)f(t)}{[g(t)]^2}, \quad g(t) \neq 0$$

$$[f(g(t))]' = f'(g(t)) \cdot g'(t)$$

Example

Given $\alpha = a + ib$ evaluate $\frac{d}{dt}e^{\alpha t}$.

$$\frac{d}{dt}e^{\alpha t} = \frac{d}{dt}e^{at}e^{ibt} = \frac{d}{dt}\left[e^{at}(\cos bt + i\sin bt)\right] =$$

$$\frac{d}{dt}(e^{at}\cos bt) + i\frac{d}{dt}(e^{at}\sin bt)$$

$$= (ae^{at}\cos bt - be^{at}\sin bt) + i(ae^{at}\sin bt + be^{at}\cos bt)$$

$$= (a + ib)(e^{at}\cos bt + ie^{at}\sin bt)$$

$$= \alpha e^{\alpha t}$$

Definition

A complex-valued function of a real variable f is said to be **piecewise continuous** on $[a, b]$ if the following hold:
(i) f exists and is continuous for all but finitely many points in (a, b).
(ii) At any point c in (a, b) where f fails to be continuous both the left limit $\lim_{t\uparrow c} f(t)$ and the right limit $\lim_{t\downarrow c} f(t)$ exist and are finite.
(iii) At the end points, the right limit $\lim_{t\downarrow a} f(t)$ and the left limit $\lim_{t\uparrow b} f(t)$ exist and are finite.
The function f is said to be **piecewise smooth** on $[a, b]$ if f and f' are both piecewise continuous on $[a, b]$.

Definition

Consider the set of points $z(t) = x(t) + iy(t),\ a \leq t \leq b$. If $z(t)$ is continuous and $z'(t)$ is piecewise continuous, then we call this set of points **a contour** (or **a path**).

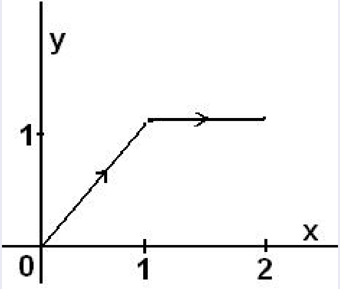
Example

$$z(t) = \begin{cases} t + it & \text{when } 0 \le t \le 1 \\ t + i & \text{when } 1 \le t \le 2 \end{cases}$$

is a contour, because (1) its component functions are continous so are $t + it$ and $t + i$, and their ends have the same value for $t = 1$, and (2)its derivative

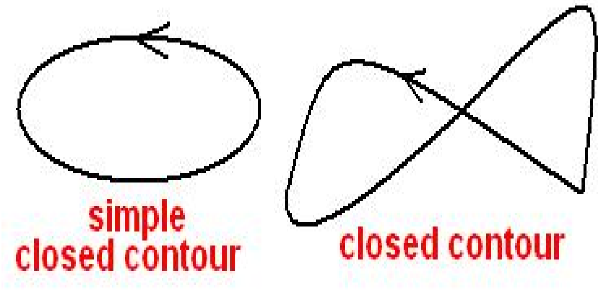
$$z(t) = \begin{cases} 1 + i1 & \text{when } 0 \le t \le 1 \\ 1 + i0 & \text{when } 1 \le t \le 2 \end{cases}$$

is piecewise continuous

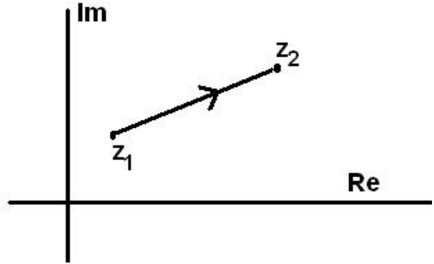


Definition

If a contour crosses itself at its endpoints, then it is called **a closed contour**. If a contour crosses itself only at its endpoints, then it is called **a simple closed contour**.



Parametrization of a straight line

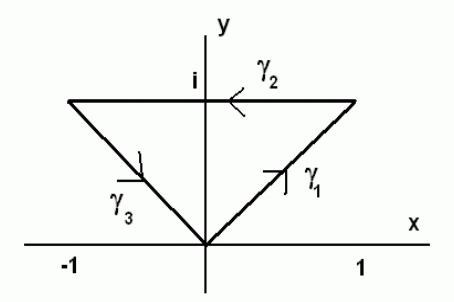


$$(1-t)z_1 + tz_2;\ 0 \le t \le 1$$

$$(1-5t)z_1 + 5tz_2;\ 0 \le t \le \frac{1}{5}$$

$$(1-(t-3))z_1 + (t-3)z_2;\ 3 \le t \le 4$$

Example



$$\gamma_1(t) = (1+i)t\,, \quad 0 \le t \le 1$$
$$\gamma_2(t) = (1-t)(1+i) + t(-1+i) = (1+i) - 2t\,, \quad 0 \le t \le 1$$
$$\gamma_3(t) = (1-t)(-1+i)\,, \quad 0 \le t \le 1$$

We can use the expressions above to parametrize γ over different intervals, say, γ_1 over $[0, 1/3]$, γ_2 over $[1/3, 2/3]$, γ_3 over $[2/3, 1]$. For γ_1 it suffices to change t by $3t$:

$$\gamma_1(t) = 3t(1+i)\ , \quad 0 \leq t \leq 1/3$$

For γ_2 over $[1/3, 2/3]$ we first scale it by changing t to $3t$:

$$\gamma_2(t) = (1+i) - 6t \ , \quad 0 \leq t \leq 1/3$$

then we shift to the right by $1/3$ units:

$$\gamma_2(t) = (1+i) - 6(t - 1/3) = 3 + i - 6t \ , \quad 1/3 \leq t \leq 2/3$$

For γ_3 we first scale it by a factor of $1/3$:

$$\gamma_3(t) = (1 - 3t)(-1+i) \ , \quad 0 \leq t \leq 1/3$$

then we shift to the right by $2/3$ units:

$$\gamma_3(t) = 1 - 3(t - 2/3)(-1+i) = (-1+i)(3-3t) \ , \quad 2/3 \leq t \leq 1$$

Paste them:

$$\gamma(t) = \begin{cases} 3t(1+i)\ , & 0 \le t \le 1/3 \\ 3+i-6t\ , & 1/3 \le t \le 2/3 \\ (-1+i)(3-3t)\ , & 2/3 \le t \le 1 \end{cases}$$

Suppose f is a real valued continuous function on $[a, b]$. Let P be a partition of $[a, b]$ s.t. $a = x_0 < x_1 < \ldots < x_m = b$. Let $\Delta x_k \triangleq x_k - x_{k-1}$. The Riemann sum corresponding to P

$$\sum_{k=1}^{m} f(x_k^*)(x_k - x_{k-1})$$

where x_k^* is a point in $[x_{k-1}, x_k]$.

When the largest interval length approaches zero, this sum is called definite integral of f and denoted by $\int_a^b f(x)dx$. If F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Suppose $f(x) = u(x) + iv(x)$ on $[a, b]$.
$P \triangleq a = x_0 < x_1 < \ldots < x_m = b$.

$$\sum_{k=1}^{m} f(x_k^*)(x_k - x_{k-1}) = \sum_{k=1}^{m} u(x_k^*)(x_k - x_{k-1}) + i \sum_{k=1}^{m} v(x_k^*)(x_k - x_{k-1})$$

$$\rightarrow \int_a^b f(x)dx = \int_a^b (u(x) + iv(x))dx = \int_a^b u(x)dx + i \int_a^b v(x)dx$$

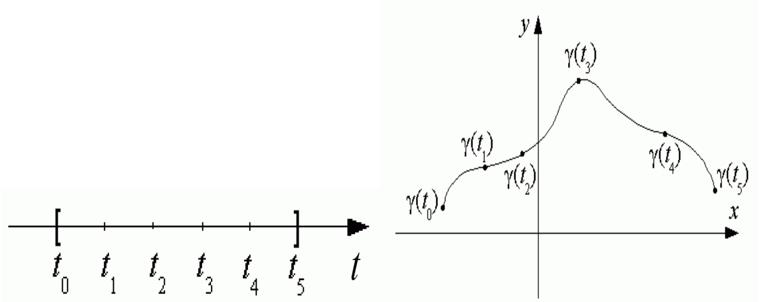
If f is a piecewise continuous complex valued function on $[a, b]$, then its integral over $[a, b]$ may be written as summation of its integrals over adjacent closed subintervals $[a_0, a_1], [a_1, a_2], \ldots, [a_{m-1}, a_m]$ such that f is continuous on each subinterval:

$$\int_a^b f(x)dx = \sum_{j=1}^{m} \int_{a_{j-1}}^{a_j} u(x)dx + i \sum_{j=1}^{m} \int_{a_{j-1}}^{a_j} v(x)dx$$

Suppose $\gamma(t)$, $a \leq t \leq b$, is the path. Suppose that f is a continuous complex valued function on the graph γ. That is

$$t \to f(\gamma(t))$$

is continuous function from $[a, b]$ into $\mathbb{C}$. We want to integrate f over γ.



Let us form a Riemann-like sum:

$$\sum_{k=1}^{m} f(\gamma(t_k^*)) \cdot (\gamma(t_k) - \gamma(t_{k-1}))$$

where $a = t_0 < t_1 < \ldots < t_m = b$. Noting that $\gamma(t) = x(t) + iy(t)$ this becomes

$$\sum_{k=1}^{m} f(\gamma(t_k^*)) \cdot (x(t_k) - x(t_{k-1})) + i \sum_{k=1}^{m} f(\gamma(t_k^*)) \cdot (y(t_k) - y(t_{k-1}))$$

By mean value theorem it becomes

$$\sum_{k=1}^{m} f(\gamma(t_k^*))\cdot x'(\alpha_k)\cdot(t_k - t_{k-1}) + i\sum_{k=1}^{m} f(\gamma(t_k^*))\cdot y'(\beta_k)\cdot(t_k - t_{k-1})$$

where $t_{k-1} < \alpha_k$, $\beta_k < t_k$. As the partition gets finer, this sum converges to

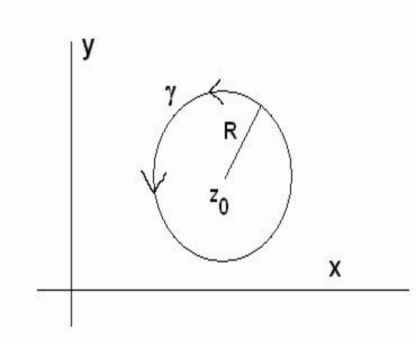
$$\int_a^b f(\gamma(t))\cdot x'(t)dt + i\int_a^b f(\gamma(t))\cdot y'(t)dt = \int_a^b f(\gamma(t))\cdot \gamma'(t)dt$$

Definition

Suppose that γ is a path over a closed interval $[a, b]$, and that f is a continuous complex valued function defined on the graph of γ. The path or contour integral of f on γ is given by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \tag{22}$$

Example



$$\int_\gamma \frac{1}{z - z_0} dz = ?$$

On the path: $\gamma(t) = z_0 + Re^{it}$, $\quad 0 \leq t \leq 2\pi$

$$\gamma(t) = z_0 + Re^{it} \to \gamma'(t) = iRe^{it}$$

Recall the definition of the contour integral

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \tag{22}$$

Using the definition of contour integral

$$\int_\gamma \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{1}{z_0 + Re^{it} - z_0} iRe^{it} dt$$

$$= \int_0^{2\pi} \frac{1}{Re^{it}} iRe^{it} dt = i \int_0^{2\pi} dt = 2\pi i$$

$$\int_\gamma (z-z_0)^n dz = \int_\gamma \frac{1}{z-z_0} dz \text{ when } n=-1$$

is already evaluated. Now, when $n \neq -1$, evaluate

$$\int_\gamma (z-z_0)^n dz$$

$$\int_\gamma (z-z_0)^n dz = \int_0^{2\pi} (Re^{it})^n \cdot iRe^{it} dt = iR^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt$$

$$= \frac{R^{n+1}}{n+1} e^{i(n+1)t} \Big|_0^{2\pi} = \frac{R^{n+1}}{n+1} (e^{2\pi(n+1)i} - e^0) = 0$$

Length of a contour

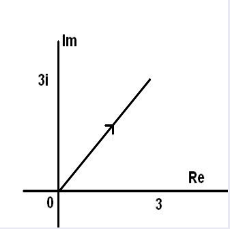
$\gamma(t)\,, \quad a \leq t \leq b,$

$$L = \int_a^b |\gamma'(t)| dt$$

Example

$$\gamma(t) = t + it\,, \quad 0 \leq t \leq 3$$

$$L = \int_0^3 |1 + i| dt = \int_0^3 \sqrt{1^2 + 1^2} dt = 3\sqrt{2}$$

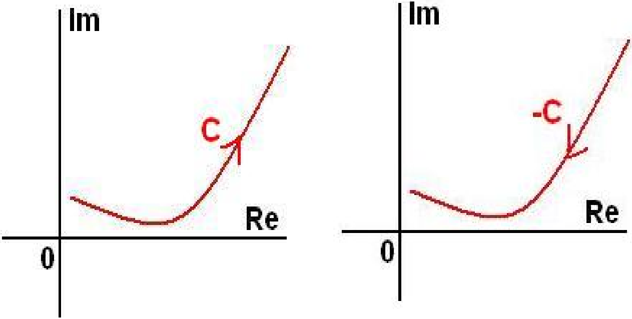


Recall the following from the Calculus course:
Theorem For the curve defined parametrically by $x = g(t)$, $y = h(t)$, $a \leq t \leq b$, if $g,\ g',\ h$ and h' are continuous on $[a, b]$ and the curve does not intersect itself (except possibly at a finite number of points), then the arc length s of the curve is given by

$$s = \int_a^b \sqrt{[g'(t)]^2 + [h'(t)]^2} dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

Exercise Find the arc length of the curve
$x = g(t) = 2\cos t + \sin 2t,\ y = h(t) = 2\sin t + \cos 2t$ for $0 \leq t \leq 2\pi$. Ans. ≈ 16

Associated with the contour C is the contour $-C$, consisting of the same set of points but with the order reversed so that the new contour extends from the point z_2 to the point z_1.



