Advanced Calculus

Part II[1]

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Notation

$$\dot{x} = \frac{dx}{dt} : \text{ first derivative of } x \text{ with respect to } t$$

$$\ddot{x} = \frac{d^2x}{dt^2} : \text{ second derivative of } x \text{ with respect to } t$$

$$x^{(n)} = \frac{d^nx}{dt^n} : \text{ n-th derivative of } x \text{ with respect to } t$$

Let f be a function of u and v.

$$f_u = \frac{\partial f}{\partial u} : \text{ partial derivative of } f \text{ with respect to } u$$

$$f_v = \frac{\partial f}{\partial v} : \text{ partial derivative of } f \text{ with respect to } v$$

Example

$$f(x) = x^3 + 2x \rightarrow \frac{df}{dx} = 3x^2 + 2$$

f is the dependent variable, and x is the independent variable.

Definition

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a **differential equation**.
A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an **ordinary differential equation**.
A differential equation involving partial derivatives of one or more dependent variables with respect to a more than one independent variable is called a **partial differential equation**.

Example

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \tag{1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{2}$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{4}$$

The first and second differential equations are ordinary, the third and fourth differential equations are partial differential equations.

Definition

The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \tag{5}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{6}$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{8}$$

In the example the first is second order, the second is fourth order, the third is first order, the fourth is second order differential equations.

Definition

A linear ordinary differential equation of order n, in the dependent variable y and the independent variable x, is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-2}(x)\frac{d^2 y}{dx^2}+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=b(x)$$

where a_0 is not identically zero.

Functions of x: $x^2, \sin(x), x+1, 5, 0$

Not functions of x: $y, 3y, y^2, \frac{dy}{dx}, (\frac{dy}{dx})^2, x+y, xy$

Definition

A nonlinear ordinary differential equation is an ordinary differential equation that is not linear.

Example

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \ldots \text{ Linear} \tag{9}$$

$$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x \ldots \text{ Linear} \tag{10}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \ldots \text{ Nonlinear} \tag{11}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6yy = 0 \ldots \text{ Nonlinear}$$

$$\frac{d^2y}{dx^2} + 5(\frac{dy}{dx})^3 + 6y = 0 \ldots \text{ Nonlinear} \tag{12}$$

$$\frac{d^2y}{dx^2} + 5(\frac{dy}{dx})^2\frac{dy}{dx} + 6y = 0 \ldots \text{ Nonlinear}$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \ldots \text{ Nonlinear} \tag{13}$$

$$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \ldots \text{Nonlinear}$$

Normal Form

Definition

The normal form of a system of n differential equations in n unknown functions $x_1, x_2, \ldots, x_n$, is in the following form:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, x_2, \ldots, x_n, t) \\ \frac{dx_2}{dt} &= f_2(x_1, x_2, \ldots, x_n, t) \\ &\vdots \\ \frac{dx_n}{dt} &= f_n(x_1, x_2, \ldots, x_n, t) \end{aligned}\right\} \tag{14}$$

Example

$$\begin{aligned} \dot{x}_1 &= x_1 + 3x_1x_2^2 + x_1 t \\ \dot{x}_2 &= x_2^3 + \sin x_1 + t^2 \\ \dot{x}_3 &= x_1x_2 + x_2x_3^2 \end{aligned}$$

Normal Form of a Linear System

Definition

The normal form, in the general case of a linear system of n differential equations in n unknown functions $x_1, x_2, \dots, x_n$, is in the following form:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + F_1(t) \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + F_2(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + F_n(t) \end{aligned}\right\} \tag{15}$$

Example

$$\begin{aligned} \dot{x}_1 &= 2tx_1 + 3x_2 + 4x_3 + t^2 \\ \dot{x}_2 &= x_1 + 6x_3 + \frac{1}{t} \\ \dot{x}_3 &= 3t^2x_1 + (4+t)x_2 + (t+t^2)x_3 \end{aligned}$$

A single n-th order linear differential equation can be converted into this form. Consider

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + a_2(t)\frac{d^{n-2}x}{dt^{n-2}} + \cdots$$

$$+a_{n-2}(t)\frac{d^2x}{dt^2} + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = F(t)$$

$$\frac{d^n x}{dt^n} + a_1(t)\underbrace{\frac{d^{n-1}x}{dt^{n-1}}}_{x_n} + a_2(t)\underbrace{\frac{d^{n-2}x}{dt^{n-2}}}_{x_{n-1}} + \cdots$$

$$+a_{n-2}(t)\underbrace{\frac{d^2 x}{dt^2}}_{x_3} + a_{n-1}(t)\underbrace{\frac{dx}{dt}}_{x_2} + a_n(t)\underbrace{x}_{x1} = F(t)$$

Notice that

$$\dot{x}_i = x_{i+1} \quad n = 1, \ldots, n-1$$

and

$$\dot{x}_n + a_1(t)x_n + a_2(t)x_{n-1} + \ldots + a_{n-1}(t)x_2 + a_n(t)x_1 = F(t)$$

$$\dot{x}_n = -a_n(t)x_1 - a_{n-1}(t)x_2 - \ldots - a_3(t)x_{n-2} - a_2(t)x_{n-1} - a_1(t)x_n + F(t)$$

$$\frac{d^n x}{dt^n} + a_1(t)\underbrace{\frac{d^{n-1}x}{dt^{n-1}}}_{x_n} + a_2(t)\underbrace{\frac{d^{n-2}x}{dt^{n-2}}}_{x_{n-1}} + \cdots$$

$$+ a_{n-2}(t)\underbrace{\frac{d^2x}{dt^2}}_{x_3} + a_{n-1}(t)\underbrace{\frac{dx}{dt}}_{x_2} + a_n(t)\underbrace{x}_{x1} = F(t)$$

Using these definitions, the normal form equivalent of (11) is

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
&\vdots \\
\frac{dx_{n-1}}{dt} &= x_n \\
\frac{dx_n}{dt} &= -a_n(t)x_1 - a_{n-1}(t)x_2 - \cdots - a_2(t)x_{n-1} - a_1(t)x_n + F(t)
\end{aligned}\right\}$$

Solution of a differential equation

Definition

Consider the n-th order ordinary differential equation

$$F[x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}] = 0 \tag{16}$$

A solution of an ordinary differential equation (16) on interval I is a function that satisfies the differential equation on the interval I.

Example

$$\underbrace{\frac{dy}{dx} + y\frac{d^2y}{dx^2} + 3x^2 + \frac{d^3y}{dx^3}\sin x}_{F[x,y,\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3}]} = 0$$

$\therefore$ **Solution** is a function.

$\therefore$ **Solution** is defined on some interval I.

$\therefore$ **Solution** satisfies the d.e. on I.

A Solution Classification

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Explicit Solutions and **Implicit** Solutions:

Explicit solution is a function f defined on interval I such that it satisfies the ordinary differential equation on interval I when f is substituted for the dependent variable.

A relation $g(x, y) = 0$ is called an **implicit solution** of the ordinary differential equation on I if this relation defines at least one function f of x on I such that this function is an explicit solution of (16) on this interval.

Both explicit and implicit solutions are called **solutions**.

Example

A function defined for all real x by

$$f(x) = 2\sin x + 3\cos x$$

is an explicit solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

for all real x. First note that f is defined and has a second derivative on the entire real interval. Next observe that

$$f'(x) = 2\cos x - 3\sin x$$

$$f''(x) = -2\sin x - 3\cos x$$

Substituting them in the differential equation we obtain

$$(-2\sin x - 3\cos x) + (2\sin x + 3\cos x) = 0$$

Example

Consider the differential equation

$$x\frac{dy}{dx} - 2y = 0$$

The function $f(x) = x^2$ on the interval $I = (-\infty, \infty)$ is an explicit solution to the d.e. above. Substitute in the d.e.:

$$xf'(x) - 2f(x) = x \cdot 2x - 2 \cdot x^2 = 0$$

for all $x \in I$. Thus f is an explicit solution to the d.e. on the interval I.

Example

Is $f(x) = e^x - x$ on the interval $I = (-\infty, \infty)$ a solution to

$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$

Example

Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} - 2y = 0$$

and the solution candidate $f(x) = x^2 - x^{-1}$ on the interval $I = (0, \infty)$.

Note that $f'(x) = 2x + x^{-2}$ and $f''(x) = 2 - 2x^{-3}$. Substitute them in the d.e.:

$$x^2 \cdot (2 - 2x^{-3}) - 2(x^2 - x^{-1}) = 0$$

for all x on the interval I. It can be shown that this function is also a solution to the differential equation on the interval $(-\infty, 0)$.

Example

The relation

$$x^2 + y^2 - 25 = 0$$

is an implicit solution of the differential equation

$$x + y\frac{dy}{dx} = 0$$

on the interval I defined by $-5 \leq x \leq 5$. It defines two functions

$$f_1(x) = \sqrt{25 - x^2}$$

and

$$f_2(x) = -\sqrt{25 - x^2}$$

for all real x on I. It can easily be shown that each of these functions is an explicit solution for the differential equation on I. Note that if one of them is an explicit solution for the differential equation on I, it suffices for being an implicit solution.

Example

It can easily be shown that each of the functions f_1 and f_2 is an explicit solution for the differential equation on I. Note that if one of them is an explicit solution for the differential equation on $I: \ -5 \leq x \leq 5$ it suffices for being an implicit solution. Indeed, at least one of them satisfies the differential equation:

$$\left[x + y\frac{dy}{dx}\right]_{y=f_1} = 0$$

$$x + \sqrt{25 - x^2} \cdot \frac{-2x}{2\sqrt{25 - x^2}} = x - x = 0$$

Example

Consider the d.e.

$$\frac{dy}{dx} + \frac{1}{2y} = 0$$

The relation $y^2 + x - 3 = 0$ on the interval $(\infty, 3)$ is an implicit solution to the d.e. above. Differentiate throughout:

$$2y\frac{dy}{dx} + 1 = 0$$

$$\rightarrow \frac{dy}{dx} + \frac{1}{2y} = 0$$

Solution generated the d.e.! Thus the relation $y^2 + x - 3 = 0$ on the interval $(\infty, 3)$ is an implicit solution to the given d.e.

Example

Consider the relation $xy^3 - xy^3 \sin x = 1$ and solve it for y for later use:

$$xy^3(1 - \sin x) = 1 \to y^3 = \frac{1}{x(1 - \sin x)}$$

$$\to y = \left[\frac{1}{x(1 - \sin x)}\right]^{\frac{1}{3}} = [x(1 - \sin x)]^{\frac{-1}{3}}$$

Differentiate this:

$$\frac{dy}{dx} = -\frac{1}{3}\,[x(1 - \sin x)]^{\frac{-4}{3}}\,[x(-\cos x) + (1 - \sin x)]$$

$$= \frac{x \cos x + \sin x - 1}{3[x(1 - \sin x)]^{\frac{4}{3}}} = \frac{x \cos x + \sin x - 1}{3[x(1 - \sin x)]} \frac{1}{[x(1 - \sin x)]^{\frac{1}{3}}}$$

$$= \frac{x \cos x + \sin x - 1}{3[x(1 - \sin x)]} \cdot y$$

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$$xy^3 - xy^3 \sin x = 1 \leftrightarrow \frac{dy}{dx} = \frac{x\cos x + \sin x - 1}{3[x(1-\sin x)]} \cdot y$$

Thus the relation

$$xy^3 - xy^3 \sin x = 1$$

is an implicit solution to the d.e.

$$\frac{dy}{dx} = \frac{x\cos x + \sin x - 1}{3[x(1-\sin x)]} \cdot y$$

Problem Find a solution f of the differential equation

$$\frac{dy}{dx} = 2x \tag{17}$$

such that at $x = 1$ this solution f has the value 4.
Equivalently Solve

$$\frac{dy}{dx} = 2x, \quad y(1) = 4$$

$$\frac{dy}{dx} = 2x \tag{17}$$

The solution must satisfy the differential equation (17).
$y = x^2 + c$ satisfies (17) for an arbitrary constant c.
The other condition $y(1) = 4$ is satisfied if $4 = 1^2 + c$, i.e., $c = 3$.
The condition in addition to the differential equation (17) is called **boundary condition**. If the boundary conditions relate to one x value, the problem is called the **initial value problem**. If the conditions relate to two different x values, the problem is called a (two point) **boundary value problem**.

Example

$$\frac{d^2y}{dx^2} + y = 0, \quad y(1) = 3, \; y'(1) = -4$$

Since the boundary conditions are given at one x value the problem is an initial value problem.

Example

$$\frac{d^2y}{dx^2} + y = 0, \quad y(0) = 1, \; y(2) = 5$$

Boundary conditions are given at two different x values; the problem is a boundary value problem.

Existence of a unique solution

Theorem

Consider the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{18}$$

where
1) the function f is a continuous function of x and y in some domain D of xy-plane, and
2) the partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D; and
3) let (x_0, y_0) be a point in D. Then there exists a unique solution of the differential equation (18) defined on some interval $|x - x_0| < h$ where h is sufficiently small.

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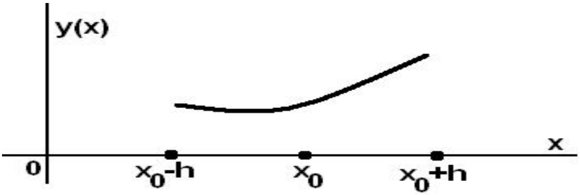
Then there exists a unique solution of the differential equation (18) defined on some interval $|x - x_0| < h$ where h is sufficiently small.



Note that this is a sufficiency theorem. $A \to B$ does not mean A is necessary for B to hold true.

Example

Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(1) = 3$$

Let us apply the existence theorem where
$f(x, y) = x^2 + y^2$, $\frac{\partial f}{\partial y} = 2y$. Both functions f and $\frac{\partial f}{\partial y}$ are continuous in every domain D of the xy-plane. The point $(1, 3)$ is in the domain D.
Thus the differential equation has a unique solution defined in the neighborhood of $x = 1$.

A first order linear differential equation in the form

$$\frac{dy}{dx} + p(x)y = g(x), \; y(x_0) = y_0$$

is a special case of the one we considered:

$$\frac{dy}{dx} = f(x, y), \;\; y(x_0) = y_0 \tag{18}$$

Example

Consider

$$(t^2 - 9)y' + 2y = \ln|20 - 4t|, \; y(4) = -3$$

In the standard form:

$$y' = -\frac{2}{(t^2 - 9)} y + \frac{\ln|20 - 4t|}{(t^2 - 9)}, \; y(4) = -3$$

$$y' = -\frac{2}{(t^2-9)}y + \frac{\ln|20-4t|}{(t^2-9)}, \; y(4) = -3$$

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comparing to the expression

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \tag{18}$$

we have

$$f(t,y) = -\frac{2}{(t^2-9)}y + \frac{\ln|20-4t|}{(t^2-9)}$$

f has discontinuities at $t = -3, +3, 5$. Discontinuities of $\frac{\partial f}{\partial y}$ are at $t = -3, +3$. The continuous interval of y is $(-\infty, \infty)$, and continuous intervals of t are

$$(-\infty, -3), (-3, 3), (3, 5), (5, \infty)$$

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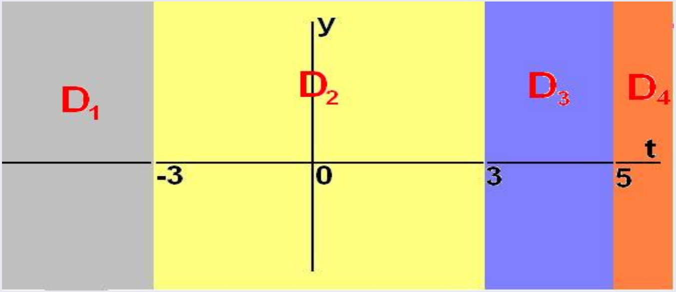

Domains for unique solution:

$$\underbrace{(-\infty,-3)\times(-\infty,\infty)}_{D_1},\ \underbrace{(-3,3)\times(-\infty,\infty)}_{D_2},$$

$$\underbrace{(3,5)\times(-\infty,\infty)}_{D_3},\ \underbrace{(5,\infty)\times(-\infty,\infty)}_{D_4}$$

The initial condition $y(4)=-3$, corresponding to the pair $(4,-3)$ in the theorem, is in the domain D_3.

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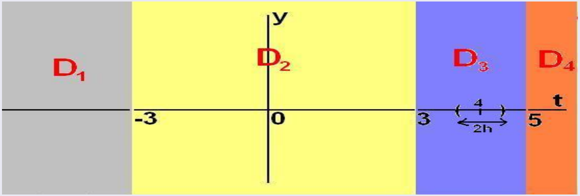

Thus, the differential equation

$$(t^2 - 9)y' + 2y = \ln|20 - 4t|, \ y(4) = -3$$

satisfies the hypotheses of the existence and uniqueness theorem as any initial condition does in domain D_3. Therefore, it has a unique solution defined for $|t - 4| < h$ for some h.
We will see in the sequel that the sufficient existence conditions are simpler for the linear differential equations.

Exercise

1) Show that $y = 4e^{2x} + 2e^{-3x}$ is a solution of the initial value problem

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0;\ y(0) = 6,\ y'(0) = 2$$

2) Do the following problems have unique solutions?

a)

$$\frac{dy}{dx} = x^2 \sin y,\ y(1) = -2$$

b)

$$\frac{dy}{dx} = \frac{y^2}{x-2},\ y(1) = 0$$

Exact differential equations

The first order differential equations to be studied may be expressed in either the derivative form

$$\frac{dy}{dx} = f(x, y)$$

or the differential form

$$M(x, y)dx + N(x, y)dy = 0$$

An equation in one of these forms may readily be written in the other form. For example

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x - y} \leftrightarrow (x^2 + y^2)dx + (y - x)dy = 0$$

$$(\sin(x) + y)dx + (x + 3y)dy = 0 \leftrightarrow \frac{dy}{dx} = -\frac{\sin(x) + y}{x + 3y}$$

Definition

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The **total differential** dF of the function F is defined by the formula

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy$$

for all $(x,y) \in D$.