Given the parametrization of C as

$$C: \; z(t), \quad a \leq t \leq b$$

then the contour $-C$ has the following parametric representation

$$-C: \; z(-t) \quad (-b \leq t \leq -a)$$

Example

Given $C: \; t + 3t^2, \; 1 \leq t \leq 3$, find $-C$.

$$-C: -t + 3(-t)^2, \; -3 \leq t \leq -1$$

A property:

$$\int_{-C} f(z)dz = -\int_{C} f(z)dz$$

Suppose that the contour C consists of C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3, then

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Some Properties

- $\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$ for any complex constant z_0.
- $\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$
- $\left| \int_C f(z)dz \right| = \left| \int_a^b f[z(t)]z'(t)dt \right| \leq \int_a^b |f[z(t)]z'(t)|dt$

Estimation Lemma

For any nonnegative constant satisfying $|f(z)| \leq M$, $\forall z$ on C

$$\left| \int_C f(z)dz \right| \leq M \underbrace{\int_a^b |z'(t)|dt}_{L} = ML$$

Proof of an Integral Inequality

Let f be a complex valued function of a real variable t. Assume that

$$\int_a^b f(t)dt$$

exists and equals the complex number $re^{i\theta}$. Thus

$$re^{i\theta} = \int_a^b f(t)dt$$

We have

$$\begin{aligned} r &= \int_a^b e^{-i\theta} f(t)dt \\ &= \int_a^b \operatorname{Re}\left(e^{-i\theta} f(t)\right) dt + i \int_a^b \operatorname{Im}\left(e^{-i\theta} f(t)\right) dt \end{aligned}$$

We know that r is real, therefore,

$$\int_a^b \operatorname{Im}\left(e^{-i\theta} f(t)\right) dt = \operatorname{Im}(r) = 0$$

$$
\begin{aligned}
r &= \int_a^b \mathrm{Re}\left(e^{-i\theta} f(t)\right) dt \\
&\leq \int_a^b \left|\mathrm{Re}\left(e^{-i\theta} f(t)\right)\right| dt \\
&\leq \int_a^b \left|e^{-i\theta} f(t)\right| dt \\
&= \int_a^b \left|e^{-i\theta}\right| \left|f(t)\right| dt \\
&= \int_a^b \left|f(t)\right| dt
\end{aligned}
$$

$$
\therefore \qquad r \leq \int_a^b |f(t)|\, dt
$$

Considering that

$$
re^{i\theta} = \int_a^b f(t)dt \rightarrow \left|re^{i\theta}\right| = r = \left|\int_a^b f(t)dt\right|
$$

We have

$$
\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt
$$

Example

Find an upper bound for

$$\left| \int_{\Gamma} \frac{1}{(z^2+1)^2}\, dz \right|$$

where Γ is the upper half circle with $|z| = a$ with radius $a > 1$ traversed once in the counterclockwise direction.
First observe that the length of the path of integration is half the circumference of a circle with radius a, hence

$$L(\Gamma) = \frac{1}{2}(2\pi a) = \pi a.$$

Next we seek an upper bound M for the integrand when $|z| = a$.
By the triangle inequality we see that

$$|z|^2 = |z^2| = |z^2 + 1 - 1| = |(z^2 + 1) + (-1)| \leq |z^2 + 1| + 1,$$

therefore ...

Continued from the previous page

$$|z|^2 = |z^2| = |z^2 + 1 - 1| = |(z^2 + 1) + (-1)| \leq |z^2 + 1| + 1,$$

therefore

$$|z^2 + 1| \geq |z|^2 - 1 = a^2 - 1 > 0$$

because $|z| = a > 1$ on Γ. Hence

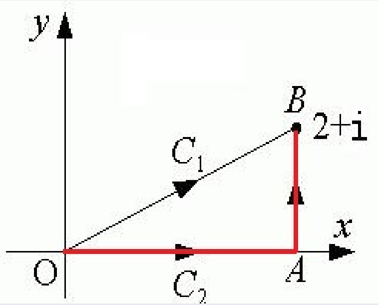
$$\left| \frac{1}{(z^2 + 1)^2} \right| \leq \frac{1}{(a^2 - 1)^2}.$$

Therefore we apply the estimation lemma with $M = 1/(a^2 - 1)^2$. The resulting bound is

$$\left| \int_C f(z) dz \right| \leq ML \to \left| \int_\Gamma \frac{1}{(z^2 + 1)^2} \, dz \right| \leq \frac{\pi a}{(a^2 - 1)^2}.$$

Example

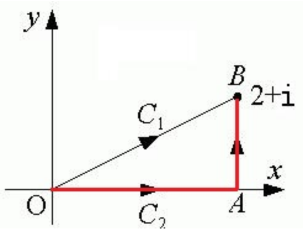
Let us compute $I_1 = \int_{C_1} z^2 dz$.



C_1 is the line segment from $z = 0$ to $z = 2 + i$.
It can be parametrized as $z = 2y + iy\,; \quad 0 \le y \le 1$.
Derivative of the contour is $z' = 2 + i$

For C_1, $z = 2y + iy$, $\quad 0 \leq y \leq 1$ and $z' = 2 + i$, $I_1 = \int_{C_1} z^2 dz$
Integral formula:

$$\int_{\gamma} f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt \tag{22}$$



$$\rightarrow I_1 = \int_0^1 (2y + iy)^2 (2 + i)dy$$

$$= \int_0^1 (2y^2 + i11y^2)dy$$

$$= \frac{2}{3} + i\frac{11}{3}$$

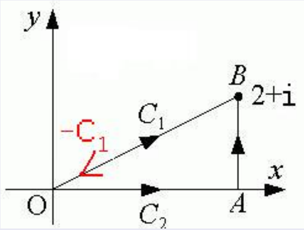
Example

Consider the example above. Let the contour start from $2+i$ and go to $0+i0$ along the points of C_1.

C_1, $z = 2y + iy, \quad 0 \le y \le 1$

To get $-C_1$ we use the formula $-C_1 = z(-y)\,, \quad -b \le y \le -a$

$$z = -2y - iy\,, \quad -1 \le y \le 0$$



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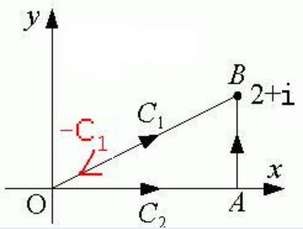
$$-C_1 = -2y - iy\,, \quad -1 \le y \le 0$$

Contour's derivative: $-2 - i$



$$I = \int_{-C_1} z^2 dz = \int_{-1}^{0} (-2y - iy)^2(-2 - i)dy$$

$$= \int_{-1}^{0} (-2y^2 - i11y^2)dy = -\frac{2}{3} - i\frac{11}{3}$$

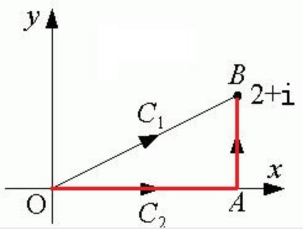
$$= -\int_{C_1} z^2 dz \quad \text{as stated before.}$$

Example

$$I_2 = \int_{C_2} z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz$$

Parametrize $\quad OA$ as $x + i0\,; \quad 0 \leq x \leq 2$

AB as $2 + iy\,; \quad 0 \leq y \leq 1$



$$I_2 = \int_0^2 x^2 dx + \int_0^1 (2 + iy)^2 i dy$$

$$= \frac{8}{3} + i\left[\int_0^1 (4 - y^2) dy + 4i \int_0^1 y dy\right]$$

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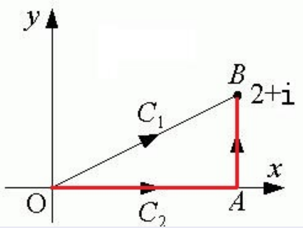
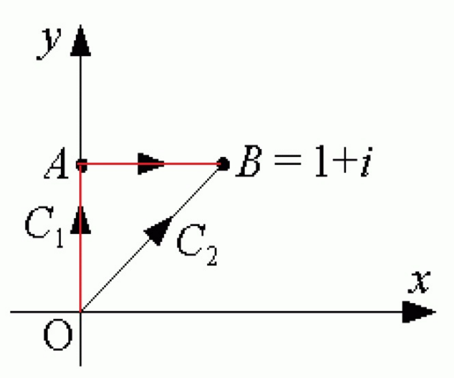


$$I_2 = \int_0^2 x^2 dx + \int_0^1 (2+iy)^2 i dy$$
$$= \frac{8}{3} + i\left[\int_0^1 (4-y^2)dy + 4i\int_0^1 ydy\right]$$
$$= \frac{2}{3} + i\frac{11}{3}$$

Alternative parametrization of C_2: $t\,;\quad 0 \le t \le 2$

$2 + i(t-2)\,;\quad 2 \le t \le 3$

Example



$$\int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$$

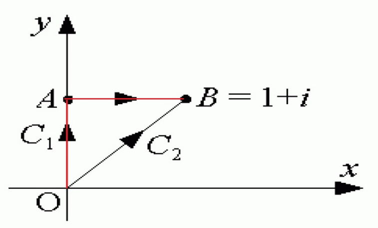
where $f(z) = y - x - i3x^2$.

Continued from the previous page

$C_1 = OA + AB$; $f(z) = y - x - i3x^2$

$OA : z = 0 + iy\,, \quad 0 \leq y \leq 1$

$$\rightarrow \int_{OA} f(z)dz = \int_0^1 yidy = i\int_0^1 ydy = \frac{i}{2}$$



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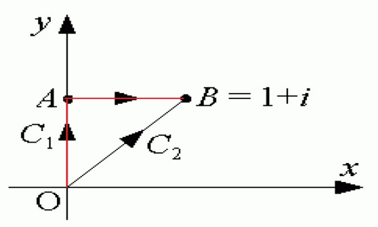
$C_1 = OA + AB$; $f(z) = y - x - i3x^2$

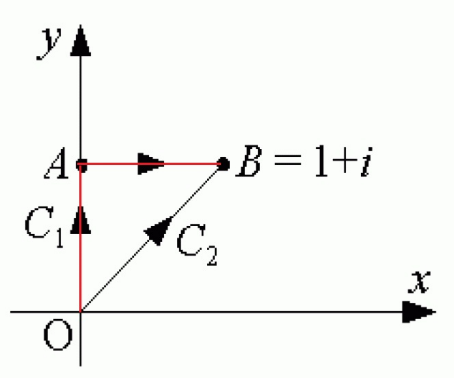


$AB : z = x + i\,, \quad 0 \leq x \leq 1$

$$\rightarrow \int_{AB} f(z)dz = \int_0^1 (1 - x - i3x^2) \cdot 1 \cdot dx$$

$$= \int_0^1 (1-x)dx - 3i \int_0^1 x^2 dx = \frac{1}{2} - i$$

Example



$$\int_{C_1} f(z)dz = \underbrace{\int_{OA} f(z)dz}_{\frac{i}{2}} + \underbrace{\int_{AB} f(z)dz}_{\frac{1}{2}-i} = \frac{1-i}{2}$$

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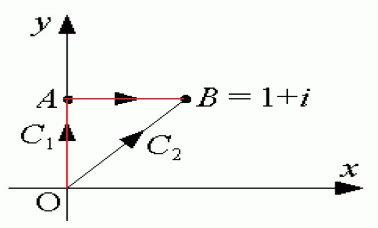
$f(z) = y - x - i3x^2$



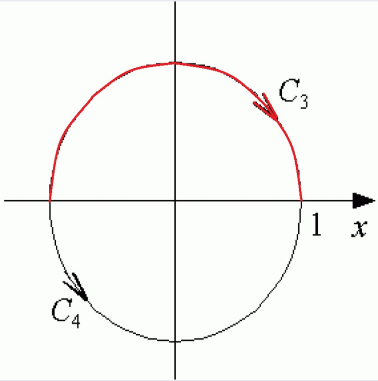
$C_2 : x + ix\,, \quad 0 \le x \le 1$

$$\int_{C_2} f(z)dz = \int_0^1 -i3x^2(1+i)dx = 3(1-i)\int_0^1 x^2 dx = 1 - i$$

The integrals along the two paths C_1 and C_2 have different values even though those paths have the same initial and the same final values.

Example

$C_3 : e^{-i\theta}, \quad -\pi \leq \theta \leq 0$



$$I_3 = \int_{C_3} \bar{z} dz \, ;$$

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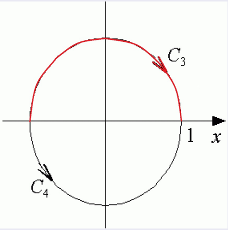
$C_3 : e^{-i\theta}, \quad -\pi \leq \theta \leq 0$



Let us use the identity

$$I_3 = -\int_{-C_3} \bar{z}\, dz$$

where $-C_3 : e^{i\theta}; \quad 0 \leq \theta \leq \pi$

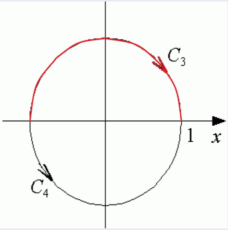
$$I_3 = -\int_0^{\pi} e^{-i\theta} i e^{i\theta} d\theta = -i\pi$$

Example

$C_4 : e^{i\theta}, \qquad \pi \leq \theta \leq 2\pi$

$$I_4 = \int_{C_4} \overline{z} dz = \int_{\pi}^{2\pi} e^{-i\theta} i e^{i\theta} d\theta = i\pi$$

Note that $I_3 \neq I_4$.



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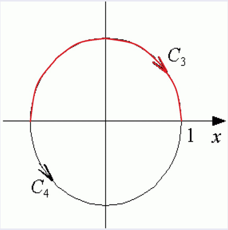


Define $\quad C_0 = C_4 - C_3$

$$I_0 = \int_{C_0} \overline{z} dz = I_4 - I_3 = 2\pi i$$

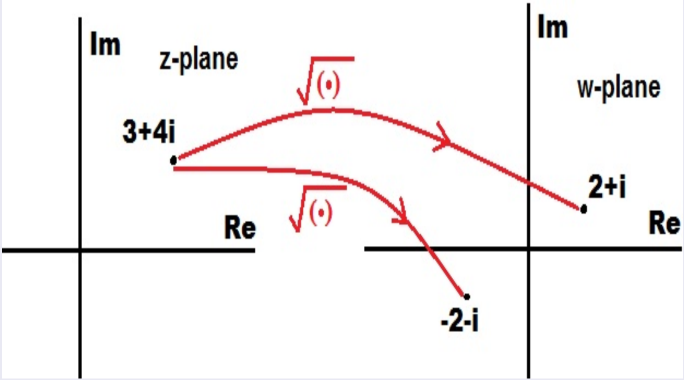
When z is on the unit circle

$$|z| = 1 \to |z|^2 = 1^2 \to z\overline{z} = 1\,; \quad \overline{z} = \frac{1}{z}$$

$$\to I_0 = \int_{C_0} \frac{1}{z} dz = 2\pi i$$

Consider the function $w = z^{\frac{1}{2}}$. This is a multivalued function. This function maps $z = re^{i\theta}$ to two points: $r^{\frac{1}{2}}e^{i\frac{\theta}{2}}$ and $-r^{\frac{1}{2}}e^{i\frac{\theta}{2}}$

Example



Example

The function $w = z^{\frac{1}{2}}$ has two branches. The branch

$$f(z) = \sqrt{r}e^{i\theta/2}\,; \quad r > 0\,; \quad 0 < \theta < 2\pi$$

is analytic. The positive real axis with 0 is excluded from the domain in order to avoid discontinuities on it.

If we consider the following function and its domain

$$f(z) = \sqrt{r}e^{i\theta/2}\,; \quad r > 0\,; \quad 0 \leq \theta < 2\pi$$

then, for instance, the neighbouring points of $^{z} = 3e^{i0}$ won't have neighbouring images under the function given above. Thus, it wouldn't be a branch of $w = z^{\frac{1}{2}}$.

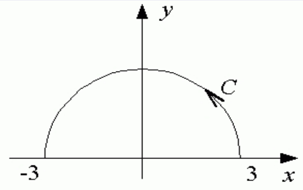
Example

Consider $f(z) = z^{1/2}$. A branch of this multivalued function is

$$f(z) = \sqrt{r}e^{i\theta/2}\,; \quad r > 0\,; \quad 0 < \theta < 2\pi$$

Consider the semicircular path C defined by $z = 3e^{i\theta}\,; \quad 0 \leq \theta \leq \pi$

The function is not defined at the point $3 + i0$, but this does not harm the existence of the integral. Integral is defined for piecewise continuous functions. Nonetheless, we may make it continuous by defining $f(3 + i0) = \sqrt{3}$.



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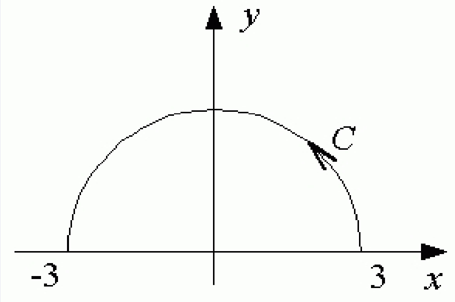


Noting $C: \; 3e^{i\theta}\,; \quad 0 \leq \theta \leq \pi$, let us compute $I = \int_C z^{1/2} dz$ for the branch of $z^{1/2}$ described.

$$I = \int_0^{\pi} \sqrt{3} e^{i\theta/2} 3ie^{i\theta} d\theta = 3\sqrt{3}i \int_0^{\pi} e^{i3\theta/2} d\theta$$

$$= 3\sqrt{3}i \left\{ \frac{-2}{3i}(1+i) \right\} = -2\sqrt{3}(1+i)$$

Example

Let the contour C be as in the previous example. Also let us use the same branch of $z^{1/2}$ as in the preceding example.

We show that

$$\left| \int_C \frac{z^{1/2}}{z^2+1} dz \right| \leq \frac{3\sqrt{3}\pi}{8}$$

Recall and use $\left| \int_C f(z) dz \right| \leq ML$.

Clearly, $|z| = 3$ on C, therefore $|z^{1/2}| = \sqrt{3}$

Also $|z^2 + 1| \geq ||z|^2 - 1| = 8$. Because $|z_1 + z_2| \geq ||z_1| - |z_2||$, pp.10 Churchill

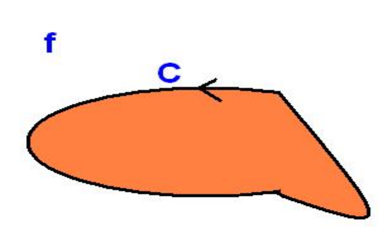
$$\rightarrow \left| \frac{z^{1/2}}{z^2+1} \right| \leq \frac{\sqrt{3}}{8}$$

Since the length of the contour is $L = 3\pi$. The conclusion follows.

Theorem

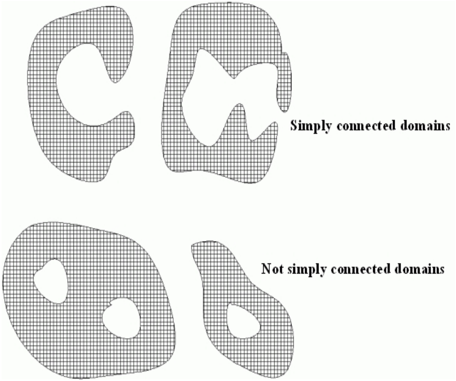
If a function f is analytic at all points interior to and on a simple closed contour C, then

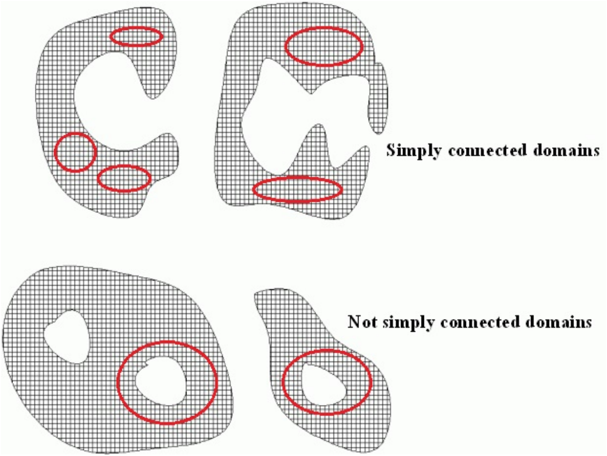
$$\int_C f(z)dz = 0$$



Definition

A **simply connected domain** D is a domain such that every simple closed contour within it encloses only points of D.



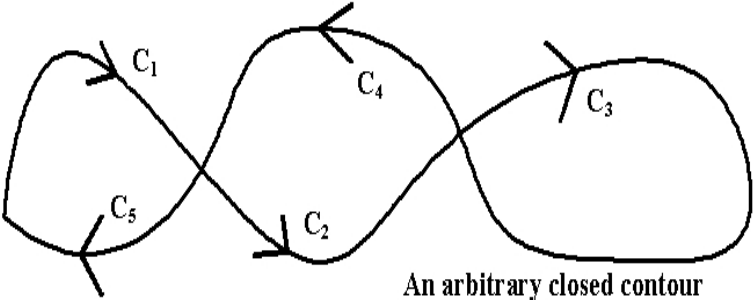

Simply connected domains
Not simply connected domains

Alternative statement of the theorem

If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)dz = 0$$

for every simple closed contour C lying in D.
The simple closed contour here can be replaced by an arbitrary closed contour C which is not necessarily simple. If C intersects itself a finite number of times, it consists of a finite number of simple closed contours. By applying the C-G theorem to each of those simple closed contours, we obtain the desired result for C.

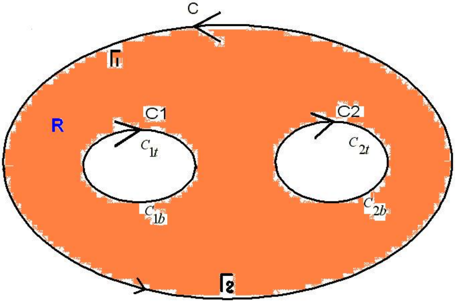


An arbitrary closed contour

$$\int_C f(z)dz := \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz + \int_{C_5} f(z)dz$$

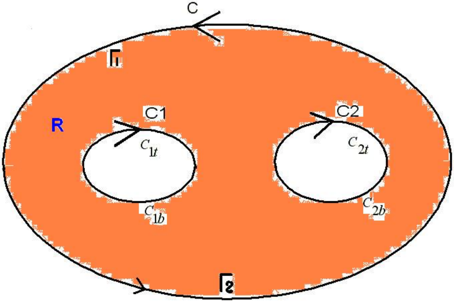
$$= \underbrace{\int_{C_1} f(z)dz + \int_{C_5} f(z)dz}_{=0 \ by \ C-G} + \underbrace{\int_{C_4} f(z)dz + \int_{C_2} f(z)dz}_{=0 \ by \ C-G} + \underbrace{\int_{C_3} f(z)dz}_{=0 \ by \ C-G}$$

Generalizing the C-G Theorem to Multiply Connected Domains



Theorem

Let C be a simple closed contour and let $C_j \quad (j = 1, \ldots, n)$ be a finite number of simple closed contours inside C such that the regions interior to each C_j have no points in common.