Example

Consider

$$F(x,y) = xy^2 + 2x^3y$$

for all real (x,y). Then

$$dF(x,y) = (y^2 + 6x^2y)dx + (2xy + 2x^3)dy$$

Definition

The expression

$$M(x,y)dx + N(x,y)dy \tag{19}$$

is called **exact differential** in a domain D if there exists a function F of two variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$. That is the expression (19) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$.
If $M(x,y)dx + N(x,y)dy$ is an exact differential then $M(x,y)dx + N(x,y)dy = 0$ is called an exact differential equation.

Example

The differential equation

$$y^2dx + 2xydy = 0$$

is an exact differential equation since $y^2dx + 2xydy$ is an exact differential. Consider $F(x, y) = xy^2$:

$$\frac{\partial F(x, y)}{\partial x} = y^2 \text{ and } \frac{\partial F(x, y)}{\partial y} = 2xy$$

Test for exactness

Theorem

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{20}$$

where M and N have continuous first partial derivatives at all points (x,y) in a rectangular domain D.
Exactness of the differential equation (20) in D is equivalent to

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

for all $(x,y) \in D$

Theorem

Suppose the differential equation $M(x, y)dx + N(x, y)dy = 0$ *is exact in a rectangular domain* D*. Then a one parameter family of solutions of this differential equation is given by* $F(x, y) = c$ *where* F *is a function such that*

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x, y) \in D$ *and* c *is an arbitrary constant.*

Justification

Thus,

$$\frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy = 0$$

is the same as

$$dF(x,y) = 0$$

which is possible if

$$F(x,y) = c$$

where c is an arbitrary constant. Namely

$$\frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy = 0 \rightarrow F(x,y) = c$$

Example

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

is exact since

$$\frac{\partial M(x,y)}{\partial y} = 4x = \frac{\partial N(x,y)}{\partial x}$$

for all real (x, y). Thus we must find F such that

$$\frac{\partial F(x,y)}{\partial x} = 3x^2 + 4xy \text{ and } \frac{\partial F(x,y)}{\partial y} = 2x^2 + 2y$$

From the first of these

$$F(x,y) = \int M(x,y)\partial x + \phi(y) = \int (3x^2 + 4xy)\partial x + \phi(y)$$

$$= x^3 + 2x^2 y + \phi(y)$$

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$$F(x,y) = x^3 + 2x^2y + \phi(y)$$

Then

$$\frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy}$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^2 + 2y$$

Thus

$$2x^2 + 2y = 2x^2 + \frac{d\phi(y)}{dy}$$

or

$$2y = \frac{d\phi(y)}{dy} \to \phi(y) = y^2 + c_0$$

Hence

$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

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$$F(x,y) = x^3 + 2x^2y + \phi(y)$$

Then

$$\frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{d\phi(y)}{dy}$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = 2x^2 + 2y$$

Thus

$$2x^2 + 2y = 2x^2 + \frac{d\phi(y)}{dy}$$

or

$$2y = \frac{d\phi(y)}{dy} \rightarrow \phi(y) = y^2 + c_0$$

Hence

$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

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$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

One parameter family of solutions:

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

or

$$x^3 + 2x^2y + y^2 = c$$

For a verification, compare total differentials of both sides:

$$d(x^3 + 2x^2y + y^2) = d(c)$$

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$$

We obtained the original equation; thus solution is verified.

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For another verification way, write the given differential equation in derivative form:

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0 \rightarrow \frac{dy}{dx} = -\frac{3x^2 + 4xy}{2x^2 + 2y}$$

Solve the solution $x^3 + 2x^2y + y^2 = c$ for y to generate an explicit solution:

$$y^2 + \underbrace{2x^2}_{B} y + \underbrace{x^3 - c}_{C} = 0$$

$$y^2 + By + C = 0 \rightarrow y_{1,2} = -\frac{B}{2} \pm \sqrt{\left[\frac{-B}{2}\right]^2 - C} = -x^2 \pm \sqrt{x^4 - x^3 + c}$$

One can show that at least one of $y_{1,2}$ satisfies the given differential equation; this is another verification of that the solution is correct.

Example

Solve the initial value problem

$$(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0, \quad y(0) = 2$$

The equation is exact:

$$\frac{\partial M(x,y)}{\partial y} = -2x\sin y + 3x^2 = \frac{\partial N(x,y)}{\partial x}$$

for all real (x, y). We must find F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = 2x\cos y + 3x^2y \text{ and}$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = x^3 - x^2\sin y - y$$

Continued from the previous page

Then

$$F(x,y) = \int M(x,y)\partial x + \phi(y) = \int (2x\cos y + 3x^2 y)\partial x + \phi(y)$$

$$= x^2\cos y + x^3 y + \phi(y)$$

$$\frac{\partial F(x,y)}{\partial y} = x^3 - x^2\sin y + \frac{d\phi(y)}{dy} = N(x,y) = x^3 - x^2\sin y - y$$

$$\frac{d\phi(y)}{dy} = -y \to \phi(y) = -\frac{y^2}{2} + c_0$$

Thus

$$F(x,y) = x^2\cos y + x^3 y - \frac{y^2}{2} + c_0$$

Continued from the previous page

Family of solutions:

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = c$$

Apply the initial conditions: $y = 2$ at $x = 0$. We find $c = -2$.
Thus the solution is:

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = -2$$

If the differential equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{21}$$

is not exact in a domain D but the differential equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is exact in D, then $\mu(x, y)$ is called an **integrating factor** of the differential equation (21).

Example

$$(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$$

is not exact. $\mu(x, y) = x^2y$ works as an integrating factor for this equation.

Multiplication of a nonexact differential equation by an integrating factor thus transforms the nonexact equation into an exact one. We refer to this resulting exact equation as *essentially equivalent* to the original. This essentially equivalent exact equation has the same one parameter family of solutions as the nonexact original. However, the multiplication of the original equation by the integrating factor may result in either

1) the loss of one or more solutions of the original, or
2) the gain of one or more functions which are solutions of the new equation but not of the original, or
3) both of these phenomena.

We should check to determine whether any solutions may have been lost or gained.

Exercises

Check whether the following are exact or not. If exact, solve them.

$$(3x+2y)dx+(2x+y)dy=0$$

$$(y^2+3)dx+(2xy-4)dy=0$$

$$(2xy+1)dx+(x^2+4y)dy=0$$

Solve the initial value problem

$$(2xy-3)dx+(x^2+4y)dy=0, \;\; y(1)=2$$

$$(3x^2y^2-y^3+2x)dx+(2x^3y-3xy^2+1)dy=0, \;\; y(-2)=1$$

Definition

An equation of the form

$$F(x)G(y)dx + f(x)g(y)dy = 0 \tag{22}$$

is called a separable equation.

Multiply (22) by the integrating factor $\frac{1}{f(x)G(y)}$:

$$\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0 \tag{23}$$

Multiply (22) by the integrating factor

$$\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0 \qquad \text{(cf. 23)}$$

This equation is exact since

$$\frac{\partial}{\partial y}\frac{F(x)}{f(x)} = 0 = \frac{\partial}{\partial x}\frac{g(y)}{G(y)}$$

Denoting $\frac{F(x)}{f(x)}$ by $M(x)$ and $\frac{g(y)}{G(y)}$ by $N(y)$, Equation (23) takes the form

$$M(x)dx + N(y)dy = 0$$

Since M is function of x only, and N is function of y only, a one parameter family of solutions is

$$\int M(x)dx + \int N(y)dy = c$$

where c is the arbitrary constant.

$$F(x)G(y)dx + f(x)g(y)dy = 0 \qquad \text{(cf. 22)}$$

Consider the original equation (22) in the following form:

$$f(x)g(y)\frac{dy}{dx} + F(x)G(y) = 0 \tag{24}$$

If there exists a real number $y = y_0$ such that $G(y_0) = 0$ then (24) reduces to

$$f(x)g(y)\frac{dy}{dx} = 0$$

which has a constant solution $y = y_0$. We next should investigate whether the constant solution $y = y_0$ of the original equation is lost or not in the process of multiplying by the integrating factor.

Example

$$(x-4)y^4dx - x^3(y^2-3)dy = 0$$

The equation above is separable. We separate the variables by dividing by x^3y^4, we obtain

$$\frac{x-4}{x^3}dx - \frac{y^2-3}{y^4}dy = 0$$

or

$$(x^{-2} - 4x^{-3})dx - (y^{-2} - 3y^{-4})dy = 0$$

Integrating we obtain the solutions

$$\frac{-1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$$

where c is any arbitrary constant.

Original equation	:	$(x-4)y^4dx - x^3(y^2-3)dy = 0$
Essentially equivalent equation	:	$\frac{x-4}{x^3}dx - \frac{y^2-3}{y^4}dy = 0$
Soln. of essentially equiv. d.e.	:	$\frac{-1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$

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In multiplying by $\frac{1}{f(x)G(y)} = \frac{1}{x^3y^4}$ in the separation process, we assumed that $x^3 \neq 0$ and $y^4 \neq 0$. We now consider the solution $y = 0$ of $G(y) = 0$, i.e., $y^4 = 0$. It is not a member of the one parameter family of solutions which we obtained. However, writing the original differential equation of the problem in the derivative form

$$\frac{dy}{dx} = \frac{(x-4)y^4}{x^3(y^2-3)}$$

it is obvious that $y = 0$ is a solution of the original equation. We conclude that it is a solution which was lost in the separation process.

Example

Consider

$$\frac{dy}{dt} = \frac{1 + \cos t}{1 + 3y^2}$$

We can write it as

$$(1 + 3y^2)dy = (1 + \cos t)dt$$

Integrating throughout yields the solution:

$$y + y^3 = t + \sin t + c$$

Example

Consider

$$\frac{dy}{dt} = -y\frac{1+2t^2}{t}, \; y(1) = 2$$

We can write it as

$$\int \frac{dy}{y} = -\int \frac{1+2t^2}{t}dt = -\int \frac{dt}{t} - \int 2tdt \; dt$$

$$\ln y = -\ln(t) - t^2 + c$$

$$y = e^{-\ln t - t^2 + c} = \frac{A}{t}e^{-t^2}$$

At $t = 1$ we have $y = 2$. So, $2 = Ae^{-1} \rightarrow A = 2e^1$. Therefore, the solution is

$$y(t) = \frac{2}{t}e^{1-t^2}$$

The first order differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be homogeneous if, when written in derivative form

$$\frac{dy}{dx} = f(x,y)$$

there exists a function g such that $f(x,y)$ can be expressed in the form $g(v)$ where $v = \frac{y}{x}$

Example

The differential equation

$$(x^2 - 3y^2)dx + 2xydy = 0$$

is homogeneous. This equation can be written as

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} = \frac{3}{2}v - \frac{1}{2}\frac{1}{v}$$

where $v := \frac{y}{x}$.

A function F is called homogeneous of degree n if $F(tx, ty) = t^n F(x, y)$.

Theorem

If

$$M(x, y)dx + N(x, y)dy = 0 \tag{25}$$

is a homogeneous equation, then the change of variables $y = vx$ transforms (25) into a separable equation in the variables v and x.

Proof

Homogeneity implies $\frac{dy}{dx} = g(\frac{y}{x})$ for some g. Let $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \to v + x\frac{dv}{dx} = g(v) \to [v - g(v)]dx + xdv = 0$$

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0$$

Integrate throughout:

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$$

where c is an arbitrary constant. Define $F(v) \triangleq \int \frac{dv}{v - g(v)}$ then the solution of the original equation is

$$F(\frac{y}{x}) + \ln|x| = c$$

Example

Consider the differential equation

$$(x^2 - 3y^2)dx + 2xydy = 0$$

We have already seen that this is homogeneous. Write this in the form

$$\frac{dy}{dx} = \frac{-x}{2y} + \frac{3y}{2x} = \frac{-1}{2v} + \frac{v}{2}$$

and let $y = vx$. Obtain

$$v + x\frac{dv}{dx} = \frac{-1}{2v} + \frac{3v}{2} \to x\frac{dv}{dx} = \frac{-1}{2v} + \frac{v}{2} \to \frac{2v}{v^2-1}dv = \frac{dx}{x}$$

Integration gives:

$$\ln|v^2 - 1| = \ln|x| + \ln|c| \to \ln|v^2 - 1| = \ln|x||c|$$

$$\to |v^2 - 1| = |cx| \to |\frac{y^2}{x^2} - 1| = |cx|$$

Definition

A first order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{26}$$

Note that:
If $P(x) = 0$, then direct integration gives the solution:
$y(x) = \int Q(x)dx$
If $Q(x) = 0$, then the equation is separable.

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{26}$$

Continued from the previous page

Equation above can be written in the form

$$[P(x)y - Q(x)]dx + dy = 0 \tag{27}$$

This has the form $M(x,y)dx + N(x,y)dy = 0$. Lets check the exactness:

$$\frac{\partial M(x,y)}{\partial y} = P(x) \text{ and } \frac{\partial N(x,y)}{\partial x} = 0$$

Equation (27) is not exact unless $P(x) = 0$, in which case Equation (26) becomes trivially simple. Let us proceed with the general case $P(x) \neq 0$.

$$[P(x)y - Q(x)]dx + dy = 0 \qquad (27)$$

Continued from the previous page

Multiply equation (27) by $\mu(x)$ to obtain

$$[\mu(x)P(x)y - \mu(x)Q(x)]dx + \mu(x)dy = 0$$

Now the equation is exact iff:

$$\frac{\partial[\mu(x)P(x)y - \mu(x)Q(x)]}{\partial y} = \frac{\partial \mu(x)}{\partial x}$$

This condition reduces to

$$\mu(x)P(x) = \frac{d}{dx}\mu(x)$$

$$\mu(x)P(x) = \frac{d}{dx}\mu(x)$$

Continued from the previous page

This can be written as a differential equation

$$\frac{d\mu}{\mu} = P(x)dx$$

$$\rightarrow \ln|\mu| = \int P(x)dx$$

$$\rightarrow \mu = e^{\int P(x)dx}$$

where it is clear that $\mu > 0$.

Continued from the previous page

Thus

$$\mu = e^{\int P(x)dx} \tag{28}$$

is the integrating factor for (26).

Recall

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{cf.26}$$

Multiply (26) throughout by the integrating factor:

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)y = e^{\int P(x)dx}Q(x) \tag{29}$$

This is equivalent to

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}Q(x) \tag{30}$$

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}Q(x) \tag{30}$$

Continued from the previous page

This results in

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx + c \tag{31}$$

$$y = e^{-\int P(x)dx}[\int e^{\int P(x)dx}Q(x)dx + c] \tag{32}$$

Example

$$\frac{dy}{dx} + \frac{2x+1}{x}y = e^{-2x} \tag{33}$$

Here $P(x) = \frac{2x+1}{x}$ and the integrating factor is

$$e^{\int \frac{2x+1}{x}dx} = e^{2x+\ln|x|} = e^{2x}e^{\ln|x|} = xe^{2x}$$

Multiply (33) by the integrating factor

$$xe^{2x}\frac{dy}{dx} + xe^{2x}\frac{2x+1}{x}y = x$$

or

$$\frac{d}{dx}(xe^{2x}y) = x$$

$$\frac{d}{dx}(xe^{2x}y) = x$$

Continued from the previous page

Integrate throughout

$$xe^{2x}y = \frac{x^2}{2} + c$$

$$y = e^{-2x}\frac{x}{2} + \frac{c}{x}e^{-2x}$$

where c is arbitrary constant.

Definition

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{34}$$

is called a Bernouilli differential equation

Clearly, for $n = 0$ and $n = 1$, the equation is linear.

Theorem

Excluding the cases $n = 0$ and $n = 1$, the transformation $v = y^{1-n}$ reduces the Bernouilli equation to a linear equation in v.

Proof

Multiply the Bernouilli equation by y^{-n} to obtain

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \tag{35}$$

Let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Now the (35) becomes

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Letting $P_1(x) = (1-n)P(x)$ and $Q_1(x) = (1-n)Q(x)$ the differential equation can be written as

$$\frac{dv}{dx} + P_1(x)v = Q_1(x)$$

which is linear in v.

Definition

A Riccati differential equation is an ordinary differential equation that has the form

$$\dot{y} = q_0(x) + q_1(x)y + q_2(x)y^2 \tag{36}$$

Theorem

The Riccati equation can always be reduced to a second order linear ODE.

Here we assume that q_2 is nonzero, otherwise (36) is a linear differential equation. If $q_0 = 0$, then (36) is a Bernouilli differential equation.

$$\dot{y} = q_0(x) + q_1(x)y + q_2(x)y^2 \tag{36}$$

Continued from the previous page

Use the transform

$$v = yq_2$$

then

$$\dot{v} = \dot{y}q_2 + y\dot{q}_2 = (q_0 + q_1 y + q_2 y^2)q_2 + v\frac{\dot{q}_2}{q_2} = q_0 q_2 + (q_1 + \frac{\dot{q}_2}{q_2})v + v^2$$

Define $Q := q_0 q_2$ and $P := q_1 + \frac{\dot{q}_2}{q_2}$ we can write

$$\dot{v} = v^2 + P(x)v + Q(x)$$

Continued from the previous page

$$\dot{v} = v^2 + P(x)v + Q(x)$$

Now use

$$v = -\frac{\dot{u}}{u}$$

This implies

$$\dot{v} = -(\frac{\dot{u}}{u})' = -(\dot{u} \times \frac{1}{u})' = -(\frac{\ddot{u}}{u}) + (\frac{\dot{u}}{u})^2 = -(\frac{\ddot{u}}{u}) + v^2$$

so that

$$\frac{\ddot{u}}{u} = v^2 - \dot{v} = -Q - Pv = -Q + P\frac{\dot{u}}{u}$$

and hence

$$\ddot{u} - P\dot{u} + Qu = 0. \qquad \text{Q.E.D.}$$

$$\dot{y} = q_0(x) + q_1(x)y + q_2(x)y^2 \tag{36}$$

Theorem

If any solution $u(x)$ of the Riccati equation (36) is known, then substitution of $y = u + \frac{1}{z}$ will transform (36) into a linear 1st order equation in z.