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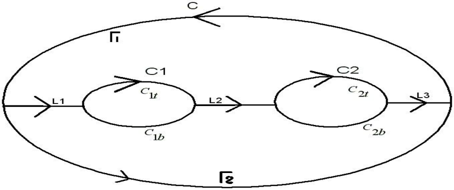
Let R be the closed region consisting of all points within and on C except for points interior to each C_j. Let B denote the entire oriented boundary of R consisting of C and all the contours C_j, described in a direction such that the interior points of R lie to the left of B. Then if f is analytic throughout R

$$\int_B f(z)dz = 0$$

Proof



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$$\int_B \dots := \int_C \dots + \int_{C_1} \dots + \int_{C_2} \dots$$

$$= \underbrace{\int_{\Gamma_1} \dots + \int_{L_1} \dots + \int_{C_{1t}} \dots + \int_{L_2} \dots + \int_{C_{2t}} \dots + \int_{L_3} \dots}_{=0 \; by \; C-G} +$$

$$\underbrace{\int_{\Gamma_2} \dots + \int_{-L_3} \dots + \int_{C_{2b}} \dots + \int_{-L_2} \dots + \int_{C_{1b}} \dots + \int_{-L_1} \dots}_{=0 \; by \; C-G}$$

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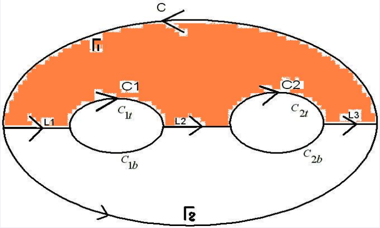

Because hypothesis says that f is analytic throughout R including its boundaries, f is analytic on and inside the contour above. Thus, Cauchy-Goursat Theorem results in

$$\int_{\Gamma_1} \ldots + \int_{L_1} \ldots + \int_{C_{1t}} \ldots + \int_{L_2} \ldots + \int_{C_{2t}} \ldots + \int_{L_3} \ldots = 0$$

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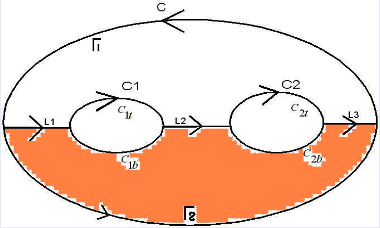


Likewise, because hypothesis says that f is analytic throughout R including its boundaries, f is analytic on and inside the contour above. Thus, Cauchy-Goursat Theorem results in

$$\int_{\Gamma_2} \cdots + \int_{-L_3} \cdots + \int_{C_{2b}} \cdots + \int_{-L_2} \cdots + \int_{C_{1b}} \cdots + \int_{-L_1} \cdots = 0$$

Continued from the previous page

Notice that we used the facts:

$$\int_{L_i} \ldots + \int_{-L_i} \ldots = 0$$

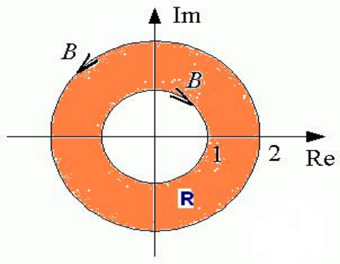
$$\int_{C_i} \ldots = \int_{C_{it}} \ldots + \int_{C_{ib}} \ldots$$

and

$$\int_{C} \ldots = \int_{\Gamma_1} \ldots + \int_{\Gamma_2} \ldots$$

Example

$$\int_B \frac{dz}{z^2(z^2+9)} = 0$$



Generalization of the C-G theorem to multiply connected domains is applicable here. The hypotheses of the theorem are satisfied: The integrand is analytic except at the points $z = 0$ and $z = \mp 3i$, all of which lie outside the annular region with boundary B. Therefore, along the oriented boundary the integral equals zero.

Antiderivatives

There are certain functions whose integrals from z_1 to z_2 are independent of the path.

Definition

The function F is said to be an antiderivative of f in a domain D if: $F'(z) = f(z)$ for all z in D.

Example

$f(z) = z$ has an antiderivative $F(z) = \frac{z^2}{2}$. Another antiderivative of $f(z)$ is $F(z) = \frac{z^2}{2} + 3$.

Theorem

Suppose that a function f is continuous in a domain D. TFAE:

a) *f has an antiderivative F in D.*

b) *The integrals of f along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value.*

c) *The integrals of f around closed contours lying entirely in D all have value zero.*

Example

The continuous function $f(z) = z^2$ has an antiderivative $F(z) = z^3/3$ throughout $\mathbb{C}$.

$$\int_0^{1+i} z^2 dz = \left.\frac{z^3}{3}\right|_0^{1+i} = \frac{2}{3}(-1+i)$$

holds true for every contour from $z = 0$ to $z = 1 + i$.

Example

The function $1/z^2$, which is continuous everywhere, except at the origin, has an antiderivative $-1/z$ in the domain $|z| > 0$. Consequently

$$\int_{z_1}^{z_2} \frac{1}{z^2} dz = -\frac{1}{z}\bigg|_{z_1}^{z_2} = \frac{1}{z_1} - \frac{1}{z_2} \quad (z_1 \neq 0\,, \quad z_2 \neq 0)$$

for any contour from z_1 to z_2 that does not pass through the origin. In particular $\int_C (1/z^2) dz = 0$ when C is the circle $z = 2e^{i\theta}\,, \quad -\pi \leq \theta \leq \pi$

Example

$D : |z| > 0\,, \quad -\pi < \mathrm{Arg} z < \pi$ (i.e. negative real axis is excluded)

$$\int_{-2i}^{2i} \frac{dz}{z} = \mathrm{Log}\, z \Big|_{-2i}^{2i} = \mathrm{Log}\,(2i) - \mathrm{Log}\,(-2i)$$

$$= \left(\ln 2 + i\frac{\pi}{2} \right) - \left(\ln 2 - i\frac{\pi}{2} \right) = \pi i$$

when the path of the integration does not cross the negative real axis.

Example

$$\int_i^{\pi/2} \cos z dz = \sin z \Big|_i^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin i = 1 - i \sinh 1 = 1 - i1.1752$$

Cauchy Integral Formula

Theorem

Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_0 is any interior point of C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0} \quad \textit{Cauchy integral formula}$$

This can be written also as

$$2\pi i f(z_0) = \int_C \frac{f(z)dz}{z - z_0}$$

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0} \quad \text{Cauchy integral formula}$$

$$2\pi i f(z_0) = \int_C \frac{f(z)dz}{z - z_0} \quad \text{Cauchy integral formula}$$

Example

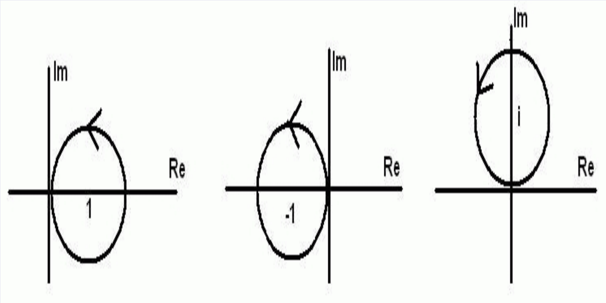
Let C be the positively oriented circle $|z| = 2$.

$$\int_C \frac{zdz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)} dz = 2\pi i \left(\frac{-i}{10}\right) = \frac{\pi}{5}$$

Note that the function $f(z) = \frac{z}{9 - z^2}$ is analytic within and on C and the point $z_0 = -i$ is interior to C.

Example

Integrate $\frac{z^2+1}{z^2-1}$ in the ccw sense around a circle of radius 1 with center at the point **a)** $z = 1$ **b)** $z = -1$ **c)** $z = i$



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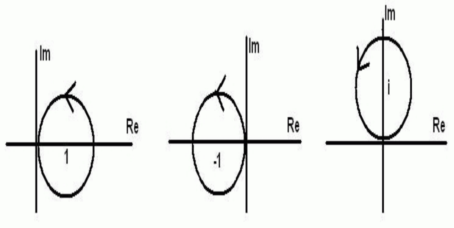

$$\text{(a)} \qquad \int_C \frac{z^2+1}{z^2-1}\,dz = \int_C \frac{\frac{z^2+1}{z+1}}{z-1}\,dz$$

The expression on the right has $z_0 = 1$ and $f(z) = \frac{z^2+1}{z+1}$
The point $z_0 = 1$ lies inside the circle C under consideration, and $f(z)$ is analytic inside and on C. The point $z = -1$ where $f(z)$ is not analytic, lies outside C. Hence by Cauchy integral formula

$$\int_C \frac{z^2+1}{z^2-1}\,dz = \int_C \frac{\frac{z^2+1}{z+1}}{z-1}\,dz = 2\pi i \left[\frac{z^2+1}{z+1}\right]_{z=1} = 2\pi i$$

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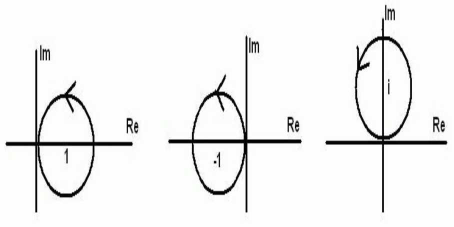

$$\text{(b)} \qquad \int_C \frac{z^2+1}{z^2-1}dz = \int_C \frac{\frac{z^2+1}{z-1}}{z+1}dz = \int_C \frac{\frac{z^2+1}{z-1}}{z-(-1)}dz$$

Now $f(z) = \frac{z^2+1}{z-1}$ is analytic within and on C and $z_0 = -1$ is an interior point of C. Therefore

$$\int_C \frac{z^2+1}{z^2-1}dz = \int_C \frac{\frac{z^2+1}{z-1}}{z-(-1)}dz = 2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1} = -2\pi i$$

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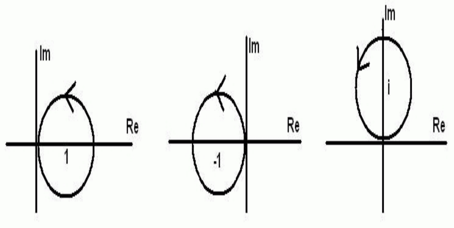


c) The given function is analytic everywhere on and inside the circle. Hence by the Cauchy-Goursat theorem, the integral

$$\int_C \frac{z^2+1}{z^2-1}dz$$

has the value zero.

Exercises

Integrate $z^2/(z^2+1)$ in the ccw sense around the circle
1) $|z+i|=1$ **2)** $|z-i|=1/2$ **3)** $|z|=2$ **4)** $|z|=1/2$
Integrate $z^2/(z^4-1)$ in the ccw sense around the circle
5) $|z-1|=1$ **6)** $|z+i|=1$ **7)** $|z-i|=1/2$ **8)** $|z|=2$
Answers
1) π **3)** 0 **5)** $i\pi/2$ **7)** $\pi/2$

9) Let C denote the boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$ where C is described in positive sense. Evaluate each of these integrals

a) $\int_C \frac{e^{-z}dz}{z-(i\pi/2)}$ Ans. 2π

b) $\int_C \frac{\cos z dz}{z(z^2+8)}$ Ans. $i\frac{\pi}{4}$

Cauchy Integral Formula

Theorem

Let f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_0 is any interior point of C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0} \quad \textit{Cauchy integral formula}$$

This can be written also as

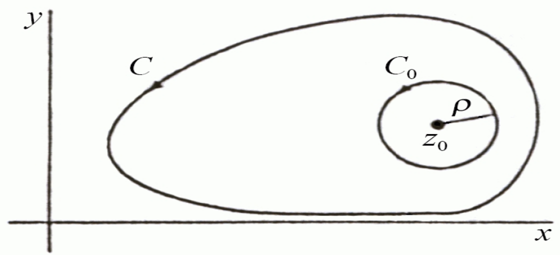
$$2\pi i f(z_0) = \int_C \frac{f(z)dz}{z - z_0}$$

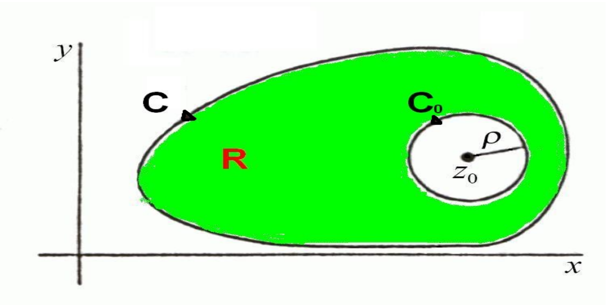
Since f is analytic everywhere within C, automatically it is continuous everywhere within C, specifically at z_0. At z_0, for a given ε there exist δ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

Select ρ less than δ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| = \rho$$





Consider $\frac{f(z)}{z-z_0}$. Numerator is analytic in R; the denominator equals zero nowhere in R. Thus $\frac{f(z)}{z-z_0}$ is analytic at all points on C and C_0 and at all points in between them, we can use the generalized C-G theorem:

$$\int_C \frac{f(z)}{z-z_0}dz - \int_{C_0} \frac{f(z)}{z-z_0}dz = 0$$

Note that the direction of C_0 is not as in the Generalized C-G theorem for the multiply connected domains, this caused us to use "-" instead of "+" in the expression above.

$$\int_C \frac{f(z)}{z-z_0}dz - \int_{C_0} \frac{f(z)}{z-z_0}dz = 0$$

This is equivalent to

$$\int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \frac{f(z)dz}{z-z_0}$$

Subtract $\int_{C_0} \frac{f(z_0)dz}{z-z_0}$ from both sides of equality to obtain

$$\int_C \frac{f(z)dz}{z-z_0} - \int_{C_0} \frac{f(z_0)dz}{z-z_0} = \int_{C_0} \frac{f(z)}{z-z_0}dz - \int_{C_0} \frac{f(z_0)}{z-z_0}dz$$

$$\int_C \frac{f(z)dz}{z-z_0} - f(z_0)\int_{C_0} \frac{dz}{z-z_0} = \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz$$

By a previous example:

$$\int_{C_0} \frac{dz}{z-z_0} = 2\pi i$$

$$\rightarrow \int_C \frac{f(z)dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}dz$$

$$\rightarrow \int_C \frac{f(z)dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0} dz$$

Since the length of C_0 is $2\pi\rho$, using the modulus inequality on integrals

$$\left| \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0} dz \right| < \underbrace{\frac{\varepsilon}{\rho}}_{M} \underbrace{2\pi\rho}_{L} = 2\pi\varepsilon$$

$$\rightarrow \left| \int_C \frac{f(z)dz}{z-z_0} - 2\pi i f(z_0) \right| < 2\pi\varepsilon$$

Since the lefthand side is constant and righthand side is arbitrarily small (because we can select ε arbitrarily small) the lefthand side must be 0. Consequently

$$\int_C \frac{f(z)dz}{z-z_0} = 2\pi i f(z_0) \qquad \text{Q.E.D.}$$

Derivatives of Analytic Functions

Theorem

If a function is analytic at a point, then its derivatives of all orders are also analytic functions at that point.

Example

$f(z) = \frac{1}{z}$ is analytic at $z_0 = 1 + i$. Each of the functions $f', f'', f''', f^{(4)}, f^{(5)}, \ldots$ is analytic at $z_0 = 1 + i$.

Corollary

If a function $f(z) = u(x, y) + iv(x, y)$ is analytic at a point $z = x + iy$, then the component functions u and v have continuous partial derivatives of all orders at that point.

Note that if f is analytic at z, then its component functions satisfy the Cauchy Riemann equations at z. This shows the existence of u_x, u_y, v_x, v_y at z. The derivative is then

$$f'(z) = u_x + iv_x \text{ or equivalently } f'(z) = v_y - iu_y$$

By the theorem, f' is also analytic at z. Therefore, the real and imaginary components of f' must satisfy the C-R equations at z.

$$f'(z) = u_x + iv_x \text{ or equivalently } f'(z) = v_y - iu_y$$

By the theorem, f' is also analytic at z. Therefore, the real and imaginary components of f' must satisfy the C-R equations at z. That is,

$$u_{xx} = v_{xy}, \quad u_{xy} = -v_{xx}$$

$$v_{yx} = -u_{yy}, \quad v_{yy} = u_{yx}$$

This shows existence of all the 2nd order partial derivatives at z. Continuing this process shows that u and v have continuous partial derivatives of all orders at z.

A Generalization of Cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots$$

Under the same hypotheses as in Cauchy integral formula.
Alternatively:

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots$$

Example

Let C be $|z| = 10$, positively oriented and let $f(z) = 1$. Then

$$\int_C \frac{dz}{(z-2)^5} = \frac{2\pi i}{4!} f^{(4)}(2) = 0$$

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots$$

Example

Evaluate

$$\int_C \frac{e^{5z}}{(z-i)^3} dz$$

where $C = \{z : \ |z| = 2\}$ oriented ccw.

Solution Noting $f(z) = e^{5z}$ is entire, and $z_0 = i$ is is an interior point of C, we have

$$\int_C \frac{e^{5z}}{(z-i)^3} dz = \frac{2\pi i}{2!} f''(i) = 25\pi i e^{5i}$$

Theorem

(Morera's Theorem) If a function f is continuous throughout a domain D and if $\int_C f(z)dz = 0$ for every closed contour C lying in D, then f is analytic throughout D.

Definition

We say that a sequence $\{a_n\}$ converges to a complex number L, or has limit L, as n tends to infinity and write

$$\lim_{n\to\infty} a_n = L$$

if given any $\varepsilon > 0$ there is an integer N such that

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n \geq N$$

If the sequence $\{a_n\}$ does not converge, then we say that it diverges.

Example

Consider the sequence

$$\left\{e^{in\frac{\pi}{4}}\right\} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ i,\ -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ -1,$$

$$-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},\ -i,\ \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},\ 1, \ldots$$

The sequence is not converging since its terms will cycle over the first eight terms indefinitely.

Example

$$\left\{ \frac{e^{in(\pi/4)}}{n} \right\}$$

Let us show that it converges to $L = 0$. Given $\varepsilon > 0$ we have

$$|a_n - L| = \left| \frac{e^{in(\pi/4)}}{n} - 0 \right| = \frac{\left| e^{in(\pi/4)} \right|}{n} = \frac{1}{n} < \varepsilon$$

for all $n > \frac{1}{\varepsilon}$, and so the sequence converges to $L = 0$.

Theorem

Suppose that $\{a_n\}$ *is a sequence of complex numbers and write* $a_n = x_n + iy_n$*, where* $x_n = \operatorname{Re} a_n$ *and* $y_n = \operatorname{Im} a_n$*. Then*

$$\lim_{n\to\infty} a_n = L = \alpha + i\beta \quad \Leftrightarrow \quad \lim_{n\to\infty} x_n = \alpha \quad \textit{and} \quad \lim_{n\to\infty} y_n = \beta$$

Definition

An infinite series of complex numbers $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$ converges to a number S, called the sum of the series, if the sequence

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_N \,; \quad (N = 1, 2, \dots)$$

of partial sums converges to S.

Define a sequence $S_1 := a_1, \; S_2 := a_1 + a_2, \; S_3 := a_1 + a_2 + a_3, \dots$. If the sequence $S_1, S_2, S_3, \dots$ converges, then it converges to the sum of the series above.

Theorem

Suppose that $a_n = x_n + iy_n \quad (n = 1, 2, \ldots)$ *then*

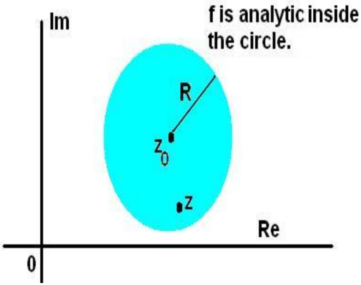
$$\sum_{n=1}^{\infty} a_n = S = X + iY \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} x_n = X \quad \textit{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

Theorem

Let f be analytic everywhere inside a circle C with center at z_0 and radius R. Then at each point inside C

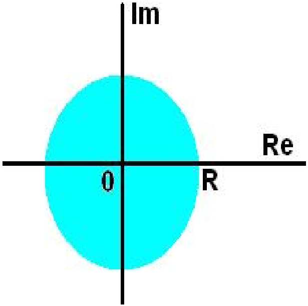
$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \qquad (23)$$

that is, the power series here converges to $f(z)$ when $|z - z_0| < R$.



(23) is called the expansion of f into a Taylor series about the point z_0. Taylor expansion for $z_0 = 0$ is called Maclaurin expansion:

$$f(z) = f(0) + \frac{f'(0)}{1!}(z) + \frac{f''(0)}{2!}(z)^2 + \ldots$$



Maclaurin expansion:

$$f(z) = f(0) + \frac{f'(0)}{1!}(z) + \frac{f''(0)}{2!}(z)^2 + \dots$$

Example

Since the function e^z is entire, it has a Maclaurin series representation which is valid for all z. Noting that

$$\left.\frac{d^n}{dz^n}e^z\right|_{z=0} = 1$$

we have

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

Example

$$f(z) = \sin z \quad \to \left\{ \begin{array}{l} f^{(2n)}(0) = 0 \\ f^{(2n+1)}(0) = (-1)^n \end{array} \right\} \quad n = 0, 1, \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \ldots$$

Similarly

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \ldots$$

Example

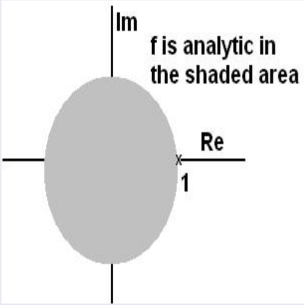
Note that $\sinh z = -i \sin(iz)$. In order to compute the Taylor series of $\sinh z$, we only need to replace z by iz in the Taylor series expansion of $\sin z$, and multiply the result by $-i$.

$$\underbrace{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \ldots}_{\sin z} \rightarrow \underbrace{-i\left[iz - \frac{(iz)^3}{3!} + \frac{(iz)^5}{5!} - \frac{(iz)^7}{7!} \ldots\right]}_{\sinh z}$$

Example

$\frac{1}{1-z}$ is analytic in $|z| < 1$. Its Maclaurin series is

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1 \tag{24}$$



Example

To find power series representation of $\frac{1}{1+z}$ we may utilize that of $\frac{1}{1-z}$ given in (24). In (24) we substitute $-z$ for z. That is,

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1 \tag{24}$$

becomes

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z| < 1$$

Note that $|z| < 1 \quad \leftrightarrow \quad |-z| < 1.$
Similarly

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \ldots = \sum_{n=0}^{\infty} z^{2n}, \quad |z| < 1$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1 \tag{24}$$

Example

Develop $\frac{1}{c-bz}$ in powers of $(z-a)$ where $c - ab \neq 0$ and $b \neq 0$.

$$\frac{1}{c - bz} = \frac{1}{c - ab - b(z-a)} = \frac{1}{(c-ab)\left[1 - \frac{b(z-a)}{c-ab}\right]}$$

$$= \frac{1}{(c-ab)} \times \frac{1}{\left[1 - \frac{b(z-a)}{c-ab}\right]} = \frac{1}{c-ab} \sum_{0}^{\infty} \left[\frac{b(z-a)}{c-ab}\right]^n$$

This series is convergent for $\left|\frac{b(z-a)}{c-ab}\right| < 1$, that is $|z-a| < \left|\frac{c-ab}{b}\right| = \left|\frac{c}{b} - a\right|$.

Example

Find Maclaurin series of $f(z) = \tan z$.

$$\tan z = \frac{\sin z}{\cos z}$$

tan has singularities at the zeros of cos, i.e., at $\frac{\pi}{2} + n\pi,\ n = 0, 1, \ldots$. Thus, for the tan function, the region of convergence around the origin is $|z| < \frac{\pi}{2}$.