Proof If u is a solution of the Riccati equation then

$$\frac{du}{dx} = q_0(x) + q_1(x)u + q_2(x)u^2 \tag{37}$$

By using the substitution $y = u + \frac{1}{z}$, we have

$$\frac{dy}{dx} = \frac{d}{dx}(u + \frac{1}{z}) = \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx} \tag{38}$$

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$$\dot{y} = q_0(x) + q_1(x)y + q_2(x)y^2 \tag{36}$$

$$\frac{dy}{dx} = \frac{d}{dx}(u + \frac{1}{z}) = \frac{du}{dx} - \frac{1}{z^2}\frac{dz}{dx} \tag{38}$$

Substitute (38) in the Riccati equation (36):

$$\begin{aligned}\tfrac{du}{dx} - \tfrac{1}{z^2}\tfrac{dz}{dx} &= q_2(u + \tfrac{1}{z})^2 + q_1(u + \tfrac{1}{z}) + q_0 \\ &= \underbrace{(q_2u^2 + q_1u + q_0)}_{\text{equals } \frac{du}{dx} \text{ by (37)}} + (\tfrac{2u}{z}q_2 + \tfrac{1}{z^2}q_2 + \tfrac{1}{z}q_1)\end{aligned}$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{2u}{z}q_2 + \frac{1}{z^2}q_2 + \frac{1}{z}q_1$$

$$\frac{dz}{dx} = -2uzq_2 - q_2 - zq_1$$

$$\frac{dz}{dx} = -(2uq_2 + q_1)z - q_2$$

which is a linear 1st order differential equation in z.

Example

Consider the Riccati equation

$$\frac{dy}{dx} + y = xy^2 - \frac{1}{x^2} \tag{39}$$

$y = \frac{1}{x}$ is a particular solution to (39). We want to find the other solution. Use the transform

$$y = \frac{1}{x} + \frac{1}{z}$$

then we have

$$y' = -\frac{z'}{z^2} - \frac{1}{x^2}$$

Substitute y and y' in (39):

$$-\frac{z'}{z^2} - \frac{1}{x^2} + \frac{1}{x} + \frac{1}{z} = x(\frac{1}{x} + \frac{1}{z})^2 - \frac{1}{x^2}$$

$$-\frac{z'}{z^2}-\frac{1}{x^2}+\frac{1}{x}+\frac{1}{z}=x(\frac{1}{x}+\frac{1}{z})^2-\frac{1}{x^2}$$

$$-\frac{z'}{z^2}-\cancel{\frac{1}{x^2}}+\cancel{\frac{1}{x}}+\frac{1}{z}=x(\cancel{\frac{1}{x^2}}+\frac{1}{z^2}+\frac{2}{xz})-\cancel{\frac{1}{x^2}}$$

Continued from the previous page

Simplification yields a 1st order linear de:

$$z'+z=-x$$

Its solution is

$$z=1-x+ce^{-x}$$

Noting that $y=\frac{1}{x}+\frac{1}{z}$, the solution to (39) is

$$y=\frac{1}{x}+\frac{1}{1-x+ce^{-x}}$$

Example

Consider the Riccati equation

$$x\frac{dy}{dx} - 3y + y^2 = 4x^2 - 4x$$

Obviously $u(x) = 2x$ is a particular solution of this differential equation. From this we can obtain a 1st order linear differential equation in z.

Orthogonal trajectories

Let

$$F(x, y, c) = 0 \tag{40}$$

be a given one parameter family of curves in xy-plane. A curve that intersects curves of the family (40) at right angles is called an orthogonal trajectory of the given family.

Example

Consider the family of curves $x^2 + y^2 = c^2$. Each straight line passing through the origin $y = kx$ is an orthogonal trajectory of the given family of circles.

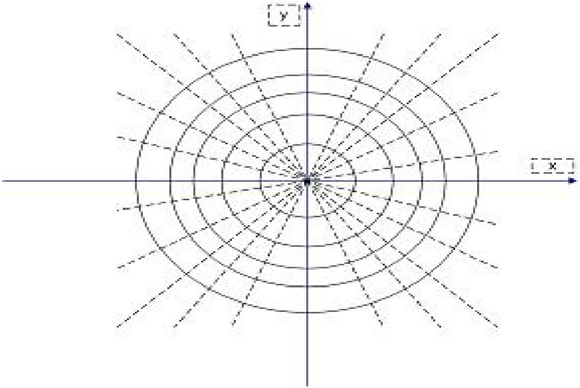


Figure: Orthogonal trajectories for $x^2 + y^2 = c$

How to find orthogonal trajectories

Step 1. Differentiate $F(x, y, c) = 0$ with respect to x to obtain

$$\frac{dy}{dx} = f(x, y) \tag{41}$$

Step 2. Solutions of $\frac{dy}{dx} = \frac{-1}{f(x,y)}$ are the orthogonal trajectories.

Step 1. Differentiate $F(x, y, c) = 0$ with respect to x to obtain

$$\frac{dy}{dx} = f(x, y) \tag{41}$$

Step 2. Solutions of $\frac{dy}{dx} = \frac{-1}{f(x,y)}$ are the orthogonal trajectories.

In $F(x, y, c) = 0$ the slope of the curve passing through the point (x, y) is $\frac{dy}{dx}$, which is $f(x, y)$. However, the slope of the curves passing through (x, y) having right angle to $F(x, y, c) = 0$ curves are $\frac{-1}{f(x,y)}$.

Caution. In step 1 finding the differential equation (41) of the given family, be sure to eliminate the parameter c during the process.

Example

$F(x, y, c) = 0$ is given by $x^2 + y^2 - c^2 = 0$. Differentiation gives

$$2x + 2y\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \underbrace{\frac{-x}{y}}_{f(x,y)}$$

We are looking for the orthogonal trajectories, so we must solve

$$\frac{dy}{dx} = \underbrace{\frac{y}{x}}_{\frac{-1}{f(x,y)}}$$

or

$$\frac{dy}{dx} = \frac{y}{x} \rightarrow \frac{dy}{y} = \frac{dx}{x} \rightarrow \ln y = \ln x + \ln k \rightarrow \ln y = \ln kx \rightarrow y = kx$$

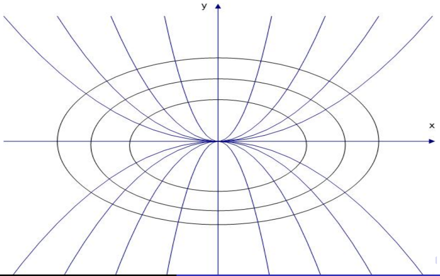
Example

Find the orthogonal trajectories of the family of parabolas $y = cx^2$.

$$y = cx^2 \to \frac{dy}{dx} = 2\underbrace{c}_{\frac{y}{x^2}}x \to \frac{dy}{dx} = 2\frac{y}{x}$$

Orthogonal trajectory finding requires solving $\frac{dy}{dx} = \frac{-x}{2y}$

$$2ydy = -xdx \to y^2 = -\frac{x^2}{2} + c \to x^2 + 2y^2 = k^2$$



Oblique trajectories

Definition

Let

$$F(x, y, c) = 0 \tag{42}$$

be a given one parameter family of curves in xy-plane. A curve that intersects curves of the family (42) at a constant angle $\alpha \neq 90^0$ is called an oblique trajectory of the given family.

Differential equation corresponding to (42) is

$$\frac{dy}{dx} = f(x, y) \tag{43}$$

Then the curve of family (42) through the point (x, y) has slope $f(x, y)$ at (x, y) and hence its tangent line has angle of inclination $\tan^{-1}[f(x, y)]$ there. The tangent line of an oblique trajectory that intersects this curve at the angle α will thus have an inclination $\tan^{-1}[f(x, y)] + \alpha$ at the point (x, y). Hence the slope of the oblique trajectory is given by

$$\tan\{\tan^{-1}[f(x, y)] + \alpha\} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha}$$

Thus the differential equation of such a family of oblique trajectories is given by

$$\frac{dy}{dx} = \frac{f(x, y) + \tan\alpha}{1 - f(x, y)\tan\alpha} \tag{44}$$

Example

Find the family of oblique trajectories that intersect the family of straight lines $y = cx$ at angle 45^0.

$$y = cx \rightarrow \frac{dy}{dx} = c \rightarrow \frac{dy}{dx} = \frac{y}{x}$$

In

$$\frac{dy}{dx} = \frac{f(x,y) + \tan\alpha}{1 - f(x,y)\tan\alpha} \tag{44}$$

use $f(x,y) = \frac{y}{x}$ and $\tan\alpha = 1$:

$$\frac{dy}{dx} = \frac{\frac{y}{x} + 1}{1 - \frac{y}{x}1} = \frac{x+y}{x-y}$$

This is a homogeneous differential equation Let $y = vx$:

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

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After simplification

$$\frac{(v-1)dv}{v^2+1} = \frac{-dx}{x}$$

Integrating

$$\frac{1}{2}\ln(v^2+1) - \tan^{-1}(v) = -\ln|x| - \ln|c|$$

$$\ln c^2x^2(v^2+1) - 2\tan^{-1} v = 0$$

$$\ln c^2(x^2+y^2) - 2\tan^{-1}\frac{y}{x} = 0$$

Definition

A linear ordinary differential equation of order n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in, the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x) \quad (45)$$

where a_0 is not identically zero.

We shall assume that $a_0,\ a_1, \ldots, a_n$ and F are continuous real functions on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$. The righthand member $F(x)$ is called the nonhomogeneous term. If F is identically zero Equation (45) reduces to

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0 \quad (46)$$

and is then called homogeneous.

Theorem

Consider the n-th order linear differential equation given by Equation (45) where $a_0,\ a_1, \ldots, a_n$ *and* F *are continuous real functions on a real interval* $a \leq x \leq b$ *and that* $a_0(x) \neq 0$ *for any* x *on* $a \leq x \leq b$*. Let* x_0 *be any point on the interval* $a \leq x \leq b$*, and let* $c_0,\ c_1, \ldots, c_{n-1}$ *be n arbitrary real constants. Then there exists a unique solution of Equation (45) such that*

$$f(x_0) = c_0,\ f'(x_0) = c_1, \ldots, f^{(n-1)}(x_0) = c_{n-1}$$

and this solution is defined over the entire interval $a \leq x \leq b$*.*

Example

Consider the initial value problem

$$\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3y = e^x, \quad y(1) = 2, y'(1) = -5$$

In the interval $-\infty < x < \infty$ the hypotheses of Theorem 6 are satisfied, so the equation has a unique solution in this interval.

Corollary

Let f be a solution of the n-th order homogeneous linear differential equation given by Equation (46) such that

$$f(x_0) = 0,\ f'(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0$$

where x_0 is a point of the interval $a \leq x \leq b$ in which the coefficients $a_0,\ a_1, \ldots, a_n$ are all continuous and $a_0(x) \neq 0$. Then $f(x) = 0$ for all $x \in [a, b]$.

Theorem

For a homogeneous linear differential equation, (a) the sum of the solutions is also a solution and (b) a constant multiple of a solution is also a solution.

Proof Consider

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = 0 \tag{47}$$

where α, β and γ are functions of t.
Let the functions x_1 and x_2 be solutions to (47). Then

$$\alpha \frac{d^2x_1}{dt^2} + \beta \frac{dx_1}{dt} + \gamma x_1 = 0 \text{ and } \alpha \frac{d^2x_2}{dt^2} + \beta \frac{dx_2}{dt} + \gamma x_2 = 0$$

We wish to prove that $x_1 + x_2$ is also a solution, that is

$$\alpha \frac{d^2(x_1 + x_2)}{dt^2} + \beta \frac{d(x_1 + x_2)}{dt} + \gamma (x_1 + x_2) = 0$$

$$\alpha \frac{d^2(x_1 + x_2)}{dt^2} + \beta \frac{d(x_1 + x_2)}{dt} + \gamma(x_1 + x_2) = 0$$

Continued from the previous page

Using the basic property of the derivatives:

$$\alpha \frac{d^2(x_1)}{dt^2} + \alpha \frac{d^2(x_2)}{dt^2} + \beta \frac{d(x_1)}{dt} + \beta \frac{d(x_2)}{dt} + \gamma x_1 + \gamma x_2 = 0$$

$$\alpha \frac{d^2(x_1)}{dt^2} + \beta \frac{d(x_1)}{dt} + \gamma x_1 + \alpha \frac{d^2(x_2)}{dt^2} + \beta \frac{d(x_2)}{dt} + \gamma x_2 = 0 + 0 = 0$$

Likewise, we wish to show that if x satisfies (47) then kx also satisfies it for any constant k.

$$\alpha \frac{d^2(kx)}{dt^2} + \beta \frac{d(kx)}{dt} + \gamma(kx) = \alpha k \frac{d^2(x)}{dt^2} + \beta k \frac{d(x)}{dt} + k\gamma x$$

$$= k(\alpha \frac{d^2(x)}{dt^2} + \beta \frac{d(x)}{dt} + \gamma x) = k \cdot 0 = 0$$

Theorem

Let $f_1, f_2, \ldots, f_m$ be any m solutions of the homogeneous linear differential equation (46). Then $c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$ is also a solution of (46), where $c_1, \ldots, c_m$ are m arbitrary constants.

Definition

If $f_1, f_2, \ldots, f_m$ are m given functions, and $c_1, c_2, \ldots, c_m$ are m constants then the expression $c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$ is called a linear combination of $f_1, f_2, \ldots, f_m$.

Theorem

(Restated) Any linear combination of solutions of the homogeneous linear differential equation (46) is also a solution of (46).

Example

$\sin x$ and $\cos x$ are solutions of

$$\frac{d^2y}{dx^2} + y = 0$$

By the theorem $5\sin x + 6\cos x$ is also a solution of the equation.

Definition

The n functions $f_1, f_2, \ldots, f_n$ are called **linearly dependent** on $a \le x \le b$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

for all x such that $a \le x \le b$.

Example

Are the functions $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = x^2 + 2x$ $f_4(x) = 3$ linearly dependent on $0 \le x \le 10$?

$$c_1 x + c_2 x^2 + c_3(x^2 + 2x) + c_4 3 = 0,\ \forall x \in [0, 10]$$

In addition to zero the solution $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$ we have a nonzero solution $c_1 = 2, c_2 = 1, c_3 = -1, c_4 = 0$.
$\therefore$ This group of functions is linearly dependent.

In particular two functions f_1 and f_2 are linearly dependent on $a \leq x \leq b$ if there exist constants c_1, c_2, not both zero, such that

$$c_1 f_1 + c_2 f_2 = 0$$

for all x such that $a \leq x \leq b$.

Example

x and $2x$ are linearly dependent on the interval $0 \leq x \leq 1$, since there exist constants c_1, c_2, not both zero, such that

$$c_1 \cdot x + c_2 \cdot 2x = 0 \tag{48}$$

for all x on the interval $0 \leq x \leq 1$. For instance, $c_1 = 2, c_2 = -1$

Notice that we found constants c_1 and c_2 that work for all x in the given interval $0 \leq x \leq 1$. If they worked for some x values only then we wouldn't say that the functions are linearly dependent. The next example illustrates this idea:

Example

Consider the functions $\cos x$, $\cos 2x$, and $\cos 3x$ on the interval $-\pi \leq x \leq \pi$. Form the linear dependence equation

$$c_1 \cos x + c_2 \cos 2x + c_3 \cos 3x = 0, \; -\pi \leq x \leq \pi \tag{49}$$

When $x = 0$ this equation holds for $c_1 = 1, c_2 = 1$ and $c_3 = -2$. But this does not make this set linearly dependent. For linear dependency on $-\pi \leq x \leq \pi$, the constants c_1, c_2 and c_3 must work for ALL x on the interval $-\pi \leq x \leq \pi$. Notice thet, for instance, when $x = \frac{\pi}{2}$, the above c_1, c_2, c_3 don't satisfy Equation (49).

Definition

The n functions $f_1, f_2, \ldots, f_n$ are called **linearly independent** on the interval $a \leq x \leq b$ if they are not linearly dependent there.

Example

Are $f_1(t) = 2t$ and $f_2(t) = t^2$ linearly dependent on $0 \leq t \leq 2$? If we can find constants c_1 and c_2, not both zero, such that

$$c_1 2t + c_2 t^2 = 0,\ 0 \leq t \leq 2 \tag{50}$$

holds, then f_1 and f_2 are linearly dependent.
Suppose for some c_1 and c_2, not both zero, Equation (50) is satisfied. Then it must hold particularly at $t = 0.5$ and $t = 1$:

$$c_1 + 0.25c_2 = 0$$
$$2c_1 + c_2 = 0$$

These two equations imply $c_1 = c_2 = 0$, that is, the functions are linearly independent. While we require Equation (50) hold at all points on $0 \leq t \leq 2$, it even does not hold at two points on that interval!
$\therefore$ This group of functions is **linearly independent**.

Alternative analysis of the previous example

Example

Are $f_1(t) = 2t$ and $f_2(t) = t^2$ linearly dependent on $0 \leq t \leq 2$? If we can find constants c_1 and c_2, not both zero, such that

$$c_1 2t + c_2 t^2 = 0,\ 0 \leq t \leq 2 \tag{50}$$

holds, then f_1 and f_2 are linearly dependent.
Note that if (50) holds on $0 \leq t \leq 2$, then so does its derivative:

$$c_1 \cdot 2 + c_2 \cdot 2t = 0,\ 0 \leq t \leq 2$$

This implies $c_1 = -c_2 t$. Substitute this in (50):
$-c_2 t \cdot 2t + c_2 t^2 = 0,\ 0 \leq t \leq 2 \rightarrow -c_2 t^2 = 0,\ 0 \leq t \leq 2. \rightarrow$ $c_2 = 0$.. Use this in (50):
$c_1 \cdot 2t = 0,\ 0 \leq t \leq 2, \rightarrow c_1 = 0.$
We have only one solution $c_1 = c_2 = 0$,
$\therefore$ the set of functions $\{f_1, f_2\}$ is linearly independent.

Theorem

The n-th order homogeneous linear differential equation (46) always possesses n solutions that are linearly independent. Further, if $f_1, f_2, \dots, f_n$ are n linearly independent solutions of (46), then every solution f of (46) can be expressed as a linear combination

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$$

of these n linearly independent solutions by proper choice of the constants $c_1, c_2, \dots, c_n$.