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$$f(0) = \tan(0) = 0$$

$$f'(z) = \sec^2 z = 1 + \tan^2 z = 1 + f^2(z) \text{ and } f'(0) = 1$$

$$f'(z) = 1 + f^2(z) \;\to\; f'' = 2ff' \quad \text{and} \quad f''(0) = 0$$

$$f''' = 2(f')^2 + 2ff'' \quad \text{and} \quad f'''(0) = 2 \to f'''(0)/3! = 1/3$$

$$f^{(4)} = 6f'f'' + 2ff''' \quad \text{and} \quad f^{(4)}(0) = 0$$

$$f^{(5)} = 6(f'')^2 + 8f'f''' + 2ff^{(4)}$$

and

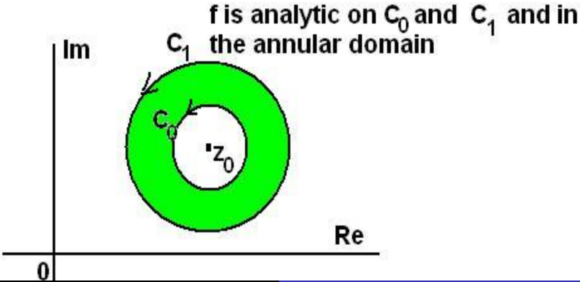
$$f^{(5)}(0) = 16\,, \quad \left(f^{(5)}(0)\right)/5! = 2/15$$

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \ldots \; |z| < \frac{\pi}{2}$$

Theorem

Let C_0 and C_1 denote two positively oriented circles centered at a point z_0 where C_0 is smaller than C_1. If a function is analytic on C_0 and C_1 and throughout the annular domain between them, then at each point z in that domain $f(z)$ is represented by the expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}$$



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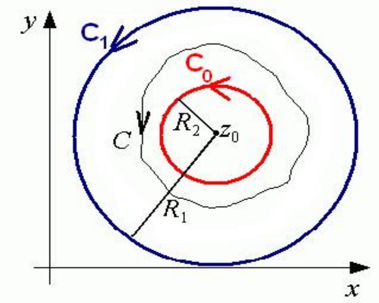
In the Laurent expansion we have

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}\,; \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}}\,; \quad n = 1, 2, \ldots$$

and C is any positively oriented simple closed contour in the annular domain encircling C_0.

The series here is called Laurent Series.



Alternative style of expression for the Laurent series formula

$$f(z) = \sum_{n=-\infty}^{\infty} d_n (z - z_0)^n$$

where

$$d_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

Example

$\int_C (z-z_0)^m dz$

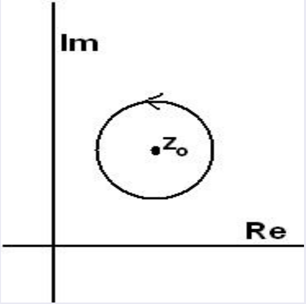
where C is the positively oriented circle $|z-z_0| = R$ with $R > 0$.

Path parametrization: $z(\theta) = z_0 + Re^{i\theta}$, $\quad 0 \le \theta \le 2\pi$

$z'(\theta) = iRe^{i\theta}$

$\rightarrow \int_C (z-z_0)^m dz = \int_0^{2\pi} R^m e^{im\theta}\, iRe^{i\theta} d\theta = iR^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} d\theta$

when $m = -1$, it is $\quad iR^{-1+1} \int_0^{2\pi} e^{i(-1+1)\theta} d\theta = i \int_0^{2\pi} d\theta = 2\pi i$



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When $m \neq -1$ it is $\quad iR^{m+1} \frac{e^{i(m+1)\theta}}{i(m+1)}\Big|_0^{2\pi} = 0$

$$\therefore \quad \int_C (z - z_0)^m dz = \begin{cases} 2\pi i & \text{when} \quad m = -1 \\ 0 & \text{when} \quad m \neq -1 \end{cases}$$

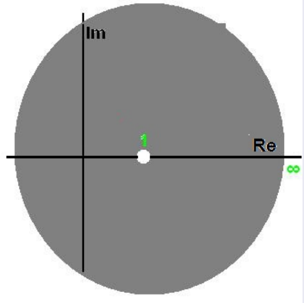
$$\therefore \quad \frac{1}{2\pi i} \int_C (z - z_0)^m dz = \begin{cases} 1 & \text{when} \quad m = -1 \\ 0 & \text{when} \quad m \neq -1 \end{cases} \tag{25}$$

Summary: C is a positively oriented circle that encircles z_0. Results above are valid for any radius $R > 0$.

Example

Consider $f(z) = \frac{1}{(z-1)^2}$, $\quad 0 < |z-1| < \infty$

We want to find Laurent series expansion of $f(z)$ about $z = 1$.



$$f(z) = \frac{1}{(z-1)^2}\,, \quad 0 < |z-1| < \infty$$

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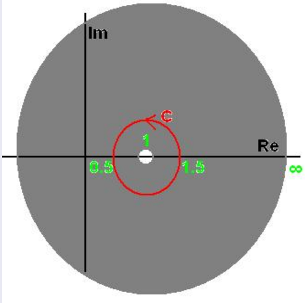
Use the formula for the coefficients d_n with a simple closed contour containing the inner circle, for instance, $|z-1| = 0.5$.



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Let us find the coefficients d_n:

$$d_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} = \frac{1}{2\pi i}\int_C \frac{1/(z-1)^2}{(z-1)^{n+1}}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{(z-1)^{n+3}}dz$$

$$= \frac{1}{2\pi i}\int_C (z-1)^{-(n+3)}dz$$

Use the result of the preceding example:

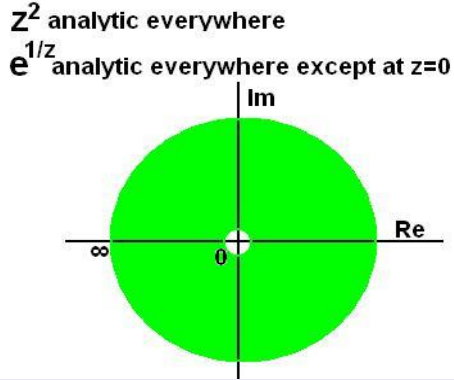
$$= \begin{cases} 1 & \text{when} \quad -(n+3) = -1 \; (i.e.,\; n=-2) \\ 0 & \text{else} \end{cases}$$

$d_{-2} = 1\,, \quad d_n = 0$ when $n \neq -2$.
The Laurent series of $f(z)$ is itself: $\frac{d_{-2}}{(z-1)^2} = \frac{1}{(z-1)^2}$

Example

The Laurent series expansion of $z^2 e^{1/z}$ with center at 0 is

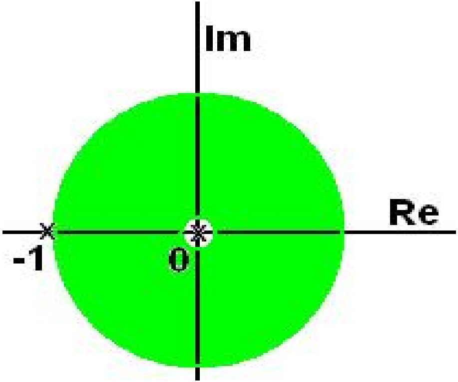
$$z^2\left(1+\frac{1}{1!z}+\frac{1}{2!z^2}+\dots\right)=z^2+z+\frac{1}{2!}+\frac{1}{3!z}+\dots \quad 0<|z|<\infty$$



Example

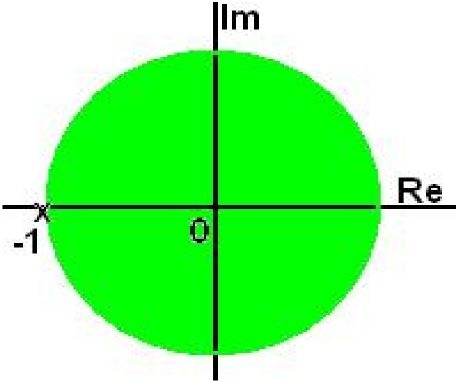
$$\frac{1+2z}{z^2+z^3} = \frac{1}{z^2}\frac{1+2z}{1+z} = \frac{1}{z^2}\left(2-\frac{1}{1+z}\right) = \frac{1}{z^2}(2-1+z-z^2+z^3-z^4+\ldots)$$

$$= \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + z^3 - \ldots \quad 0 < |z| < 1$$



Example

$$\frac{e^z}{1+z} = (1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots)(1 - z + z^2 - z^3 + \ldots)$$
$$= 1 + \frac{1}{2}z^2 - \frac{1}{3}z^3 \ldots \quad |z| < 1$$

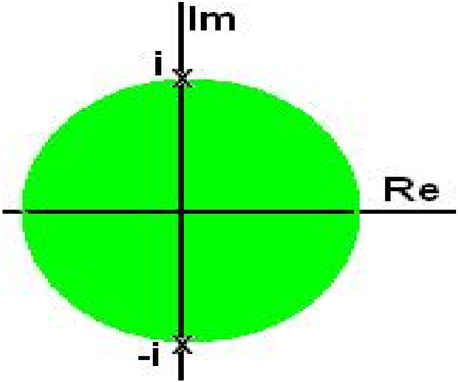


Example

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots \quad |z| < 1$$

Integrate both sides:

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots \quad |z| < 1$$



Example

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \quad |z-1| < 1$$

Note that

$$\frac{1}{z} = \frac{1}{1-1+z} = \frac{1}{1+\underbrace{(z-1)}} = \cdots, \ |z-1| < 1$$

Differentiate each side:

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty} (-1)^n n (z-1)^{n-1}, \quad |z-1| < 1$$

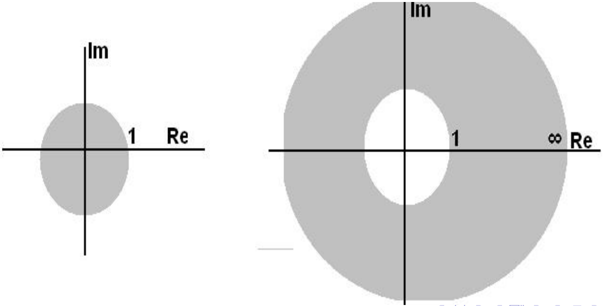
$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (-1)^n (n+1)(z-1)^n$$

Example

Find the Laurent series expansion of $f(z) = \frac{1}{1-z}$ for $|z| > 1$.

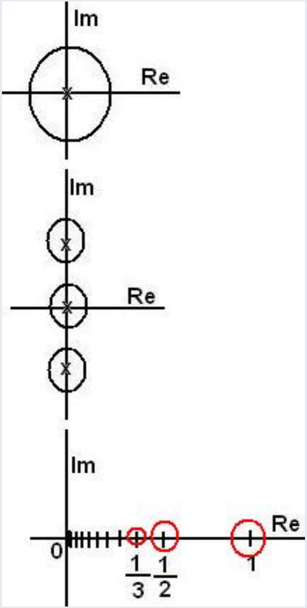
$$\frac{1}{1-z} = \frac{1}{z}\frac{1}{\frac{1}{z}-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{-1}{z^{n+1}}, \quad |z| > 1$$

$$\therefore \quad \frac{1}{1-z} = \sum_{n=1}^{\infty}\frac{-1}{z^n} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots \quad |z| > 1$$



Recall: z_0 is called a **singular point** of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. That is, f is not analytic at the singular point z_0, and in every neighborhood of z_0 we are able find a point such that f is analytic there. A singular point z_0 is said to be **isolated** if, in addition, there is some neighborhood of z_0 throughout which f is analytic except at the point itself. That is, if z_0 is an isolated singular point of f, we can find a neighborhood of z_0 such that z_0 is the only singular point in that neighborhood.

Example



$\frac{1}{z}$ has isolated singular point at $z = 0$

$\frac{z+1}{z^3(z^2+1)}$ has three isolated singular points at $z = 0, z = i, z = -i$.

$\frac{1}{\sin(\pi/z)}$ has singular points at $z = 0$ and $z = 1/n, \quad n = \pm 1, \pm 2, \ldots$. Each singular point except $z = 0$ is isolated.

When z_0 is an isolated singular point of a function f, there is a positive number R such that f is analytic at each point z for which $0 < |z - z_0| < R$. Consequently the function is represented by a Laurent series

$$f(z) = \sum_{0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_n}{(z - z_0)^n} + \ldots \tag{26}$$

valid in $0 < |z - z_0| < R$.

Recall that

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}\,; \quad n = 1, 2, \ldots$$

where C is any positively oriented simple closed contour around z_0 and lying in the domain $0 < |z - z_0| < R$. When n=1, this expression for b_n can be written

$$\int_C f(z)dz = 2\pi i b_1 \tag{27}$$

The complex number b_1, which is the coefficient of $1/(z - z_0)$ in expansion (26) is called the **residue** of f at the isolated singular point z_0. Residues are useful in evaluating certain integrals around simple closed contours.

Recap

Hypotheses

z_0 is an isolated singular point of f. (Then we can find R such that f is analytic in $0 < |z - z_0| < R$).
C is in $0 < |z - z_0| < R$ and encloses z_0.