Example

$\sin x$ and $\cos x$ are solutions of

$$\frac{d^2y}{dx^2} + y = 0 \tag{51}$$

for all x, $-\infty < x < \infty$. Further one can show that these two solutions are linearly independent. Now suppose f is any solution of (51), then by the theorem f can be expressed as a certain linear combination $c_1 \sin x + c_2 \cos x$ of the two linearly independent solutions $\sin x$ and $\cos x$ by proper choice of c_1 and c_2.

Definition

If $f_1, f_2, \ldots, f_n$ are n linearly independent solutions of the n-th order homogeneous linear differential equation (46) on $a \leq x \leq b$, then the set $f_1, f_2, \ldots, f_n$ is called a fundamental set of solutions of (46) and the function

$$f(x) = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n, \quad a \leq x \leq b$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants, is called a general solution of (46) on $a \leq x \leq b$.

Example

$\sin x$ and $\cos x$ are linearly independent solutions of

$$\frac{d^2y}{dx^2} + y = 0 \tag{52}$$

for all x, $-\infty < x < \infty$. So, $\{\sin x, \cos x\}$ is a fundamental set of solutions for the differential equations (52). Thus $c_1 \sin x + c_2 \cos x$ is a general solution for (52). One can verify that $3 \sin x$ and $2 \sin x + \cos x$ are linearly independent solutions of (52). Therefore, $\{3 \sin x, 2 \sin x + \cos x\}$ is another fundamental set of solutions for (52). This implies that $c_1 3 \sin x + c_2(2 \sin x + \cos x)$ is also a general solution for (52). That is, expressing the general solution is not unique. The two general solution expressions represent the same set.

Definition

Let $f_1, f_2, \ldots, f_n$ be n real functions each of which has an $(n-1)$st derivative on a real interval $a \leq x \leq b$. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called Wronskian of these n functions.

Theorem

The n solutions $f_1, f_2, \ldots, f_n$ of the n-th order homogeneous linear differential equation (46) are linearly independent on $a \leq x \leq b$ if and only if the Wronskian of $f_1, f_2, \ldots, f_n$ is different from zero for some x on the interval $a \leq x \leq b$.

Theorem

The Wronskian of n solutions $f_1, f_2, \ldots, f_n$ of equation (46) is either identically zero on $a \leq x \leq b$ or else is never zero on $a \leq x \leq b$.

Example

Let us show that $\sin x$ and $\cos x$ are linearly independent for all real x:

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

Example

The solutions e^x, e^{-x}, and e^{2x} of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

are linearly independent on every real interval:

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = -6e^{2x} \neq 0$$

for all real x.
The general solution to the d.e. is, therefore,

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$$

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Do the solutions e^x, e^{-x}, and $e^x + e^{-x}$ of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

linearly independent on every real interval? Can we write the general solution to the d.e. as

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3(e^x + e^{-x})$$

Example

Are the functions $\sin x$ and $|\sin x|$ linearly independent on (a) $0 \leq x \leq \pi$ (b) $0 \leq x \leq 2\pi$ (c) $0 \leq x \leq 4\pi$

Theorem

Let v be any solution of the given n-th order nonhomogeneous linear differential equation (45). Let u be any solution of the corresponding homogeneous equation. Then $u + v$ is also a solution of the given nonhomogeneous linear differential equation (45).

Example

$y = x$ is a solution of the nonhomogeneous differential equation $\frac{d^2y}{dx^2} + y = x$ and that $y = \sin x$ is a solution of the corresponding homogeneous differential equation $\frac{d^2y}{dx^2} + y = 0$. By the theorem, the sum $y = x + \sin x$ is also a solution of the nonhomogeneous equation.

Theorem

Let y_p be a given solution of the n-th order nonhomogeneous linear differential equation (45) involving no arbitrary constants. Let $y_c = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ be the general solution of the corresponding homogeneous equation (46). Then every solution ϕ of the n-th order nonhomogeneous linear differential equation (45)can be expressed in the form

$$y_c + y_p$$

that is

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$$

for suitable choice of n arbitrary constants $c_1, c_2, \ldots, c_n$.

Definition

Consider the n-th order nonhomogeneous linear differential equation (45) and the corresponding homogeneous equation (46). The general solution of (46) is called the complementary function of (45). We shall denote this by y_c. Any particular solution of (45) involving no arbitrary constants is called a particular integral of (45). The solution $y_c + y_p$ is called the general solution of (45).

Example

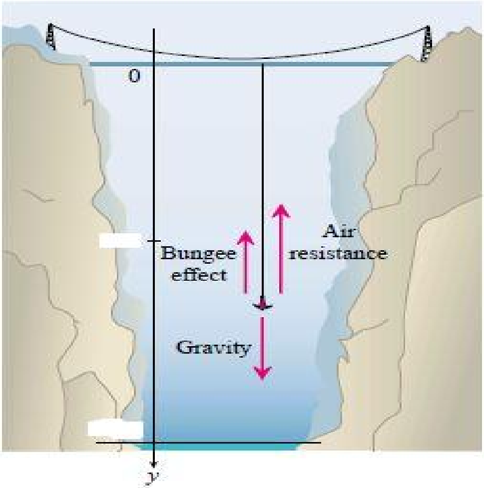
Consider

$$\frac{d^2y}{dx^2} + y = x$$

$$y_c = c_1 \sin x + c_2 \cos x, \quad y_p = x$$

General solution:

$$y = c_1 \sin x + c_2 \cos x \; + x$$


0
Bungee
effect
Air
resistance
Gravity
y

Example

A man $1.8m$ tall and weighing $80kg$ bungee jumps off a bridge over a river.

The bridge is $200m$ above the water surface and the unstretched bungee cord is $30m$ long.

The spring constant of the bungee cord is $K_s = 11N/m$, meaning that, when the cord is stretched, it resists the stretching with a force of 11 newtons per meter of stretch.

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When the man jumps off the bridge he goes into free fall until the bungee cord is extended to its full unstretched length.

This occurs when the man's feet are at $30m$ below the bridge.

His initial velocity and position are zero. His acceleration is $9.8m/s^2$ until he reaches 30 m below the bridge.

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His position is the integral of his velocity and his velocity is the integral of his acceleration.

So, during the initial free-fall time, his velocity is $9.8 \times t$ m/s, where t is time in seconds and his position is $4.9 \times t^2$ m below the bridge.

Solving for the time of full unstretched bungee-cord extension we get $2.47s$. At that time his velocity is 24.25 meters per second, straight down. At this point the analysis changes because the bungee cord starts having an effect.

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There are two forces on the man:

1. The downward pull of gravity mg where m is the man's mass and g is the acceleration caused by the earth's gravity

2. The upward pull of the bungee cord $K_s(y(t) - 30)$ where $y(t)$ is the vertical position of the man below the bridge as a function of time.

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Then, using the principle that force equals mass times acceleration and the fact that acceleration is the second derivative of position, we can write

$$mg - K_s(y(t) - 30) = m\ddot{y}(t)$$

or

$$m\ddot{y}(t) + K_s y(t) = mg + 30K_s$$

This is a second-order, linear, constant-coefficient, inhomogeneous, ordinary differential equation. Its total solution is the sum of its homogeneous solution and its particular solution.

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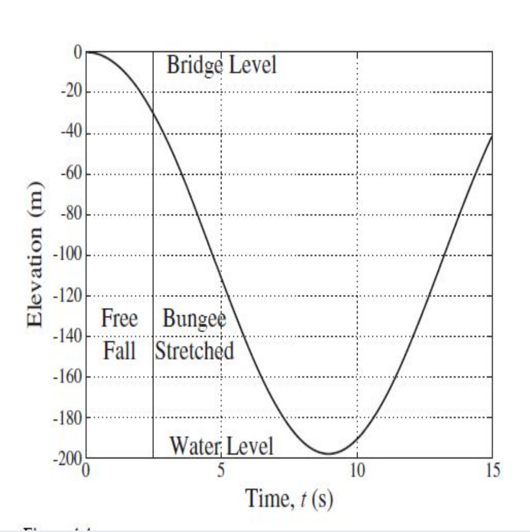

An order reduction technique

$$x^3 + 4x^2 + 2x + 6 = 0$$

(Hypothetically) I don't know how to solve 3rd degree polynomial equations. If one tells me that one of its root is at $x = -3.8829$, will it help me to find the others?

$$\frac{x^3 + 4x^2 + 2x + 6}{x + 3.8829} = x^2 + 0.1172x + 1.5453 \text{ a second degree polynomial}$$

Theorem

Let f be a nontrivial solution of the n-th order homogeneous linear differential equation given by Equation (46). Then the transformation $y = f(x)v$ reduces Equation (46) to an $(n-1)$st order homogeneous linear differential equation in the dependent variable $w = \frac{dv}{dx}$.

Suppose f is a known nontrivial solution of the second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{53}$$

Let a solution to the equation above be

$$y = f(x)v \tag{54}$$

where f is the known solution of (53) and v is a function of x that will be determined.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{53}$$

$$y = f(x)v \rightarrow \tag{54}$$

$$\frac{dy}{dx} = f(x)\frac{dv}{dx} + f'(x)v \tag{55}$$

$$\frac{d^2y}{dx^2} = f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v \tag{56}$$

Substituting (54), (55), and (56) in (53) we obtain

$$a_0(x)[f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v]$$

$$+a_1(x)[f(x)\frac{dv}{dx} + f'(x)v] + a_2(x)f(x)v = 0$$

$$a_0(x)[f(x)\frac{d^2v}{dx^2} + 2f'(x)\frac{dv}{dx} + f''(x)v]$$

$$+a_1(x)[f(x)\frac{dv}{dx} + f'(x)v] + a_2(x)f(x)v = 0$$

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx}$$

$$+[a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)]v = 0$$

Since f is a solution of (53), the coefficient of v is zero, and so that the last equation reduces to

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx} = 0$$

$$a_0(x)f(x)\frac{d^2v}{dx^2} + [2a_0(x)f'(x) + a_1(x)f(x)]\frac{dv}{dx} = 0$$

Letting $w = \frac{dv}{dx}$, this becomes

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0$$

This is a first order homogeneous linear differential equation in the dependent variable w. The equation is separable, thus by the assumptions $f(x) \neq 0$ and $a_0(x) \neq 0$, we may write

$$\frac{dw}{w} = -[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}]dx$$

$$\frac{dw}{w} = -[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}]dx$$

Integrating we obtain

$$\ln|w| = -\ln[f(x)]^2 - \int \frac{a_1(x)}{a_0(x)}dx + \ln|c|$$

$$\ln\frac{|w|[f(x)]^2}{c} = -\int \frac{a_1(x)}{a_0(x)}dx$$

$$w = \frac{ce^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2} \rightarrow v = \int \frac{ce^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}dx \rightarrow y(x) = f(x)\int \frac{ce^{-\int \frac{a_1(x)}{a_0(x)}dx}}{[f(x)]^2}$$

It can be shown that the new solution and f are linearly independent. Thus the linear combination $c_1 f + c_2 fv$ is the general solution of (53).

Example

$y = x$ is a solution of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \tag{57}$$

Find a linearly independent solution by reducing the order.
Let $y = vx$, then $\frac{dy}{dx} = x\frac{dv}{dx} + v$ and $\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$. Substitute them in (57):

$$(x^2+1)(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}) - 2x(x\frac{dv}{dx} + v) + 2xv = 0$$

or

$$x(x^2+1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0$$

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$$x(x^2+1)\frac{d^2v}{dx^2}+2\frac{dv}{dx}=0$$

Letting $w=\frac{dv}{dx}$ we obtain

$$x(x^2+1)\frac{dw}{dx}+2w=0$$

$$\frac{dw}{w}=-2\frac{dx}{x(x^2+1)}$$

$$\frac{dw}{w}=(\frac{-2}{x}+\frac{2x}{x^2+1})dx$$

$$\ln|w|=-2\ln|x|+\ln(x^2+1)+\ln|c|\rightarrow$$

$$\ln|w|=-\ln x^2+\ln(x^2+1)+\ln|c|\rightarrow$$

$$\ln|w|=\ln\frac{c(x^2+1)}{x^2}$$

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$$w = \frac{c(x^2+1)}{x^2}$$

Use $\frac{dv}{dx} = w$:

$$v(x) = c\left[x - \frac{1}{x}\right]$$

$$y(x) = cx\left[x - \frac{1}{x}\right] = c(x^2-1)$$

Theorem

Let x_1 and x_2 respectively be the solutions of

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_1(t) \tag{58}$$

and

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_2(t) \tag{59}$$

where α, β and γ are functions of t. Then $x_1 + x_2$ is a solution of

$$\alpha \frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + \gamma x = f_1(t) + f_2(t) \tag{60}$$

Proof

$$\alpha \frac{d^2(x_1 + x_2)}{dt^2} + \beta \frac{d(x_1 + x_2)}{dt} + \gamma (x_1 + x_2) = \cdots\cdots$$

$$\alpha\frac{d^2(x_1+x_2)}{dt^2}+\beta\frac{d(x_1+x_2)}{dt}+\gamma(x_1+x_2)$$

$$=\alpha\frac{d^2x_1}{dt^2}+\alpha\frac{d^2x_2}{dt^2}+\beta\frac{dx_1}{dt}+\beta\frac{dx_2}{dt}+\gamma x_1+\gamma x_2$$

$$=\underbrace{\alpha\frac{d^2x_1}{dt^2}+\beta\frac{dx_1}{dt}+\gamma x_1}_{f_1(t)}+\underbrace{\alpha\frac{d^2x_2}{dt^2}+\beta\frac{dx_2}{dt}+\gamma x_2}_{f_2(t)}=f_1(t)+f_2(t)$$

Indeed, knowing solutions corresponding to f_1 and f_2 we get the solution corresponding to the forcing function f_1+f_2.

Theorem

Let f_1 be a solution of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_1(x)$$

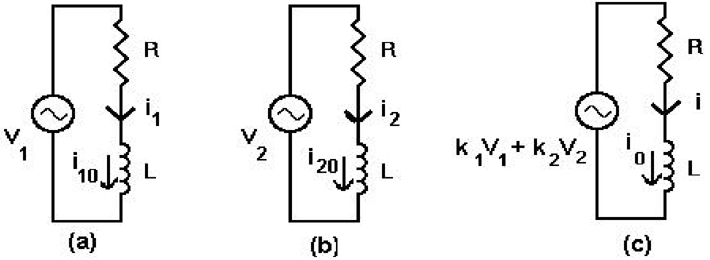
Let f_2 be a solution of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F_2(x)$$

Then $k_1 f_1 + k_2 f_2$ is a solution of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = k_1 F_1(x) + k_2 F_2(x)$$

where k_1 and k_2 are arbitrary constants.



Let $i_{10} = i_{20} = 0$. And let v_1 results in the current i_1, and v_2 results in the current i_2. Then for the input $k_1 v_1 + k_2 v_2$ with $i_0 = 0$, the current i will be $k_1 i_1 + k_2 i_2$.

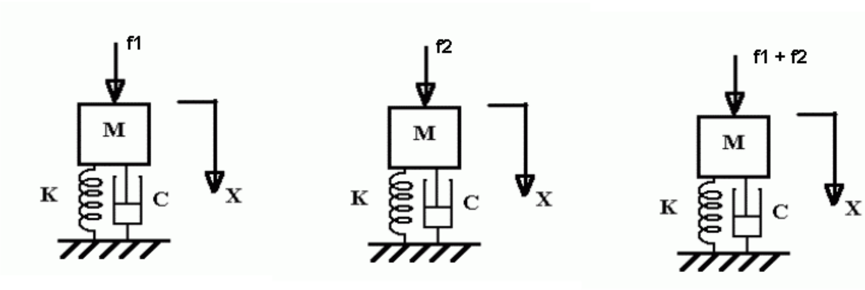


Figure: Mass-damper-spring system with inputs

Preliminaries

Quadratic formula If

$$ax^2 + bx + c = 0 \tag{61}$$

then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{62}$$

Qubic formula If

$$x^3 + px^2 + qx + r = 0 \tag{63}$$

then use the transformation

$$x = u - \frac{p}{3} \tag{64}$$

to obtain

$$u^3 + au + b = 0 \tag{65}$$

where $a = q - \frac{p^2}{3}$ and $b = r - \frac{pq}{3} + \frac{2p^3}{27}$. For the solution of (65) evaluate

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

$$B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

The roots of (65) are:

$$u = A + B$$

$$u = -\frac{1}{2}(A + B) + \sqrt{-\frac{3}{4}}(A - B)$$

$$u = -\frac{1}{2}(A + B) - \sqrt{-\frac{3}{4}}(A - B)$$

Consider

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \tag{66}$$

where $a_0, a_1, \ldots, a_n$ are real constants. Consider the solution candidate:

$$y = e^{mx}$$

Then we have:

$$\frac{dy}{dx} = me^{mx}, \quad \frac{d^2 y}{dx^2} = m^2 e^{mx}, \quad \ldots, \frac{d^n y}{dx^n} = m^n e^{mx}$$

Substitute in (66):

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

or

$$e^{mx}(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) = 0$$

Since $e^{mx} \neq 0$, for the satisfaction of the equation we must have

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \tag{67}$$

This equation is called **auxiliary equation** or the **characteristic equation** of the given differential equations (66).

Coefficients of the auxiliary equation

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \qquad (66)$$

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \qquad (67)$$

Theorem

Consider the n-th order homogeneous linear differential equations (66) with constant coefficients. If the auxiliary equation (67) has the n real-distinct roots $m_1, m_2, \ldots, m_n$ then the general solution of (66) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Consider

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

Hence $m_1 = 1$ and $m_2 = 2$. The roots are real and distinct. Thus e^x and e^{2x} are solutions. The general solution is then

$$y = c_1 e^x + c_2 e^{2x}$$

where c_1, c_2 are arbitrary constants.

Theorem

Consider the n-th order homogeneous linear differential equations (66) with constant coefficients. If the auxiliary equation (67) has the real root m occurring k times, then the part of the general solution of (66) corresponding to this k-fold repeated root is

$$(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})e^{mx}$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants.

Example

Find the general solution of

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0.$$

The auxiliary equation

$$m^3 - 4m^2 - 3m + 18 = 0$$

has the roots $3, 3, -2$. The general solution is then

$$y = (c_1 + c_2x)e^{3x} + c_3e^{-2x}$$

where c_1, c_2, c_3 are arbitrary constants.

Example

Let a constant coefficient homogeneous linear differential in the independent variable x have the characteristic equation

$$(m-4)^3(m-2)^2(m-5)=0$$

The general solution is

$$(c_1+c_2x+c_3x^2)e^{4x}+(c_4+c_5x)e^{2x}+c_6e^{5x}$$

where $c_1, c_2, \ldots, c_6$ are arbitrary constants.

Theorem

Consider the n-th order homogeneous linear differential equations (66) with constant coefficients. If the auxiliary equation (67) has the conjugate complex roots $a+bi$ and $a-bi$, neither repeated, then the corresponding part of the general solution of (66) may be written as

$$y = e^{ax}(c_1 \sin(bx) + c_2 \cos(bx))$$

where c_1, c_2 are arbitrary constants.
If, however, $a+bi$ and $a-bi$ are each k-fold roots of the auxiliary equation (67) then the corresponding part of the general solution of (66) may be written as

$$y = e^{ax}[(c_1 + c_2 x + \cdots + c_k x^{k-1}) \sin(bx)$$

$$+(c_{k+1} + c_{k+2} x + \cdots + c_{2k} x^{k-1}) \cos(bx)]$$

where $c_1, c_2, \ldots, c_{2k}$ are arbitrary constants.

Example

$$\frac{d^2y}{dx^2} + y = 0 \rightarrow m^2 + 1 = 0 \rightarrow m = 0 \pm i$$

$$\rightarrow y = e^{0x}[c_1 \sin(1 \cdot x) + c_2 \cos(1 \cdot x)] = [c_1 \sin x + c_2 \cos x]$$

where c_1, c_2 are arbitrary constants.

Example

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \to m = 3 \pm 4i \to y = e^{3x}[c_1 \sin(4x) + c_2 \cos(4x)]$$

where c_1, c_2 are arbitrary constants.

Example

Let a constant coefficient homogeneous linear differential in the independent variable x have the characteristic equation

$$(m-4-i3)^3(m-4+i3)^3(m-5)=0$$

The general solution is

$$e^{4x}[(c_1+c_2x+c_3x^2)\sin 3x+(c_4+c_5x+c_6x^2)\cos 3x]+c_7e^{5x}$$

where $c_1, c_2, \ldots, c_7$ are arbitrary constants.

Example

Solve the initial value problem

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \quad y(0) = -3, \; y'(0) = -1$$

Its general solution is

$$y = e^{3x}[c_1 \sin(4x) + c_2 \cos(4x)]$$

where c_1, c_2 are arbitrary constants. From this we find:

$$\frac{dy}{dx} = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x]$$

Continued from the previous page

$$y = e^{3x}[c_1 \sin(4x) + c_2 \cos(4x)]$$

$$\frac{dy}{dx} = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos 4x]$$

Apply the initial conditions:

$$-3 = e^{3\cdot 0}[c_1 \sin(4\cdot 0) + c_2 \cos(4\cdot 0)] \to c_2 = -3$$

$$-1 = e^{3\cdot 0}[(3c_1 - 4c_2)\sin(4\cdot 0) + (4c_1 + 3c_2)\cos(4\cdot 0)]$$

$$\to 4c_1 + 3c_2 = -1 \to c_1 = 2$$

The solution is

$$y = e^{3x}[2\sin(4x) - 3\cos(4x)]$$

Consider

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{4x}$$

A solution candidate for this system is $y_p = Ae^{4x}$. Hope that for some value of A, this candidate satisfies the differential equation. Substitute the candidate and its derivatives

$$\rightarrow y_p' = 4Ae^{4x}, \;\; y_p'' = 16Ae^{4x}$$

in the differential equation:

$$16Ae^{4x} - 2(4Ae^{4x}) - 3(Ae^{4x}) = 2e^{4x}$$

Simplification yields: $A = \frac{2}{5} \rightarrow y_p = \frac{2}{5}e^{4x}$.

Now consider

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^{3x}$$

Let this time the particular solution be $y_p = Ae^{3x}$. Substitute this and its derivatives in the differential equation:

$$9Ae^{3x} - 2(3Ae^{3x}) - 3(Ae^{3x}) = 2e^{3x}$$

This results in:

$$0 = 2e^{3x}$$

This equality does not hold. Therefore, this candidate does not work for any A. The reason that $y_p = Ae^{3x}$ does not work is that e^{3x} is also the solution of the homogeneous part. Now try: $y_p = Axe^{3x}$. Substitute this and its derivatives in the differential equation to find that $A = \frac{1}{2}$. Thus $y_p = \frac{1}{2}xe^{3x}$ is the solution.

Definition

UC functions are x^n, where n is a positive integer or zero, e^{ax}, $\sin(bx+c)$, $\cos(bx+c)$ and finite product of these four types.

Example

$$x^3,\ e^{3x},\ \sin(2x),\ e^x\sin(2x+\frac{\pi}{2}),\ e^x x^3\cos(4x)$$

Definition

Given a UC function $f(\mathrm{x})$, its **UC set** is the set of all UC functions consisting of
(1) $f(x)$ itself and
(2) all linearly independent functions whose linear combinations are the successive derivatives of $f(x)$.
For convenience in UC methods procedure, UC sets are standardized. See Table 1.

Example

For the UC function $f(x) = x^5$, the set $\{f, f', f'', \ldots\}$ is $\{x^5, 5x^4, 20x^3, 60x^2, 120x, 120, 0\}$. We use the UC set of x^5 as $S = \{x^5, x^4, x^3, x^2, x, 1\}$. Notice that, constant multiples or linear combinations of the linearly independent functions $x^5, x^4, x^3, x^2, x, 1$ yield all successive derivatives of $f(x)$.

Example

Given $f(x) = sin2x$, we use the UC set $\{\sin 2x, \cos 2x\}$. Note that, derivatives of $f(x)$ are $f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$, $f'''(x) = -8\cos 2x$, $f^{(4)}(x) = 16cos2x, \ldots$ which are multiples of either $\sin 2x$ or $\cos 2x$.

Example

Given $f(x) = e^{ax}$, we use the UC set $S = \{e^{ax}\}$. Note that, derivatives of $f(x)$ are:
$\dot{f}(x) = ae^{ax}, \ddot{f}(x) = a^2e^{ax}, \ldots, f^{(n)}(x) = a^ne^{ax}$. These are all multiples of e^{ax}.

Example

Let $f(x) = x^3$ and $g(x) = \cos 2x$, then
$h(x) = f(x)g(x) = x^3 \cos 2x$. UC set of x^3 is $S_1 = \{x^3, x^2, x, 1\}$,
UC set of $\cos 2x$ is $S_2 = \{\cos 2x, \sin 2x\}$. Then we use UC set of $x^3 \cos 2x$ as
$S = \{x^3 \cos 2x, x^3 \sin 2x, x^2 \cos 2x, x^2 \sin 2x, x \cos 2x, x \sin 2x, \cos 2x, \sin 2x\}$.

For some UC functions, the canonical UC sets are presented in Table 1.

UC function	UC set
x^n	$\{x^n, x^{n-1}, x^{n-2}, \dots, x, 1\}$
e^{ax}	$\{e^{ax}\}$
$\sin(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
$\cos(bx+c)$	$\{\sin(bx+c), \cos(bx+c)\}$
$x^n e^{ax}$	$\{x^n e^{ax}, x^{n-1}e^{ax}, \dots, xe^{ax}, e^{ax}\}$
$x^n \sin(bx+c)$	$\{x^n \sin(bx+c), x^n \cos(bx+c), x^{n-1}\sin(bx+c),$ $x^{n-1}\cos(bx+c), \dots, x\sin(bx+c), x\cos(bx+c)$ $\sin(bx+c), \cos(bx+c)\}$
$x^n \cos(bx+c)$	$\{x^n \sin(bx+c), x^n \cos(bx+c), x^{n-1}\sin(bx+c),$ $x^{n-1}\cos(bx+c), \dots, x\sin(bx+c), x\cos(bx+c)$ $\sin(bx+c), \cos(bx+c)\}$
$e^{ax}\sin(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$
$e^{ax}\cos(bx+c)$	$\{e^{ax}\sin(bx+c), e^{ax}\cos(bx+c)\}$

Table: 1 Some UC functions and the corresponding UC sets

Problem statement

We want to find a particular solution of

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = F(x)$$

where F is a finite linear combination of UC functions $u_1, u_2, \ldots, u_m$:

$$F = k_1 u_1 + k_2 u_2 + \cdots + k_m u_m$$

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=k_1u_1+k_2u_2+\cdots+k_mu_m$$

1. Obtain UC sets $S_1, S_2, \ldots, S_m$ for the UC functions $u_1, u_2, \ldots, u_m$ as in Table 1.
2. If $S_i \subseteq S_j$ for some $i, j \in \{1, 2, \ldots, m\}$, then omit S_i from further consideration.
3. Consider the UC sets remaining after step 2. If any element of S_i is a solution for the homogeneous part, then multiply S_i by the lowest integer power of x so that the resulting set S_i' does not contain solution of homogeneous part anymore. If any set is revised, then omit its original form from further consideration.
4. Multiply every element of the available sets by an undetermined coefficient and add them up. It is a valid particular solution candidate. Substitute the candidate in the differential equation and solve it for the undetermined coefficients.

Example

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 e^x$$

Let us find the general solution of the homogeneous part. Homogeneous part of the d.e. is as follows:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Its characteristic equation is $m^2 - 3m + 2 = 0$. This has the roots 1 and 2, therefore, the general solution is:

$$y_c = c_1 e^x + c_2 e^{2x}$$

Step 1
UC set of $x^2 e^x$ is $S = \{x^2 e^x, xe^x, e^x\}$.
Step 2
Since we have only one UC set, this step is omitted.

Continued from the previous page

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2e^x$$

$$y_c = c_1e^x + c_2e^{2x}$$

UC set of x^2e^x is $S = \{x^2e^x, xe^x, e^x\}$.

Step 3

e^x is a member of y_c, therefore we multiply S by x.

$$S' = \{x^3e^x, x^2e^x, xe^x\}$$

Multiplication by x^2, or x^3 also result in a set that does not contain a solution of homogeneous part. But the algorithm says "Prefer the lowest integer power of x"

Step 4

A particular solution candidate is:

$$y_p = Ax^3e^x + Bx^2e^x + Cxe^x$$

Example

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^x$$

$$y_c = c_1e^x + c_2xe^x$$

Step 1 UC set of x^2e^x is $S = \{x^2e^x, xe^x, e^x\}$.
Step 2 Omitted. Because we have only one UC set, this step is not applicable to this problem.
Step 3 e^x is a member of y_c, however, if we multiply S by x the resulting set will contain xe^x which is also member of y_c. Hence, we multiply the set by x^2.

$$S' = \{x^4e^x, x^3e^x, x^2e^x\}$$

Step 4 A particular solution candidate is:

$$y_p = Ax^4e^x + Bx^3e^x + Cx^2e^x$$

Example

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 2e^x - 10\sin x$$

$$y_c = c_1e^{3x} + c_2e^{-x}$$

Step 1 UC sets: $S_1 = \{e^x\}$, $S_2 = \{\sin x, \cos x\}$
Step 2 Note that neither of these sets is identical with nor included in the other, hence both are retained.
Step 3 None of the functions e^x, $\sin x$, $\cos x$ in either of these sets is a solution of the corresponding homogeneous equation. Hence neither sets needs to be revised.
Step 4 Form the linear combination:

$$y_p = Ae^x + B\sin x + C\cos x$$

Substitute this and its derivatives in the differential equation to obtain $A = -\frac{1}{2}$, $B = 2$, and $C = -1$.

Example

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$$

$$y_c = c_1e^{2x} + c_2e^x$$

Step 1
UC sets: $S_1 = \{x^2, x, 1\}$, $S_2 = \{e^x\}$, $S_3 = \{xe^x, e^x\}$,$S_4 = \{e^{3x}\}$
Step 2 $S_2 \subset S_3 \rightarrow$ Delete the set S_2.
Now we have the sets S_1, S_3 and S_4 remaining.
Step 3 e^x of S_3 is a member of y_c. Multiply S_3 by x:

$$S_3' = \{x^2e^x, xe^x\}$$

Now we have S_1, S_3' and S_4 to consider.

Continued from the previous page

Step 1

UC sets: $S_1 = \{x^2, x, 1\}$, $S_2 = \{e^x\}$, $S_3 = \{xe^x, e^x\}$,$S_4 = \{e^{3x}\}$

Step 2 $S_2 \subset S_3 \to$ Delete the set S_2.

Now we have the sets S_1, S_3 and S_4 remaining.

Step 3 e^x of S_3 is a member of y_c. Multiply S_3 by x:

$$S_3' = \{x^2e^x, xe^x\}$$

Now we have S_1, S_3' and S_4 to consider.

Step 4

Form the linear combination by using the members of S_1, S_4, and S_3':

$$y_p = Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x$$

Substitute this and its derivatives in the differential equation to obtain

$$y_p = x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x$$

Example

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} = 3x^2 + 4\sin x - 2\cos x$$

$$y_c = c_1 + c_2x + c_3\sin x + c_4\cos x$$

Step 1 UC sets: $S_1 = \{x^2, x, 1\}$, $S_2 = \{\sin x, \cos x\}$, $S_3 = \{\sin x, \cos x\}$
Step 2 S_2 and S_3 are identical; delete the set S_3.
Step 3 Multiply S_1 by x^2. The revised set is $S_1' = \{x^4, x^3, x^2\}$.
Multiply S_2 by x. The revised set is $S_2' = \{x\sin x, x\cos x\}$
Form the linear combination by using the members of S_1' and S_2':

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx\sin x + Ex\cos x$$

Step 4 Substitute this and its derivatives in the differential equation to obtain

$$y_p = \frac{1}{4}x^4 - 3x^2 + x\sin x + 2x\cos x$$

Consider

$$y''(x) + P(x)y'(x) + Q(x)y(x) = f(x) \tag{68}$$

We want to find a particular solution in cases where undetermined coefficients method cannot be applied to produce y_p.
Suppose

$$y_c = c_1 y_1 + c_2 y_2$$

is a known general solution to

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0. \tag{69}$$

Then it is possible to find a y_p of the form

$$y_p = Ay_1 + By_2$$

where A and B are some functions of x to be determined (at the present moment they are unknowns).

We need to substitute this form of y_p in (68) and try to find A and B. To do this, we need to find y_p' and y_p''.

$$y_p = Ay_1 + By_2 \to y_p' = Ay_1' + A'y_1 + By_2' + B'y_2$$

To avoid dealing with second derivatives of A and B we will look for A and B satisfying the following condition:

$$A'y_1 + B'y_2 = 0 \tag{70}$$

Now we need to find a solution that satisfies both (68) and (70). We shall see that imposing an additional condition would not cause any additional trouble in finding a solution.

$$\to y_p' = Ay_1' + By_2'$$

Thus

$$y_p'' = Ay_1'' + A'y_1' + By_2'' + B'y_2'$$

We substitute them in (68):

$$\underbrace{Ay_1''} + A'y_1' + \overbrace{By_2''} + B'y_2' + \underbrace{PAy_1'}$$

$$+ \overbrace{PBy_2'} + \underbrace{QAy_1} + \overbrace{QBy_2} = f \tag{71}$$

Recall that each of y_1 and y_2 is a solution to the d.e.'s homogeneous part:

$$y''(x) + P(x)y'(x) + Q(x)y(x) = 0. \tag{69}$$

Thus, the sum of the underbraced terms $A(y_1'' + Py_1' + Qy_1)$ equals zero. The sum of the overbraced terms above $B(y_2'' + Py_2' + Qy_2)$ also equals zero. Thus (71) becomes

$$A'y_1' + B'y_2' = f \tag{72}$$

To find A and B we need to solve (70) and (72):

$$A'y_1 + B'y_2 = 0$$

$$A'y_1' + B'y_2' = f$$

In matrix notation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

In matrix notation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Cramer's rule may be used:

$$A' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 f}{W(y_1, y_2)} \to A = \int \frac{-y_2 f}{W(y_1, y_2)} dx$$

$$B' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f}{W(y_1, y_2)} \to B = \int \frac{y_1 f}{W(y_1, y_2)} dx$$

Since the particular solution has the form

$$y_p = A(x)y_1 + B(x)y_2$$

we have

$$y_p = -y_1 \int \frac{y_2 f}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f}{W(y_1, y_2)} dx$$

Example

Determine the general solution for

$$\frac{d^2y}{dx^2} + y = \tan x$$

$$y_c = c_1 \cos x + c_2 \sin x \to y_p = A(x) \cos x + B(x) \sin x$$

$$A' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \cos x - \sec x \to A = \sin x - \ln|\sec x + \tan x| + c_3$$

$$B' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \sin x \to B = -\cos x + c_4$$

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$$\rightarrow y_p = \cos x(\sin x - \ln|\sec x + \tan x| + c_3) + \sin x(-\cos x + c_4)$$

Particular solution, by definition, is free of arbitrary constants. So take $c_3 = 0$ and $c_4 = 0$:

$$y_p = \cos x(\sin x - \ln|\sec x + \tan x|) + \sin x(-\cos x)$$

Thus the general solution to the differential equation is

$$y = c_1 \sin x + c_2 \cos x + \cos x(\sin x - \ln|\sec x + \tan x|) + \sin x(-\cos x)$$

Example

Consider the differential equation

$$\ddot{y} - 2\dot{y} - 3y = xe^{-x}$$

One may solve it by undetermined coefficients method. We solve it by the variation of parameters method. The homogeneous part has the general solution:

$$y_c(x) = c_1 e^{-x} + c_2 e^{3x}$$

The particular solution will have the form:

$$y_p = A(x)y_1 + B(x)y_2$$

or more explicitly

$$y_p = -y_1 \int \frac{y_2 f}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 f}{W(y_1, y_2)} dx$$

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$$y_c(x) = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{e^{3x}}_{y_2}$$

$$y_p = y_1 \cdot \underbrace{-\int \frac{y_2 f}{W(y_1, y_2)} dx}_{A(x)} + y_2 \cdot \underbrace{\int \frac{y_1 f}{W(y_1, y_2)} dx}_{B(x)}$$

$$A(x) = -\int \frac{y_2 f}{W(y_1, y_2)} dx = -\int \frac{e^{3x} x e^{-x}}{\begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix}} dx$$

$$= -\int \frac{xe^{2x}}{4e^{2x}} dx = -\int \frac{x}{4} dx = -\frac{x^2}{8}$$

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$$y_c(x) = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{e^{3x}}_{y_2}$$

$$y_p = y_1 \cdot \underbrace{-\int \frac{y_2 f}{W(y_1, y_2)} dx}_{A(x)} + y_2 \cdot \underbrace{\int \frac{y_1 f}{W(y_1, y_2)} dx}_{B(x)}$$

$$B(x) = \int \frac{y_1 f}{W(y_1, y_2)} dx = \int \frac{e^{-x} x e^{-x}}{4e^{2x}} dx = \int \frac{xe^{-4x}}{4} dx$$

$$= -\frac{x}{16} e^{-4x} - \frac{1}{64} e^{-4x}$$

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$$y_c(x) = c_1 \underbrace{e^{-x}}_{y_1} + c_2 \underbrace{e^{3x}}_{y_2}$$

$$y_p = y_1 A(x) + y_2 B(x)$$

$$A(x) = -\frac{x^2}{8}$$

$$B(x) = -\frac{x}{16} e^{-4x} - \frac{1}{64} e^{-4x}$$

Thus

$$y_p(x) = -\frac{x^2}{8} e^{-x} + e^{3x}(-\frac{x}{16} e^{-4x} - \frac{1}{64} e^{-4x})$$

$$y_p(x) = -\frac{x^2}{8} e^{-x} + e^{x}(-\frac{x}{16} - \frac{1}{64})$$

General Solution:

$$y(x) = y_c(x) + y_p(x) = c_1 e^{-x} + c_2 e^{3x} - \frac{x^2}{8} e^{-x} + e^{x}(-\frac{x}{16} - \frac{1}{64})$$

First order Case

Consider

$$y' + P(x)y = f(x) \tag{73}$$

Suppose y_1 is a solution to

$$y' + P(x)y = 0 \tag{74}$$

Look for $y_p = A(x)y_1$. Substitute in (73) yields:

$$\underbrace{Ay_1'} + A'y_1 + \underbrace{PAy_1} = f$$

Since y_1 is a solution to (74) the sum of the underbraced terms, i.e., $A(y_1' + Py_1)$ equals zero, so

$$A'y_1 = f \rightarrow A' = \frac{f}{y_1} \rightarrow A = \int \frac{f}{y_1}dx \rightarrow y_p = y_1 \int \frac{f}{y_1}dx$$

Theorem

The transformation $x = e^t$ *reduces the equation*

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x)$$

to a linear differential equation with constant coefficients.