Conclusion

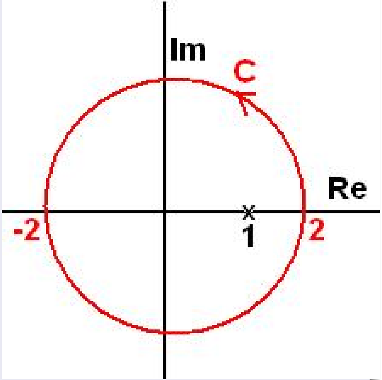
$$\int_C f(z)dz = 2\pi i b_1$$

where b_1 is the coefficient of $1/(z - z_0)$ in Laurent series of f written for $0 < |z - z_0| < R$.

Example

Consider $\int_C \frac{e^{-z}}{(z-1)^2} dz$ where C is the circle $|z| = 2$ described in the positive sense. Let us check whether the hypotheses of previous recap page are satisfied: The integrand f has an isolated singular point at $z = 1$. R is ∞, that is, f is analytic in $0 < |z-1| < \infty$. The contour C is in the annular domain $0 < |z - z_0| < \infty$ and encloses $z_0 = 1$. $\because$ Hypotheses are satisfied.

To evaluate the integral we determine the residue b_1 at $z = 1$ and use $\int_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i b_1$.



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The formula $e^z = \sum_0^\infty \frac{z^n}{n!}$ is useful below:

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}e^{-(z-1)}}{(z-1)^2} = \frac{e^{-1}}{(z-1)^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(z-1)^n$$

$$= e^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(z-1)^{n-2} = \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{z-1} + \frac{e^{-1}}{2} - \frac{e^{-1}}{6}(z-1) + \cdots,$$

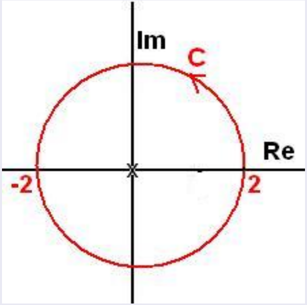
The coefficient of $1/(z-1)$ is $-1/e$. In other words, residue of f at $z=1$ is $-1/e$. Hence

$$\int_C \frac{e^{-z}}{(z-1)^2}dz = 2\pi i\frac{-1}{e}$$

Example

Evaluate $\int_C e^{(1/z^2)} dz$ for the same C as in the previous example.

Let us check whether the hypotheses for the residue formula are satisfied: The integrand f has an isolated singular point at $z = 0$. R is ∞, that is, f is analytic in $0 < |z - 0| < \infty$. The contour C is in the annular domain $0 < |z - 0| < \infty$ and encloses $z_0 = 0$. $\ddot\smile$ Hypotheses are satisfied.



To evaluate the integral we determine the residue b_1 of f at $z = 0$ and use $\int_C e^{(1/z^2)} dz = 2\pi i b_1$.

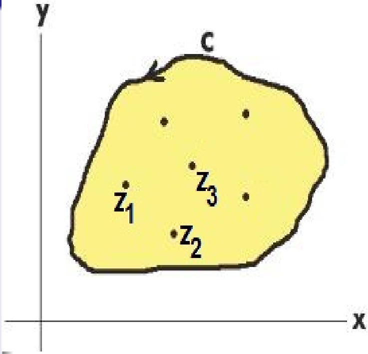
$$e^{1/z^2} = 1 + \frac{1}{1! z^2} + \frac{1}{2! z^4} + \frac{1}{3! z^6} + \dots, \quad 0 < |z| < \infty$$

Since the residue of f at $z = 0$ is 0 (i.e., $b_1 = 0$) the integral is 0.

Theorem

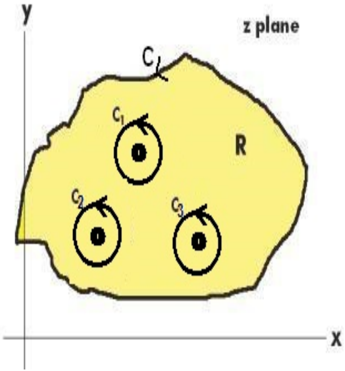
Let C be a positively oriented simple closed contour within and on which a function f is analytic except for a finite number of singular points $z_1, z_2, \ldots, z_n$ interior to C. If $B_1, B_2, \ldots, B_n$ denote the residues of f at those respective points, then

$$\int_C f(z)dz = 2\pi i(B_1 + B_2 + \ldots + B_n)$$



Proof

Let the singular points $z_1, \ldots, z_n$ be centers of positively oriented circles $C_1, \ldots, C_n$ which are interior to C and are so small that no two of the circles have points in common. The circles C_j together with the simple closed contour C form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain. According to C-G theorem



$$\int_C f(z)dz - \underbrace{\int_{C_1} f(z)dz}_{2\pi i B_1 \ by (27)} - \underbrace{\int_{C_2} f(z)dz}_{2\pi i B_2 \ by (27)} - \ldots - \underbrace{\int_{C_n} f(z)dz}_{2\pi i B_n \ by (27)} = 0 \,,$$

$$\rightarrow \quad \int_C f(z)dz = 2\pi i (B_1 + B_2 + \ldots + B_n)$$

Example

$\int_C \frac{5z-2}{z(z-1)}dz$ where C is the circle $|z|=2$, described counterclockwise.

Singular points in the contour are $z=0$ and $z=1$.

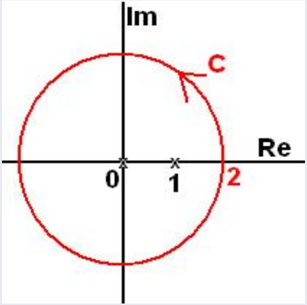
B_1: residue at $z=0$

$$\frac{5z-2}{z(z-1)} = \left(\frac{5z-2}{z}\right)\left(\frac{-1}{1-z}\right)$$

$$= \left(5-\frac{2}{z}\right)(-1-z-z^2-\dots)$$

$$= \frac{2}{z} - 3 - 3z - \dots \quad 0<|z|<1$$

$$\rightarrow \quad B_1 = 2$$



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B_2: residue at $z = 1$

$$\frac{5z-2}{z(z-1)}$$

$$= \Big(\frac{5(z-1)+3}{z-1}\Big)\Big(\frac{1}{1+(z-1)}\Big)$$

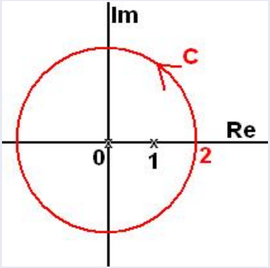
$$= \Big(5+\frac{3}{z-1}\Big)\big(1-(z-1)+(z-1)^2-\dots\big)$$

$$= \dots + \frac{B_2}{z-1} + \dots$$

$$0 < |z-1| < 1$$

$$\rightarrow \quad B_2 = 3$$

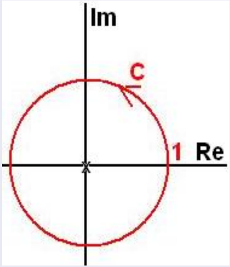
$$\int_C \frac{5z-2}{z(z-1)}dz = 2\pi i(2+3) = 10\pi i$$



Example

$\int_C \frac{\sin z}{z^4} dz$ where C is the unit circle oriented counterclockwise.

The integrand has an isolated singular point at the origin. So we can write the Laurent series for the integrand that is valid for $0 < |z| < \infty$. The integration path is a simple contour that lies in the domain of the Laurent series and it encloses the inner circle of the domain. So, the residue theorem is applicable to this problem:



$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} \dots$$

Residue at $z = 0$ is $-1/3! = -1/6$

$$\rightarrow \int_C \frac{\sin z}{z^4} dz = 2\pi i(\frac{-1}{6}) = \frac{-\pi i}{3}$$

Principal Part of a Function

Consider the Laurent series of f centered at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots, \quad 0 < |z-z_0| < R$$

for some positive R. The portion of the series involving the negative powers of $z - z_0$ is called the **principal part** of f at z_0. Let the principal part contain finite number of terms, that is,

$$f(z) = \sum_{0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_m}{(z-z_0)^m}$$

$$0 < |z-z_0| < R$$

where $b_m \neq 0$. In this case the isolated singular point z_0 is called a **pole of order** m. A pole of order $m = 1$ is called a **simple pole**.

Example

$$\frac{z^2 - 2z + 3}{z - 2} = z + \frac{3}{z-2} = 2 + (z-2) + \frac{3}{z-2}, \quad 0 < |z-2| < \infty$$

The function above has a simple pole at $z = 2$. Its residue is 3.

Example

$$\frac{\sinh z}{z^4} = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots\right)$$

$$= \frac{1}{z^3} + \frac{1}{3!z} + \frac{1}{5!}z + \frac{1}{7!}z^3 + \dots \quad 0 < |z| < \infty$$

This has a pole of order 3 at $z = 0$, with residue $1/6$.

When the principal part of f at z_0 has an infinite number of terms, that point is called an **essential singular point**.

Theorem

(Picard's Theorem) In each neighborhood of an essential singular point a function assumes every finite value, with one possible exception, an infinite number of times.

Example

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}, \quad 0 < |z| < \infty$$

This has an essential singular point at $z = 0$. Therefore, in the neighborhood of the essential singular point, say $|z| < 0.1$, the function $e^{1/z}$ can take every finite value, say 45. Thus, according to the theorem, there are infinitely many z values in the neighborhood $|z| < 0.1$ satisfying $e^{1/z} = 45$.

Example

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}, \quad 0 < |z| < \infty$$

This has an essential singular point at $z = 0$. Its residue at $z = 0$ is 1.

We can show that $e^{\frac{1}{z}}$ has the value -1 an infinite number of times in each neighborhood of the origin. Recall that: $e^z = -1$ when $z = (2n+1)\pi i$, $n = 0, \pm 1, \ldots$ This means that $e^{\frac{1}{z}} = -1$ when $z = \frac{1}{(2n+1)\pi i} \frac{i}{i} = \frac{-i}{(2n+1)\pi}$, $n = 0, \pm 1, \ldots$. In a set notation, solutions are:

$$\{-0.1061i, -0.0637i, -0.0455i, -0.0354i, -0.0289i, -0.0245i, \ldots\}$$

Clearly, infinite number of these points lie in any given neighborhood of the origin.

Let

$$f(z) := \frac{\phi(z)}{z - z_0} \tag{28}$$

with ϕ analytic at z_0 and $\phi(z_0) \neq 0$. The Taylor series

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z - z_0) + \frac{\phi''(z_0)}{2!}(z - z_0)^2 + \cdots \tag{29}$$

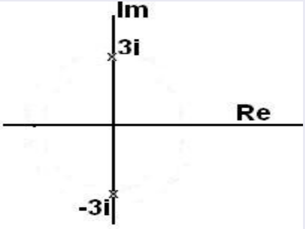
is valid in $|z - z_0| < R$ for some R. Substitute this in (28):

$$f(z) = \frac{\phi(z_0)}{z - z_0} + \frac{\phi'(z_0)}{1!} + \frac{\phi''(z_0)}{2!}(z - z_0) + \frac{\phi'''(z_0)}{3!}(z - z_0)^2 \cdots \tag{30}$$

Residue of f at z_0 is the coefficient of $\frac{1}{z - z_0}$, that is, $\phi(z_0)$.

Example

Example Consider $f(z) = \frac{z+1}{z^2+9}$. f has isolated singular points at $3i$ and $-3i$. Find the residue at $3i$. We can write $f(z) = \frac{\phi(z)}{z-3i}$ with $\phi(z) = \frac{z+1}{z+3i}$. Note that ϕ is analytic at $3i$ and $\phi(3i) \neq 0$. Thus $3i$ is a simple pole of f. The residue at $3i$ is $\phi(3i) = \frac{3-i}{6}$.