We shall show it for the second order differential equation

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = F(x)$$

Letting $x = e^t$ assuming $x > 0$, we have $t = \ln x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} \to x \frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dt}\right) + \frac{dy}{dt} \frac{d}{dx} \frac{1}{x} = \frac{1}{x} \left(\frac{d^2y}{dt^2} \frac{dt}{dx}\right) - \frac{1}{x^2} \frac{dy}{dt}$$

$$= \frac{1}{x^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \to x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Note that

$$\frac{d}{dx}(u) = \frac{du}{dt} \frac{dt}{dx} \to \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2} \frac{dt}{dx}$$

Substituting in the differential equation

$$a_0\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + a_1\frac{dy}{dt} + a_2y = F(e^t)$$

or

$$a_0\frac{d^2y}{dt^2} + (a_1 - a_0)\frac{dy}{dt} + a_2y = F(e^t)$$

Compare to:

$$a_0x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = F(x)$$

Remark

1. The leading coefficient $a_0x^n = 0$ for $x = 0$, therefore, $x = 0$ is not included in the domain. We take the domain as $x > 0$.
2. If the domain is $x < 0$, then the correct transformation is $x = -e^t$.

Example

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$$

Let $x = e^t$, assume $x > 0$. Noting that $a_0 = 1,\ a_1 = -2,\ a_2 = 2$, we obtain

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

The general solution will be

$$y = c_1e^t + c_2e^{2t} + \frac{1}{2}e^{3t}$$

In terms of the original independent variable x:

$$y = c_1x + c_2x^2 + \frac{1}{2}x^3$$

Consider a second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{75}$$

or equivalently

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{76}$$

where $P_1(x) = \frac{a_1(x)}{a_0(x)}$ and $P_2(x) = \frac{a_2(x)}{a_0(x)}$. Assume that Equation (75) does not have a solution expressible as a finite linear combination of known elementary functions. Assume that it has a solution in the form of infinite series:

$$c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-x_0)^n \tag{77}$$

where $c_0, c_1, \ldots$ are constants. (77) is known as power series in $(x - x_0)$.

Definition

A function f is said to be analytic at x_0 if its Taylor series about x_0

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

exists and converges to $f(x)$ for all x in some interval including x_0.

Definition

The point x_0 is called **an ordinary point** of the differential equation (75) if both of the functions P_1 and P_2 in the equivalent normalized equation (76) are analytic at x_0. If either (or both) of the functions is not analytic at x_0, then x_0 is called **a singular point** of the differential equation (75).

Example

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 + 2)y = 0$$

Here $P_1(x) = x$ and $P_2(x) = x^2 + 2$. Both functions are analytic everywhere. Thus all the points are ordinary points.

Example

$$(x-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \frac{1}{x}y = 0$$

or equivalently,

$$\frac{d^2y}{dx^2} + \frac{x}{(x-1)}\frac{dy}{dx} + \frac{1}{x(x-1)}y = 0$$

Here $P_1(x) = \frac{x}{(x-1)}$ and $P_2(x) = \frac{1}{x(x-1)}$. P_1 is analytic everywhere except at $x = 1$. P_2 is analytic everywhere except at $x = 0$ and $x = 1$. Thus $x = 0$ and $x = 1$ are singular points of the differential equation.

Theorem

Hypothesis:
The point x_0 is an ordinary point of the differential equation (75).
Conclusion:
The differential equation (75) has two nontrivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

and these power series converge in some interval $|x - x_0| < R$ (where $R > 0$) about x_0.

The method of solution

Assume that the solution y is

$$y = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - x_0)^n$$

Then

$$\frac{dy}{dx} = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x - x_0)^{n-1}$$

$$\frac{d^2y}{dx^2} = 2c_2+6c_3(x-x_0)+12c_4(x-x_0)^2+\cdots = \sum_{n=2}^{\infty} n(n-1)c_n(x-x_0)^{n-2}$$

We substitute y and its derivatives in the differential equation. We then simplify the resulting equation

$$K_0 + K_1(x - x_0) + K_2(x - x_0)^2 + \cdots = 0$$

In order that this equation be valid for all x in the interval of convergence $|x - x_0| < R$, we must set

$$K_0 = K_1 = K_2 = \cdots = 0$$

Example

Consider

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$$

We want to find power series solution of this equation about $x_0 = 0$. Solution has the form: $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$
Equivalently, $y = \sum_{n=0}^{\infty} c_n x^n$. This implies:

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} nc_n x^{n-1}, \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$$

Substituting in the differential equation we obtain

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + x^2\sum_{n=0}^{\infty} c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

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$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} + x^2\sum_{n=0}^{\infty} c_n x^n + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} + 2\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{1} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{2} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+2}}_{3} + \underbrace{2\sum_{n=0}^{\infty} c_n x^n}_{4} = 0 \tag{78}$$

Consider the first term and use $n = m + 2$ transformation

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)c_{m+2}x^m = \sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

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Consider the third term and use $n = m - 2$ transformation

$$\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{m=2}^{\infty} c_{m-2} x^m = \sum_{n=2}^{\infty} c_{n-2} x^n$$

Now Equation (78) becomes

$$\underbrace{\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n}_{1} + \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{2} + \underbrace{\sum_{n=2}^{\infty} c_{n-2} x^n}_{3} + 2 \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{4} = 0 \tag{79}$$

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Now Equation (78) becomes

$$\underbrace{\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n}_{1} + \underbrace{\sum_{n=1}^{\infty} nc_n x^n}_{2} + \underbrace{\sum_{n=2}^{\infty} c_{n-2}x^n}_{3} + \underbrace{2\sum_{n=0}^{\infty} c_n x^n}_{4} = 0 \tag{80}$$

Obtain useful appearances of the terms:

$$\text{1st term: } 2c_2 + 6c_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n$$

$$\text{2nd term: } c_1 x + \sum_{n=2}^{\infty} nc_n x^n$$

$$\text{4th term: } 2c_0 + 2c_1 x + 2\sum_{n=2}^{\infty} c_n x^n$$

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Now Equation (80) becomes

$$2c_2 + 6c_3x + \sum_{n=2}^{\infty}(n+2)(n+1)c_{n+2}x^n + c_1x + \sum_{n=2}^{\infty} nc_nx^n$$

$$+\sum_{n=2}^{\infty} c_{n-2}x^n + 2c_0 + 2c_1x + 2\sum_{n=2}^{\infty} c_nx^n = 0$$

$$\rightarrow (2c_0 + 2c_2) + (3c_1 + 6c_3)x$$

$$+\sum_{n=2}^{\infty}[(n+2)(n+1)c_{n+2} + (n+2)c_n + c_{n-2}]x^n = 0$$

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$$(2c_0 + 2c_2) + (3c_1 + 6c_3)x$$

$$+\sum_{n=2}^{\infty}[(n+2)(n+1)c_{n+2} + (n+2)c_n + c_{n-2}]x^n = 0$$

Equating every power of x to zero we have:

$$c_2 = -c_0$$

$$c_3 = -\frac{1}{2}c_1$$

$$c_{n+2} = -\frac{(n+2)c_n + c_{n-2}}{(n+1)(n+2)}, \quad n \geq 2$$

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$$c_{n+2} = -\frac{(n+2)c_n + c_{n-2}}{(n+1)(n+2)}, \quad n \geq 2$$

Hence

$$c_4 = -\frac{4c_2 + c_0}{12} = \frac{1}{4}c_0$$

$$c_5 = -\frac{5c_3 + c_1}{20} = \frac{3}{40}c_1$$

The general solution is:

$$y = c_0 + c_1 x - c_0 x^2 - \frac{1}{2}c_1 x^3 + \frac{1}{4}c_0 x^4 + \frac{3}{40}c_1 x^5 + \dots$$

$$y = c_0(1 - x^2 + \frac{1}{4}x^4 + \cdots) + c_1(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots)$$

Consider a second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \tag{75}$$

and assume that x_0 is a singular point of (75). We are not assured of a power series solution in positive powers of $x - x_0$. However, under certain conditions we may assume the solution of the form

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{81}$$

where r is a certain (real or complex) constant.

Let us classify the singular points. For this, normalize (75):

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \tag{76}$$

where $P_1(x) = \frac{a_1(x)}{a_0(x)}$ and $P_2(x) = \frac{a_2(x)}{a_0(x)}$.

Definition

Consider the d.e. (75) and assume at least one of the functions P_1 and P_2 in the equivalent normalized equation (76) is not analytic at x_0, so that x_0 is a singular point of (75). If the functions defined by the products

$$(x - x_0)P_1(x) \text{ and } (x - x_0)^2 P_2(x)$$

are both analytic at x_0, then x_0 is called **regular singular point** of (75). Otherwise we call it **irregular**.

Example

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

Normalized form:

$$\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} + \frac{x-5}{2x^2}y = 0$$

Here $P_1(x) = -\frac{1}{2x}$ and $P_2(x) = \frac{x-5}{2x^2}$. Clearly $x_0 = 0$ is a singular point of the d.e.
The products $xP_1(x) = -\frac{1}{2}$ and $x^2P_2(x) = \frac{x-5}{2}$ are analytic at $x = 0$, so $x = 0$ is a regular singular point of the d.e.

Example

$$x^2(x-2)^2\frac{d^2y}{dx^2} + 2(x-2)\frac{dy}{dx} + (x+1)y = 0$$

Normalized form:

$$\frac{d^2y}{dx^2} + \frac{2}{x^2(x-2)}\frac{dy}{dx} + \frac{x+1}{x^2(x-2)^2}y = 0$$

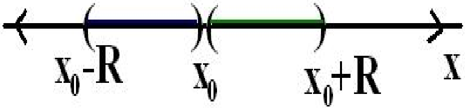
Here $P_1(x) = \frac{2}{x^2(x-2)}$ and $P_2(x) = \frac{x+1}{x^2(x-2)^2}$ have the singular points at $x = 0$ and $x = 2$.
At $x = 0$, $xP_1(x) = \frac{2}{x(x-2)}$ and $x^2P_2(x) = \frac{x+1}{(x-2)^2}$ we see that $xP_1(x)$ is not analytic at $x = 0$, so $x = 0$ is an irregular singular point of the d.e.
At $x = 2$, both $(x-2)P_1(x) = \frac{2}{x^2}$ and $(x-2)^2P_2(x) = \frac{x+1}{x^2}$ are analytic, so $x = 2$ is a regular singular point of the d.e.

Theorem

Given that x_0 is a regular singular point of the d.e. (75), the d.e. (75) has at least one nontrivial solution of the form

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{81}$$

where r is a definite (real or complex) constant which may be determined, and this solution is valid in some deleted interval $0 < |x - x_0| < R$ about x_0.



Example

We saw in a previous example that $x = 0$ is a regular singular point of the d.e.

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

By the theorem, this equation has a nontrivial solution in the form

$$|x|^r \sum_{n=0}^{\infty} c_n x^n$$

valid in some deleted interval $0 < |x| < R$ about $x = 0$.

1. Let x_0 be a regular singular point of the d.e. (75). We seek a solution of the form
$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ valid for $0 < x - x_0 < R$. Note that for $0 < x - x_0 < R$ the term $|x - x_0|^r$ becomes $(x - x_0)^r$. When $-R < x - x_0 < 0$ the following procedure may be repeated by replacing $x - x_0$ by $-(x - x_0)$.
2. Term by term differentiation:

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \rightarrow \frac{dy}{dx} = \sum_{n=0}^{\infty} (n + r) c_n (x - x_0)^{n+r-1}$$

$$\frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} (n + r)(n + r - 1) c_n (x - x_0)^{n+r-2}$$

We substitute $y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}$ in (75).

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3. Substitution results in an expression of the form

$$K_0(x-x_0)^{r+k} + K_1(x-x_0)^{r+k+1} + K_2(x-x_0)^{r+k+2} + \cdots = 0$$

4. For a solution we must set

$$K_0 = K_1 = K_2 = \cdots = 0$$

5. Equating K_0 to zero we obtain a quadratic expression in r, called indicial equation of the d.e. (75). The roots of this quadratic expression is often called the exponents of the d.e. (75). Denote the solutions r_1 and r_2 where $Re(r_1) > Re(r_2)$.
6. Now equate the remaining coefficients to zero. This leads to a set of conditions involving r.
7. We substitute r_1 for r in the conditions of step 6, and choose c_n satisfying the conditions. If c_n are so chosen, the resulting series (81) with $r = r_1$ is a solution.

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8. If $r_1 \neq r_2$, we may repeat the procedure of Step 7 using the root r_2. In this way we may obtain a linearly independent solution of the d.e. (81). When r_1 and r_2 are real and unequal, the second solution may or may not be linearly independent from the one obtained in Step 7. Also, when r_1 and r_2 are real and equal we do not get a new solution. These are exceptional cases and treated later.

Example

Solve

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

in some interval $0 < x < R$. We assume

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

where $c_0 \neq 0$. Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0$$

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Substitute $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in the differential equation:

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r}$$

$$+\sum_{n=0}^{\infty} c_n x^{n+r+1} - 5\sum_{n=0}^{\infty} c_n x^{n+r}$$

Let us simplify this:

$$\sum_{n=0}^{\infty}[2(n+r)(n+r-1)-(n+r)-5]c_n x^{n+r} + \sum_{n=1}^{\infty} c_{n-1}x^{n+r} = 0$$

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$$\sum_{n=0}^{\infty}[2(n+r)(n+r-1)-(n+r)-5]c_n x^{n+r}+\sum_{n=1}^{\infty}c_{n-1}x^{n+r}=0$$

or

$$[2r(r-1)-r-5]c_0x^r$$

$$+\sum_{n=1}^{\infty}\{[2(n+r)(n+r-1)-(n+r)-5]c_n+c_{n-1}\}x^{n+r}=0$$

The lowest power of x has the factor (indicial equation)

$$2r(r-1)-r-5=0.$$

Equating this to zero yields $r_1=\frac{5}{2}$ and $r_2=-1$. These are the exponents of the the d.e. Notice that these numbers are real and unequal.

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The coefficients of the higher power x's are equated to zero. This gives a recurrence formula:

$$[2(n+r)(n+r-1)-(n+r)-5]c_n + c_{n-1} = 0, \quad n \geq 1$$

Letting $r = r_1 = \frac{5}{2}$ yields:

$$[2(n+\frac{5}{2})(n+\frac{3}{2})-(n+\frac{5}{2})-5]c_n - c_{n-1} = 0, \quad n \geq 1$$

This simplifies to:

$$n(2n+7)c_n + c_{n-1} = 0, \quad n \geq 1$$

or

$$c_n = -\frac{c_{n-1}}{n(2n+7)}, \quad n \geq 1$$

$$c_n = -\frac{c_{n-1}}{n(2n+7)}, \quad n \geq 1$$

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$$c_1 = -\frac{c_0}{9}, \quad c_2 = -\frac{c_1}{22} = \frac{c_0}{198}, \quad c_3 = -\frac{c_2}{39} = -\frac{c_0}{7722}, \ldots$$

So the solution corresponding to $r = \frac{5}{2}$ is

$$\begin{aligned} y &= c_0\left(x^{\frac{5}{2}} - \tfrac{1}{9}x^{\frac{7}{2}} + \tfrac{1}{198}x^{\frac{9}{2}} - \tfrac{1}{7722}x^{\frac{11}{2}} + \cdots\right) \\ &= c_0 x^{\frac{5}{2}}\left(1 - \tfrac{1}{9}x + \tfrac{1}{198}x^2 - \tfrac{1}{7722}x^3 + \cdots\right) \end{aligned}$$

Recall that the general form of the solution is:

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} = c_0 x^r + c_1 x^{1+r} + c_2 x^{2+r} + c_3 x^{3+r} + \cdots$$

Continued from the previous page

Now let $r = -1$ and obtain the corresponding recurrence formula

$$[2(n-1)(n-2) - (n-1) - 5]c_n + c_{n-1} = 0, \quad n \geq 1$$

This simplifies to:

$$n(2n-7)c_n + c_{n-1} = 0, \quad n \geq 1$$

or

$$c_n = -\frac{c_{n-1}}{n(2n-7)}, \quad n \geq 1$$

This yields:

$$c_1 = \frac{1}{5}c_0, \quad c_2 = \frac{1}{6}c_1 = \frac{1}{30}c_0, \quad c_3 = \frac{1}{3}c_2 = \frac{1}{90}c_0, \ldots$$

$$c_1 = \frac{1}{5}c_0, \quad c_2 = \frac{1}{6}c_1 = \frac{1}{30}c_0, \quad c_3 = \frac{1}{3}c_2 = \frac{1}{90}c_0, \dots$$

Continued from the previous page

The solution corresponding to $r = -1$ is

$$\begin{aligned} y &= c_0(x^{-1} + \tfrac{1}{5} + \tfrac{1}{30}x + \tfrac{1}{90}x^2 + \cdots) \\ &= c_0 x^{-1}(1 + \tfrac{1}{5}x + \tfrac{1}{30}x^2 + \tfrac{1}{90}x^3 + \cdots) \end{aligned}$$

The two solution corresponding to $r_1 = \frac{5}{2}$ and $r_2 = -1$ are linearly independent. Thus the general solution could be written as

$$y = C_1 x^{\frac{5}{2}}(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \cdots)$$

$$+C_2 x^{-1}(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \cdots)$$

It is claimed in the beginning of this section that when r_1 and r_2 are real and unequal we may or may not find a second linearly independent solution in the form of (81).

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n \tag{81}$$

The following theorem states an existence condition for the linearly independent solutions.

Theorem

Let the point x_0 be a regular singular point of the d.e. (75). Let r_1 and r_2 [where $Re(r_1) \geq Re(r_2)$] be the roots of the indicial equation associated with x_0. We can conclude that:

1. Suppose $r_1 - r_2 \neq N$, where N is a nonnegative integer (that is, $r_1 - r_2 \neq 0, 1, 2, \ldots$). Then the d.e. (75) has two nontrivial linearly independent solutions y_1 and y_2 of the form (81) given respectively by

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where $c_0 \neq 0$, and

$$y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n$$

where $d_0 \neq 0$.

Continued from the previous page

2. Suppose $r_1 - r_2 = N$, where N is a positive integer. Then the d.e. (75) has two nontrivial linearly independent solutions y_1 and y_2 given respectively by

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where $c_0 \neq 0$, and

$$y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n + C y_1(x) \ln |x - x_0|$$

where $d_0 \neq 0$, and C is a constant which may or may not be zero.

The general linear system of two first order differential equations in two unknown functions x and y is of the form

$$\left.\begin{array}{l} a_1(t)\frac{dx}{dt} + a_2(t)\frac{dy}{dt} + a_3(t)x + a_4(t)y = F_1(t) \\ b_1(t)\frac{dx}{dt} + b_2(t)\frac{dy}{dt} + b_3(t)x + b_4(t)y = F_2(t) \end{array}\right\} \tag{82}$$

Solution of the system is an ordered pair (f, g) such that $x = f(t)$ and $y = g(t)$ simultaneously satisfy both equations in some interval $a \leq t \leq b$.

Definitions

$$Dx \triangleq \frac{dx}{dt}$$

$$D^n x \triangleq \frac{d^n x}{dt^n}$$

$$(2D+5)x = 2\frac{dx}{dt} + 5x$$

When $x = t^3 + \sin t$, this becomes

$$(2D+5)(t^3+\sin t) = 2\frac{d(t^3+\sin t)}{dt} + 5(t^3+\sin t)$$

$$= 2(3t^2+\cos t) + 5(t^3+\sin t)$$

A linear combination of x and its first n derivatives

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_{n-1} \frac{dx}{dt} + a_n x$$

can be written in operators notation as

$$\underbrace{\left(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n\right)}_{\text{Linear operator with constant coefficients}} \quad x$$

The operator $a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$ is denoted by L, i.e.,

$$L \stackrel{\triangle}{=} a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$$

Assume that f_1 and f_2 are both n times differentiable functions of t, and c_1 and c_2 are constants. Then

$$L[c_1 f_1 + c_2 f_2] = c_1 L[f_1] + c_2 L[f_2]$$

Example

$L = 3D^2 + 5D - 2$ applies to $3t^2 + 2\sin t$, then

$$L[3t^2 + 2\sin t] = 3L[t^2] + 2L[\sin t]$$

LHS: $(3D^2 + 5D - 2)(3t^2 + 2\sin t)$

$$= (18 - 6\sin t) + (30t + 10\cos t) + (-6t^2 - 4\sin t)$$

$$= -6t^2 + 30t + 18 - 10\sin t + 10\cos t$$

RHS: $3L[t^2] + 2L[\sin t] = 3(3D^2 + 5D - 2)t^2 + 2(3D^2 + 5D - 2)\sin t$

$$= 3(3\frac{d^2}{dt^2}t^2 + 5\frac{d}{dt}t^2 - 2t^2) + 2(3\frac{d^2}{dt^2}\sin t + 5\frac{d}{dt}\sin t - 2\sin t)$$

$$3(6 + 10t - 2t^2) + 2(-3\sin t + 5\cos t - 2\sin t)$$

$$= -6t^2 + 30t + 18 - 10\sin t + 10\cos t$$

Suppose two linear operators L_1 and L_2 apply to f successively. If f has sufficiently many derivatives

$$L_1 L_2 f = L_2 L_1 f = Lf$$

where L is the product of L_1 and L_2 using the rules of the polynomial product.

Example

$$(D+1)(D+3)\sin t = (D+3)(D+1)\sin t = (D^2+4D+3)\sin t$$

Example

Consider

$$\left.\begin{array}{l} 2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t \\ 2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2 \end{array}\right\} \tag{83}$$

In the operator notation

$$\left.\begin{array}{l} (2D-3)x - 2Dy = f_1 \\ (2D+3)x + (2D+8)y = f_2 \end{array}\right\}$$

Continued from the previous page

$$\left.\begin{aligned}(2D-3)x-2Dy&=f_1\\(2D+3)x+(2D+8)y&=f_2\end{aligned}\right\}$$

$$\left.\begin{aligned}L_1x+L_2y&=f_1, \text{ multiply by } L_4\\L_3x+L_4y&=f_2, \text{ multiply by } L_2\end{aligned}\right\}$$

$$\left.\begin{aligned}L_4L_1x+L_4L_2y&=L_4f_1\\L_2L_3x+L_2L_4y&=L_2f_2\end{aligned}\right\} \text{ subtract 2nd from the 1st}$$

$$(L_4L_1-L_2L_3)x=L_4f_1-L_2f_2$$

$$L_5x=g_1$$

Continued from the previous page

$$(L_4L_1 - L_2L_3)x = L_4f_1 - L_2f_2$$

$$[(2D+8)(2D-3) - (-2D)(2D+3)]x = (2D+8)t - (-2D)2$$

$$[8D^2 + 16D - 24]x = 2 + 8t$$

$$[D^2 + 2D - 3]x = t + \frac{1}{4}$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = t + \frac{1}{4} \tag{84}$$

$$\to x = c_1e^t + c_2e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

Continued from the previous page

Reconsider

$$\left.\begin{aligned} L_1x + L_2y &= f_1, \\ L_3x + L_4y &= f_2, \end{aligned}\right\}$$

$$\left.\begin{aligned} L_1x + L_2y &= f_1, \text{ multiply by } L_3 \\ L_3x + L_4y &= f_2, \text{ multiply by } L_1 \end{aligned}\right\}$$

$$\left.\begin{aligned} L_3L_1x + L_3L_2y &= L_3f_1 \\ L_1L_3x + L_1L_4y &= L_1f_2 \end{aligned}\right\} \text{ subtract the 1st from the 2nd}$$

$$(L_4L_1 - L_2L_3)y = L_1f_2 - L_3f_1$$

$$L_5y = g_2$$

$$[D^2 + 2D - 3]y = -\frac{3}{8}t - 1 \tag{85}$$

$$\rightarrow y = k_1e^t + k_2e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

Continued from the previous page

Solutions to (84) and (85) are:

$$\rightarrow x = c_1 e^t + c_2 e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$\rightarrow y = k_1 e^t + k_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

In x, for arbitrarily selected constants (c_1, c_2), (84) is satisfied
In y, for arbitrarily selected constants (k_1, k_2), (85) is satisfied
Recall

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = t + \frac{1}{4} \tag{84}$$

$$[D^2 + 2D - 3]y = -\frac{3}{8}t - 1 \tag{85}$$

However, arbitrarily selected constants (c_1, c_2, k_1, k_2) do not work for simultaneous solution of (83):

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However, arbitrarily selected constants (c_1, c_2, k_1, k_2) do not work for simultaneous solution of (83):

$$\left.\begin{array}{l} 2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t \\ 2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2 \end{array}\right\} \tag{83}$$

Let us substitute the solutions of (84) and (85) into the original equation (83) to resolve the issue of arbitrary constants. Generally substitution in one d.e.of the d. e. set is sufficient for resolving the arbitrary constants.

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Here we randomly chose the first equation of (83) to substitute the solutions in it:

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$

$$[2c_1e^t - 6c_2e^{-3t} - \frac{2}{3}] - [2k_1e^t - 6k_2e^{-3t} + \frac{1}{4}] - [3c_1e^t + 3c_2e^{-3t} - t - \frac{11}{12}] = t$$

or

$$(-c_1 - 2k_1)e^t + (-9c_2 + 6k_2)e^{-3t} = 0$$

Thus we must have

$$k_1 = -\frac{1}{2}c_1, \quad k_2 = \frac{3}{2}c_2$$

Continued from the previous page

Solution

$$\left.\begin{aligned} x &= c_1 e^t + c_2 e^{-3t} - \tfrac{1}{3} t - \tfrac{11}{36} \\ y &= -\tfrac{1}{2} c_1 e^t + \tfrac{3}{2} c_2 e^{-3t} + \tfrac{1}{8} t + \tfrac{5}{12} \end{aligned}\right\} \quad c_1, c_2 \text{ arbitrary constants}$$

or

$$\left.\begin{aligned} x &= -2k_1 e^t + \tfrac{2}{3} k_2 e^{-3t} - \tfrac{1}{3} t - \tfrac{11}{36} \\ y &= k_1 e^t + k_2 e^{-3t} + \tfrac{1}{8} t + \tfrac{5}{12} \end{aligned}\right\} \quad k_1, k_2 \text{ arbitrary constants}$$

Definition

Let f be a real valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t) dt \tag{86}$$

for all values of s for which this integral exists. The function F defined by the integral (86) is called the Laplace transform of the function f. We will denote the Laplace transform F of f by $\mathcal{L}\{f\}$.

Example

$$f(t) = 1,\ t > 0 \leftrightarrow \mathcal{L}\{1\} = \int_0^{\infty} e^{-st} 1 dt = \lim_{R\to\infty} \int_0^{R} e^{-st} 1 dt$$

$$= \lim_{R\to\infty} \left[\frac{-e^{-st}}{s}\right]_0^R = \lim_{R\to\infty} \left[\frac{1}{s} - \frac{e^{-sR}}{s}\right] = \frac{1}{s}$$

for all $s > 0$.

Example

$$f(t) = t,\ \ t > 0 \leftrightarrow \mathcal{L}\{t\} = \int_0^\infty e^{-st} t dt = \frac{1}{s^2}$$

for all $s > 0$.

Example

$$f(t) = e^{at},\ t > 0 \leftrightarrow \mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at} dt = \lim_{R\to\infty} \int_0^{R} e^{(a-s)t} dt$$

$$= \lim_{R\to\infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^R = \lim_{R\to\infty} \left[\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s} \right] = \frac{1}{s-a}$$

for all $s > a$.

Example

$$f(t) = \sin bt, \;\; t > 0 \leftrightarrow \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2 + b^2}, \;\; s > 0$$

Example

$$f(t) = \cos bt, \;\; t > 0 \leftrightarrow \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2}, \;\; s > 0$$

Some functions, such as $f(t) = e^{t^2}$, do not have Laplace transforms. For a function to have a Laplace transform, the following integral must exist:

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt \tag{86}$$

When do such integrals exist? To answer this we need to define **piecewise continuity** and **being of exponential order** first.

Definition

A function f of t is said to be piecewise continuous on a finite interval $a \leq t \leq b$ if this interval can be divided into a finite number of subintervals such that (1) f is continuous in the interior of each of these subintervals, and (2) f approaches finite limits as t approaches either endpoint of each of the subintervals from its interior.

Example

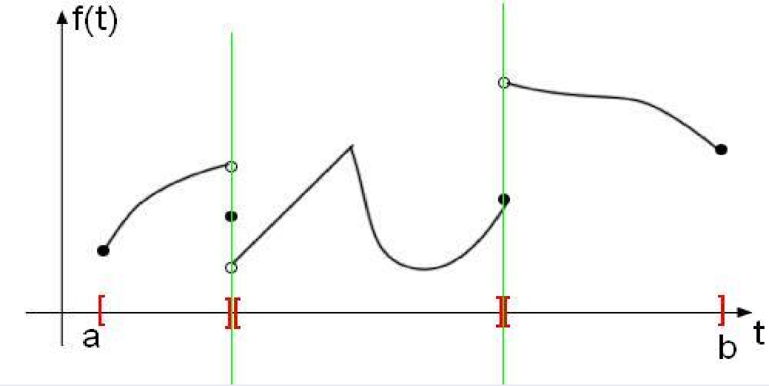

Figure: Piecewise Continuous Function on [a,b]

Example

$f(t) = \frac{1}{t-3}$ is discontinuous at $t = 3$. This function is not piecewise continuous on any interval containing $t = 3$, because neither $\lim_{t\to 3+}$ nor $\lim_{t\to 3-}$ exists.

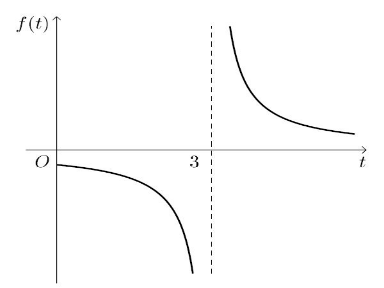

Figure: $f(t) = \frac{1}{t-3}$

Example

$f(t) = \begin{cases} 0 & t < 0 \\ \cos(\frac{1}{t}) & t > 0 \end{cases}$ is discontinuous at $t = 0$. This function is not piecewise continuous on any interval containing $t = 0$, because $\lim_{t \to 0+}$ does not exist.

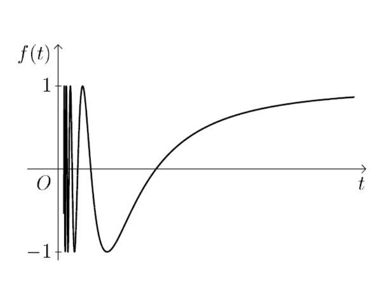

Figure: $f(t) = 0$ for $t < 0$ and $\cos \frac{1}{t}$ for $t > 0$

Definition

A function f of t is said to be of exponential order if there exist a constant α and positive constants t_0 and M such that

$$e^{-\alpha t}|f(t)| < M \tag{87}$$

for all $t > t_0$ at which f is defined. More explicitly, if f is of exponential order corresponding to some definite constant α in (87), then we say that f is of exponential order $e^{\alpha t}$.

Example

Every bounded function is of exponential order, for instance $\sin(bt)$
t^n is of exponential order
e^{at} is of exponential order
e^{t^2} is not of exponential order.

Example

$f(t) = \sin(t)$ is of exponential order. Because we can find, for instance, $\alpha = 2,\ t_0 = 1,\ M = 5$ so that

$$e^{-\alpha t}|f(t)| < M$$

is satisfied for all $t > t_0$.

Example

$f(t) = t^2$ is of exponential order. Because we can find, for instance, $\alpha = 3,\ t_0 = 2,\ M = 5$ so that

$$e^{-\alpha t}|f(t)| < M$$

is satisfied for all $t > t_0$.

Theorem

Let f be a real function that has the following properties:
1) f is piecewise continuous in every finite closed interval $0 \leq t \leq b$, $(b > 0)$
2) f is of exponential order $e^{\alpha t}$. Then the Laplace transform

$$\int_0^{\infty} e^{-st} f(t)dt$$

of f exists for $s > \alpha$.

Proof Since f is of exponential order, there exist α, t_0 and M such that

$$|f(t)| < Me^{\alpha t}, \quad \text{for } t \geq t_0$$

We can write

$$\int_0^{\infty} e^{-st} f(t)dt = \int_0^{t_0} e^{-st} f(t)dt + \int_{t_0}^{\infty} e^{-st} f(t)dt$$

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$$\int_0^\infty e^{-st} f(t)dt = \underbrace{\int_0^{t_0} e^{-st} f(t)dt}_{Part1} + \underbrace{\int_{t_0}^\infty e^{-st} f(t)dt}_{Part2}$$

Part 1 exists because the integral has finite limits and the function $f(t)$ is piecewise continuous.

For the second part, for $t \geq t_0$ note that

$$|f(t)| < Me^{\alpha t} \to |e^{-st} f(t)| < Me^{-(s-\alpha)t}$$

$$\to \int_{t_0}^\infty |e^{-st} f(t)|dt < M \int_{t_0}^\infty e^{-(s-\alpha)t} dt \leq M \int_{\color{red}{0}}^\infty e^{-(s-\alpha)t} dt = \frac{M}{s-\alpha}$$

for $s > \alpha$.

This shows that the integral $\color{blue}{\int_{t_0}^\infty |e^{-st} f(t)|dt}$ exists. This implies that $\color{blue}{\int_{t_0}^\infty e^{-st} f(t)dt}$ exists.