Consider the function

$$f(z) := \frac{\phi(z)}{(z - z_0)^m}, \ m = 2, 3, \ldots \tag{31}$$

with ϕ analytic at z_0 and $\phi(z_0) \neq 0$. Use (29)

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z - z_0) + \frac{\phi''(z_0)}{2!}(z - z_0)^2 + \cdots \tag{29}$$

in (31):

$$f(z) = \frac{\phi(z_0)}{(z - z_0)^m} + \frac{\frac{\phi'(z_0)}{1!}}{(z - z_0)^{m-1}} + \frac{\frac{\phi''(z_0)}{2!}}{(z - z_0)^{m-2}} + \cdots + \frac{\frac{\phi^{(m-1)}(z_0)}{(m-1)!}}{z - z_0} + \cdots$$

f has a pole of order m at z_0 with residue

$$b_1 = \frac{\phi^{(m-1)}(z_0)}{(m - 1)!}$$

Example

$$f(z) = \frac{z^3 + 2z}{(z-i)^3}$$

can be written as

$$f(z) = \frac{\phi(z)}{(z-i)^3}$$

where $\phi(z) = z^3 + 2z$. The function ϕ is entire and $\phi(i) \neq 0$. Hence f has a pole of order 3 at $z = i$. The residue is

$$b_1 = \frac{\phi''(i)}{2!} = 3i$$

Zeros and poles of order m

When two functions p and q are analytic at a point z_0 and $p(z_0) \neq 0$, the quotient $\frac{p(z)}{q(z)}$ has a pole of order m at z_0 if and only if q has a zero of order m there.

Example

If $f(z) = (z-7)^4$, then $f(7) = f'(7) = f''(7) = f'''(7) = 0$ and $f^{(iv)}(7) \neq 0$. Thus f has a zero of order 4 at $z_0 = 7$.

Example

$$\frac{p(z)}{q(z)} = \frac{z}{(z-7)^4}$$

p is analytic at $z_0 = 7$. And q has a zero of order 4 at $z_0 = 7$. So, the quotient $\frac{p}{q}$ has a pole of order 4 at $z_0 = 7$.

Example

Let

$$\frac{p(z)}{q(z)} = \frac{1}{z(e^z - 1)}.$$

Notice that $p(z)$ and $q(z)$ are entire functions.
$q(0) = 0,\ q'(0) = [(e^z - 1) + (e^z - 1)z]_{z=0} = 0,\ q''(0) = [2e^z + ze^z]_{z=0} = 2 \neq 0$, thus, q has a zero of order 2. Hence $\frac{p(z)}{q(z)}$ has a pole of order 2 at $z = 0$.

We find the residue of f at 0 in the next slide.

Continued from the previous page

$$f(z) = \frac{p(z)}{q(z)} = \frac{1}{z[(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots) - 1]}$$

$$= \frac{1}{z(\frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots)} = \frac{1}{z^2(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots)}$$

$$= \frac{\frac{1}{(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots)}}{z^2} = \frac{\phi(z)}{z^2}$$

We have a second order pole at 0. Also $\phi(0) \neq 0$. Thus

$$\phi'(z)|_{z=0} = \frac{-(\frac{1}{2!} + \frac{2z}{3!} + \cdots)}{(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots)^2} = -\frac{1}{2} = b_1$$

For instance, given C be $|z| = 2$ described ccw

$$\int_C \frac{1}{z(e^z - 1)} dz = 2\pi i(-\frac{1}{2}) = -\pi i$$

A special case

If two functions p and q are analytic at a point z_0 and $p(z_0) \neq 0$, $q(z_0) = 0,\ q'(z_0) \neq 0$, then z_0 is a simple pole of the quotient $\frac{p(z)}{q(z)}$ and the residue there is $b_1 = \frac{p(z_0)}{q'(z_0)}$

Example

$$\frac{p(z)}{q(z)} = \frac{z}{z^2 + 3z + 2}$$

At $z_0 = -1$, $p(-1) \neq 0$, $q(-1) = 0$, $q'(-1) = [2z + 3]_{z=-1} \neq 0$
So, the residue at $z = -1$ is

$$b_1 = \frac{p(-1)}{q'(-1)} = \left[\frac{z}{2z+3}\right]_{z=-1} = -1$$

Digression If $f(z_0) = 0$ then $f(z) = (z - z_0)g(z)$ for some g. Furthermore, if $f'(z_0) \neq 0$ then $g(z_0) \neq 0$. **EOD**

Reasoning for the special case

Since $q(z_0) = 0$ and $q'(z_0) \neq 0$; z_0 is a zero of order 1 of q. That is, $q(z) = (z - z_0)g(z)$ for some g. g is analytic at z_0 and nonzero at z_0. Thus

$$\frac{p(z)}{q(z)} = \frac{p(z)}{(z - z_0)g(z)}$$

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{z - z_0}$$

Evidently, the residue of $\frac{p(z)}{q(z)}$ at z_0 is $\frac{p(z_0)}{g(z_0)}$. Noticing that $q'(z_0) = g(z_0)$, the residue is $\frac{p(z_0)}{q'(z_0)}$

Example

Let $f(z) = \frac{\cos z}{\sin z}$. Isolated singular points are at $n\pi, \ n = 0, \pm 1, \ldots,$ $p(n\pi) \neq 0, \ q(n\pi) = 0, \ q'(n\pi) = (-1)^n \neq 0$, therefore, each singular point $z = n\pi$ is a simple pole with residue $b_1 = \frac{p(n\pi)}{q'(n\pi)} = 1$

Example

Let $f(z) = \frac{z}{z^4+4}$. Let us find the residue at the isolated singular point $z_0 = \sqrt{2}e^{i\frac{\pi}{4}} = 1 + i$. Observe that $p(z_0) \neq 0, \ q(z_0) = 0, \ q'(z_0) \neq 0$. Therefore f has a simple pole at z_0. The residue is

$$b_1 = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = \frac{-i}{8}$$

Definition Let f be continuous. Then

$$\int_0^\infty f(x)dx := \lim_{R\to\infty}\int_0^R f(x)dx \tag{32}$$

$$\int_{-\infty}^\infty f(x)dx := \lim_{R_1\to\infty}\int_{-R_1}^0 f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx \tag{33}$$

Cauchy Principal Value of $\int_{-\infty}^{\infty} f(x)dx$ is defined by

$$\text{PV} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{34}$$

If integral (33) converges, it converges to Cauchy principal value. Recall that

$$\int_{-\infty}^{\infty} f(x)dx := \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty} \int_{0}^{R_2} f(x)dx \tag{33}$$

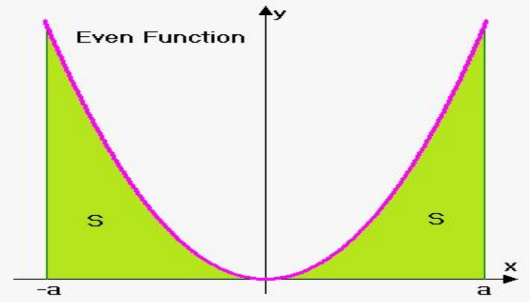
But existence of CPV does not imply existence of the limits in (33).

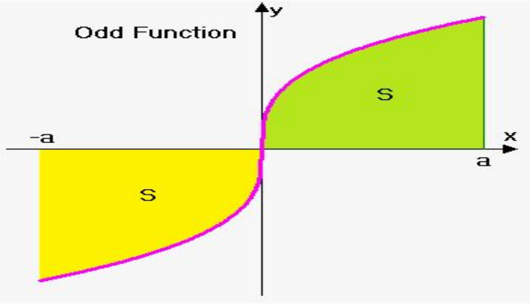
If f is an even function (i.e., $f(x) = f(-x)$) then existence of (34) implies convergence of (33). Also for even f, if either of the integrals (32) or (33)i.e.,

$$\int_0^\infty f(x)dx \text{ or } \int_{-\infty}^\infty f(x)dx$$

converges, then

$$\int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^\infty f(x)dx$$





Note that when $f(x) = x$, CPV equals zero however (33) does not converge.

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx \neq \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty}\int_{0}^{R_2} f(x)dx$$

Example

Evaluate
$$\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}dx$$
Note that f is even above. Equality below holds if the integrals converge.
$$\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{2x^2-1}{x^4+5x^2+4}dx$$
Let us define
$$f(z) = \frac{2z^2-1}{z^4+5z^2+4} = \frac{2z^2-1}{(z^2+1)(z^2+4)}$$
f has isolated singularities at $z = \pm i$ and $z = \pm 2i$.

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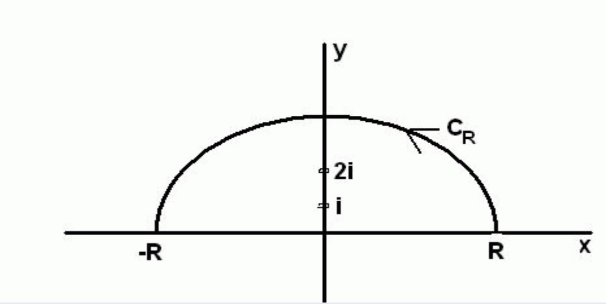
Consider the semicircular path C_R depicted in the figure.



Noting that B_1 and B_2 are the residues of f at i and $2i$ respectively, we have

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i(B_1 + B_2) \tag{35}$$

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$$f(z) = \frac{2z^2 - 1}{(z-i)(z+i)(z^2+4)}$$

Define $f(z) = \frac{\phi(z)}{z-i}$ with $\phi(z) = \frac{2z^2-1}{(z+i)(z^2+4)}$. Noting that ϕ is analytic at $z_0 = i$, and $\phi(i) \neq 0$ we may use the residue formula for the functions having a simple pole. This yields $B_1 = \phi(i) = \frac{-1}{2i}$.

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$$f(z) = \frac{2z^2 - 1}{(z^2+1)(z+2i)(z-2i)}$$

Also defining $f(z) = \frac{\phi(z)}{z-2i}$ with $\phi(z) = \frac{2z^2-1}{(z^2+1)(z+2i)}$ and noting that ϕ is analytic at $z_0 = 2i$, and $\phi(2i) \neq 0$ we have $B_2 = \phi(2i) = \frac{3}{4i}$. Now we can use the residue formula

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i(B_1 + B_2) = 2\pi i(\frac{-1}{2i} + \frac{3}{4i}) = \frac{\pi}{2}$$

$$\to \int_{-R}^{R} f(x)dx = \frac{\pi}{2} - \int_{C_R} f(z)dz$$

We will show that $\int_{C_R} f(z)dz$ equals zero. This will make the result $\frac{\pi}{2}$.

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$$\int_{-R}^{R} f(x)dx = \frac{\pi}{2} - \int_{C_R} f(z)dz$$

We will show that $\int_{C_R} f(z)dz$ equals zero. This will make the result $\frac{\pi}{2}$. Recall that:
For any nonnegative constant satisfying $|f(z)| \leq M$, $\forall z$ on C

$$\left| \int_{C} f(z)dz \right| \leq ML$$

where L is length of C.
In the present problem, length is πR, that is, the length of C_R.
Next we find an upper bound M for $|f|$ on C.

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Noting that

$$f(z) = \frac{2z^2 - 1}{(z^2+1)(z^2+4)}$$

we find upper bound for the numerator $|2z^2 - 1|$ and a lower bound for the denominator $|(z^2+1)(z^2+4)|$. Using these, we obtain an upper bound for $|f|$ on C_R.

$$|2z^2 - 1| \leq 2|z|^2 + 1 = 2R^2 + 1$$

and

$$|z^4+5z^2+4| = |z^2+1||z^2+4| \geq ||z|^2-1|||z|^2-4| = (R^2-1)(R^2-4)$$

$$\rightarrow \left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} dz \right| \leq \underbrace{\frac{2R^2+1}{(R^2-1)(R^2-4)}}_{M} \times \underbrace{\pi R}_{L}$$

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$$\rightarrow \left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \underbrace{\frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}}_{M} \times \underbrace{\pi R}_{L}$$

$$\rightarrow \left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz \right| \leq \frac{(2R^2 + 1)\pi R}{(R^2 - 1)(R^2 - 4)}$$

The RHS above goes to zero as $R \to \infty$, so is $\int_{C_R} f(z)dz$.

$$\rightarrow \int_{-R}^{R} f(x)dx = \frac{\pi}{2} - \int_{C_R} f(z)dz$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)dx = \frac{\pi}{2} - \underbrace{\lim_{R\to\infty} \int_{C_R} f(z)dz}_{0}$$

$$\rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{2x^2 - 1}{x^4 + 5x^2 + 4} dx = \frac{\pi}{2}$$

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$$\lim_{R\to\infty}\int_{-R}^{R}\frac{2x^2-1}{x^4+5x^2+4}dx=\frac{\pi}{2}$$

Because the function f is even, we have

$$\int_{-\infty}^{\infty}\frac{2x^2-1}{x^4+5x^2+4}dx=\frac{\pi}{2}$$

$$\to\int_{0}^{\infty}\frac{2x^2-1}{x^4+5x^2+4}dx=\frac{\pi}{4}$$