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$$\int_0^{\infty} e^{-st} f(t)dt = \underbrace{\int_0^{t_0} e^{-st} f(t)dt}_{Part1} + \underbrace{\int_{t_0}^{\infty} e^{-st} f(t)dt}_{Part2}$$

Integrals exist for part 1 and part 2. This shows that the integral

$$\int_0^{\infty} e^{-st} f(t)dt$$

exists.

Theorem

Let f_1 and f_2 be functions whose Laplace transforms exist, and c_1, c_2 be constants. Then $\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$.

Theorem

Let f be a real valued function that is continuous for $t \geq 0$ and of exponential order $e^{\alpha t}$. Let f' be piecewise continuous in every finite closed interval $0 \leq t \leq b$. Then $\mathcal{L}\{f'\}$ exists for $s > \alpha$ and $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$.

Example

It is known that $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$.
This implies $\mathcal{L}\{(\sin bt)'\} = s\frac{b}{s^2+b^2} - \sin(b \cdot t)_{t=0} = \frac{bs}{s^2+b^2}$
By direct computation:
$\mathcal{L}\{b\cos bt\} = \frac{bs}{s^2+b^2}$

Example

$\mathcal{L}\{t\} = \frac{1}{s^2} \to \mathcal{L}\{(t)'\} = s\frac{1}{s^2} - t|_{t=0} = \frac{1}{s}$
By direct computation:
$\mathcal{L}\{1\} = \frac{1}{s}$

Theorem

Let f be a real valued function having a continuous $(n-1)$st derivative $f^{(n-1)}$ for $t \geq 0$; and assume that $f,\ f',\ f'', \ldots, f^{(n-1)}$ are all of exponential order $e^{\alpha t}$. Suppose $f^{(n)}$ is piecewise continuous in every finite closed interval $0 \leq t \leq b$. Then $\mathcal{L}\{f^{(n)}\}$ exists for $s > \alpha$ and

$$\mathcal{L}\{f^{(n)}\} = s^n \mathcal{L}\{f\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0)$$

Example

$$\mathcal{L}\ddot{f}(t) = s^2 F(s) - sf(0) - \dot{f}(0)$$

$$\mathcal{L}\dddot{f}(t) = s^3 F(s) - s^2 f(0) - s\dot{f}(0) - \ddot{f}(0)$$

Theorem

For a given f *let* $\mathcal{L}\{f\}$ *exist for* $s > \alpha$. *Then for any constant* a, $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ *for* $s > \alpha + a$, *where* F *denotes* $\mathcal{L}\{f\}$.

Theorem

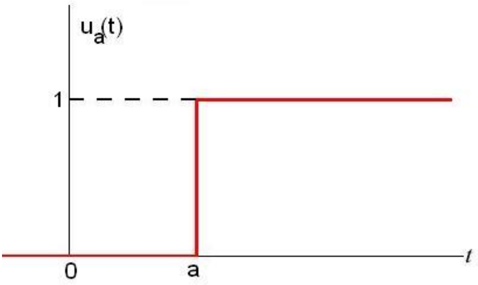
Suppose f has Laplace transform F. Then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Definition

For each real number $a \geq 0$, unit step function u_a is defined for nonnegative t by

$$u_a(t) = \begin{cases} 0; & t < a \\ 1, & t > a \end{cases}$$

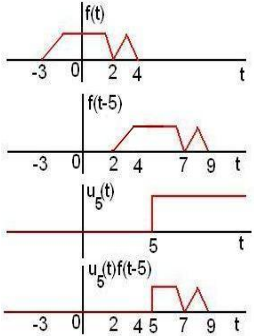


Theorem

Suppose f has Laplace transform F, and consider the translated function defined by

$$u_a(t)f(t-a) = \begin{cases} 0, & 0 < t < a \\ f(t-a), & t > a \end{cases}$$

Then $\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s)$



Example

$$g(t) = \begin{cases} 0, & 0 < t < 5 \\ t-3, & t > 5 \end{cases}$$

Before applying the theorem to this translated function, we must express the functional values $t-3$ for $t > 5$ in terms of $t-5$. That is we express $t-3$ as $(t-5)+2$ and write

$$g(t) = \begin{cases} 0, & 0 < t < 5 \\ (t-5)+2, & t > 5 \end{cases}$$

$$u_5(t)f(t-5) = \begin{cases} 0, & 0 < t < 5 \\ (t-5)+2, & t > 5 \end{cases}$$

where $f(t) = t+2, \; t > 0$. Hence we apply Theorem 29 with $f(t) = t+2$. $F(s) = \mathcal{L}\{t+2\} = \mathcal{L}\{t\} + 2\mathcal{L}\{1\} = \frac{1}{s^2} + \frac{2}{s}$. Therefore,

$$\mathcal{L}\{u_5(t)f(t-5)\} = e^{-5s}F(s) = e^{-5s}(\frac{1}{s^2} + \frac{2}{s})$$

Theorem

Let f and g be two functions that are continuous for $t \geq 0$ and that have the same Laplace transform F. Then $f(t) = g(t)$ for all $t \geq 0$.

Example

Find the inverse Laplace transform $\mathcal{L}^{-1}\{\frac{1}{s^2+6s+13}\}$.

$$\frac{1}{s^2+6s+13} = \frac{1}{(s+3)^2+2^2} = \frac{1}{2} \times \frac{2}{(s+3)^2+2^2} \leftrightarrow \frac{1}{2} e^{-3t} \sin 2t$$

Example

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s}\frac{(s^2+1)}{(s^2+1)} + \frac{Bs+C}{s^2+1}\frac{s}{s}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$$1 = (A+B)s^2 + Cs + A$$

$$\rightarrow A+B=0,\ C=0,\ A=1$$

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{s}{s^2+1}\} = 1 - \cos t$$

Example

$$\mathcal{L}^{-1}\{e^{-4s}(\frac{2}{s^2}+\frac{5}{s})\} = u_4(t)f(t-4)$$

with $f(t) = 2t + 5.$

Definition

Let f and g be two functions that are piecewise continuous on every finite closed interval $0 \leq t \leq b$ and of exponential order. The function denoted by $f * g$ and defined by

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

is called the convolution of the functions f and g.

Let $u = t - \tau$
$\rightarrow f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = -\int_t^0 f(t-u)g(u)du$
$= \int_0^t g(u)f(t-u)du = g(t) * f(t)$

Theorem

Let the functions f and g be piecewise continuous on every finite closed interval $0 \le t \le b$ and of exponential order. Then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$

Example

$$\frac{dy}{dt} - 2y = e^{5t}, \; y(0) = 3$$

Take the Laplace transform of both sides. Let the Laplace transform of the unknown function y be Y which is also unknown meanwhile.

$$sY - y(0) - 2Y = \frac{1}{s-5} \rightarrow (s-2)Y - 3 = \frac{1}{s-5}$$

$$Y = \frac{3s-14}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5}$$

To find A, multiply both sides by $(s-2)$ and evaluate at $s=2$:

$$\frac{3s-14}{(s-2)(s-5)} \times (s-2) = \frac{A}{s-2} \times (s-2) + \frac{B}{s-5} \times (s-2)$$

$$\left[\frac{3s-14}{(s-5)} = A + \frac{B}{s-5} \times (s-2)\right]_{s=2}$$

Continued from the previous page

$$\left[\frac{3s-14}{(s-5)} = A + \frac{B}{s-5} \times (s-2)\right]_{s=2}$$

$$\frac{3 \times 2 - 14}{(2-5)} = A + \frac{B}{2-5} \times (2-2) \rightarrow A = \frac{8}{3}$$

To find B, multiply both sides by $(s-5)$ and evaluate at $s=5$. This gives B as $\frac{1}{3}$. Thus

$$Y = \frac{3s-14}{(s-2)(s-5)} = \frac{\frac{8}{3}}{s-2} + \frac{\frac{1}{3}}{s-5} \leftrightarrow \frac{8}{3}e^{2t} + \frac{1}{3}e^{5t}$$

Example

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 8y = 0,\ \ y(0) = 3,\ \ y'(0) = 6$$

$$\{s^2Y - sy(0) - y'(0)\} - 2\{sY - y(0)\} - 8Y = 0$$

$$[s^2 - 2s - 8]Y - 3s = 0$$

$$Y = \frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

$$A = \left[\frac{3s}{(s-4)(s+2)} \times (s-4)\right]_{s=4} = 2$$

$$B = \left[\frac{3s}{(s-4)(s+2)} \times (s+2)\right]_{s=-2} = 1$$

$$Y = \frac{3s}{(s-4)(s+2)} = \frac{2}{s-4} + \frac{1}{s+2} \leftrightarrow 2e^{4t} + e^{-2t}$$

Example

$$\frac{d^2y}{dt^2} + y = e^{-2t}\sin t,\ \ y(0) = 0,\ \ y'(0) = 0$$

$$\{s^2Y - sy(0) - y'(0)\} + Y = \frac{1}{[(s+2)^2+1]}$$

$$\{s^2Y - s0 - 0\} + Y = \frac{1}{[(s+2)^2+1]}$$

$$Y = \frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+1}$$

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{As+B}{s^2+1}\,\frac{(s+2)^2+1}{(s+2)^2+1} + \frac{Cs+D}{(s+2)^2+1}\,\frac{s^2+1}{s^2+1}$$

Continued from the previous page

$$\frac{1}{(s^2+1)[(s+2)^2+1]} = \frac{(As+B)}{s^2+1}\frac{(s+2)^2+1}{(s+2)^2+1} + \frac{(Cs+D)}{(s+2)^2+1}\frac{s^2+1}{s^2+1}$$

$$1 = (As+B)(s^2+4s+5) + (Cs+D)(s^2+1)$$

$$1 = (A+C)s^3 + (4A+B+D)s^2 + (5A+4B+C)s + (5B+D)$$

$$\left.\begin{array}{lcl} A+C & = & 0 \\ 4A+B+D & = & 0 \\ 5A+4B+C & = & 0 \\ 5B+D & = & 1 \end{array}\right\} A = \frac{-1}{8},\ B = \frac{1}{8},\ C = \frac{1}{8},\ D = \frac{3}{8}$$

Continued from the previous page

$$
\begin{aligned}
Y &= \frac{\frac{-1}{8}s+\frac{1}{8}}{s^2+1}+\frac{\frac{1}{8}s+\frac{3}{8}}{(s+2)^2+1} \\
&= \frac{\frac{-1}{8}s}{s^2+1}+\frac{\frac{1}{8}}{s^2+1}+\frac{\frac{1}{8}s}{(s+2)^2+1} \\
&+\frac{\frac{2}{8}}{(s+2)^2+1}-\frac{\frac{2}{8}}{(s+2)^2+1}+\frac{\frac{3}{8}}{(s+2)^2+1} \\
&= \frac{\frac{-1}{8}s}{s^2+1}+\frac{\frac{1}{8}}{s^2+1}+\frac{\frac{1}{8}(s+2)}{(s+2)^2+1}+\frac{\frac{1}{8}}{(s+2)^2+1} \\
y(t) &= \frac{-1}{8}\cos t+\frac{1}{8}\sin t+\frac{1}{8}e^{-2t}\cos t+\frac{1}{8}e^{-2t}\sin t
\end{aligned}
$$

Example

$$\frac{d^3y}{dt^3}+4\frac{d^2y}{dt^2}+5\frac{dy}{dt}+2y=10\cos t,\ y(0)=0,\ y'(0)=0,\ y''(0)=3$$

$$\{s^3Y-s^2y(0)-sy'(0)-y''(0)\}+4\{s^2Y-sy(0)-y'(0)\}$$
$$+5\{sY-y(0)\}+2Y=10\frac{s}{s^2+1}$$

$$\{s^3Y-s^2 0-s0-3\}+4\{s^2Y-s0-0\}+5\{sY-0\}+2Y=10\frac{s}{s^2+1}$$

$$\{s^3Y-3\}+4\{s^2Y\}+5\{sY\}+2Y=10\frac{s}{s^2+1}$$

$$Y=\frac{3s^2+10s+3}{(s^2+1)(s+1)^2(s+2)}$$

$$=\frac{-1}{s+2}+\frac{2}{s+1}-\frac{2}{(s+1)^2}-\frac{s}{s^2+1}+\frac{2}{s^2+1}$$

$$y(t)=-e^{-2t}+2e^{-t}-2te^{-t}-\cos t+2\sin t$$

Example

$$\begin{aligned} \frac{dx}{dt} - 6x + 3y &= 8e^t \\ \frac{dy}{dt} - 2x - y &= 4e^t \end{aligned}$$

$$x(0) = -1, \quad y(0) = 0$$

In Laplace domain :

$$\begin{aligned} sX + 1 - 6X + 3Y &= \frac{8}{s-1} \\ sY - 2X - Y &= \frac{4}{s-1} \end{aligned}$$

$$\begin{aligned} (s-6)X + 3Y &= \frac{-s+9}{s-1} \\ -2X + (s-1)Y &= \frac{4}{s-1} \end{aligned}$$

Continued from the previous page

$$\begin{aligned} (s-6)X + 3Y &= \tfrac{-s+9}{s-1} \\ -2X + (s-1)Y &= \tfrac{4}{s-1} \end{aligned}$$

In matrix notation:

$$\begin{bmatrix} s-6 & 3 \\ -2 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{-s+9}{s-1} \\ \frac{4}{s-1} \end{bmatrix}$$

$$X = \frac{-s+7}{(s-1)(s-4)},\ Y = \frac{2}{(s-1)(s-4)}$$

$$\leftrightarrow x(t) = -2e^{t} + e^{4t},\ y(t) = \frac{-2}{3}e^{t} + \frac{2}{3}e^{4t}$$

Partial Fraction Decompositions. We will be concerned with the quotient of two polynomials, namely a rational function

$$F(s) = \frac{P(s)}{Q(s)}$$

where the degree of $Q(s)$ is greater than the degree of $P(s)$, and $P(s)$ and $Q(s)$ have no common factors. Then $F(s)$ can be expressed as a finite sum of partial fractions.

(i) For each linear factor of the form $as+b$ of $Q(s)$, there corresponds a partial fraction of the form

$$\frac{A}{as+b} \qquad A \text{ constant.}$$

(ii) For each repeated linear factor of the form $(as+b)^n$, there corresponds a partial fraction of the form

$$\frac{A_1}{as+b}+\frac{A_2}{(as+b)^2}+\cdots+\frac{A_n}{(as+b)^n} \qquad A_1, A_2, \ldots, A_n \text{ constants.}$$

(iii) For every quadratic factor of the form $as^2 + bs + c$, there corresponds a partial fraction of the form

$$\frac{As + B}{as^2 + bs + c} \qquad A, B \text{ constants.}$$

(iv) For every repeated quadratic factor of the form $(as^2+bs+c)^n$, there corresponds a partial fraction of the form

$$\frac{A_1 s + B_1}{as^2 + bs + c} + \frac{A_2 s + B_2}{(as^2 + bs + c)^2} + \cdots + \frac{A_n s + B_n}{(as^2 + bs + c)^n},$$
$$A_1, \ldots, A_n, B_1, \ldots, B_n \text{ constants.}$$

Time Domain Function	Laplace Transform
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$t^n (n = 1, 2, \ldots)$	$\frac{n!}{s^{n+1}}$
$t^n e^{at} (n = 1, 2, \ldots)$	$\frac{n!}{(s-a)^{n+1}}$
$t \sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
$t \cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
$e^{-at} \sin(bt)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at} \cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
$u_a(t)$	$\frac{e^{-as}}{s}$

Table: Laplace Transforms table

Example

Consider

$$t\ddot{y} + \dot{y} + 2y = 0, \;\; y(0) = 1$$

Let us transform the equation into the Laplace domain. We first do it for the first term. The properties

$$\ddot{y} \leftrightarrow s^2 Y - sy(0) - \dot{y}(0)$$

and

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

imply

$$t\ddot{y} \leftrightarrow (-1)^1 \frac{d^1}{ds^1}[s^2 Y - sy(0) - \dot{y}(0)] \rightarrow -s^2 \dot{Y} - 2sY + 1$$

The given d.e. thus have the Laplace domain representation:

$$(-s^2 \dot{Y} - 2sY + 1) + (sY - 1) + 2Y = 0$$

Continued from the previous page

The given d.e. thus have the Laplace domain representation:

$$(-s^2\dot{Y} - 2sY + 1) + (sY - 1) + 2Y = 0$$

$$\dot{Y} + \left(\frac{1}{s} - \frac{2}{s^2}\right) Y = 0$$

This is a 1st order linear differential equation in independent variable s. Its integrating factor is $\mu(s) = e^{\int \left(\frac{1}{s} - \frac{2}{s^2}\right) ds} = se^{\frac{2}{s}}$
Recall that for the 1st order linear d.e.'s we have

$$[e^{\int P(x)dx} y]' = e^{\int P(x)dx} Q(x) \qquad \text{(cf. 30)}$$

Thus

$$\left[Y(s)se^{\frac{2}{s}}\right]' = 0 \to Y(s) = \frac{Ce^{-\frac{2}{s}}}{s}$$

Continued from the previous page

$$Y(s) = \frac{Ce^{-\frac{2}{s}}}{s}$$

Use Maclaurin expansion of the exponential term to obtain:

$$Y(s) = C\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n! s^{n+1}} = C\left(\frac{1}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{4}{3s^4} + \dots\right)$$

Now take the inverse Laplace transform:

$$y(t) = C\sum_{n=0}^{\infty} \frac{(-1)^n 2^n t^n}{(n!)^2} = C\left(1 - 2t + t^2 - \frac{2}{9}t^3 + \dots\right)$$

The condition $y(0) = 1$ gives $C = 1$. Thus the result is

$$y(t) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n t^n}{(n!)^2} = 1 - 2t + t^2 - \frac{2}{9}t^3 + \dots$$

The matrix method

Consider the linear system of the form

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned} \tag{88}$$

Define

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; \quad \frac{dX}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

Now (88) can be written as

$$\frac{dX}{dt} = AX \tag{89}$$

Definition

By a solution of the system (88), that is, of the vector differential equation (89), we mean an $n \times 1$ column vector function

$$\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}$$

whose components $\phi_1, \phi_2, \ldots, \phi_n$ have a continuous derivative on the real interval $a \leq t \leq b$, and which is such that

$$\begin{aligned}
\frac{d\phi_1}{dt} &= a_{11}\phi_1 + a_{12}\phi_2 + \cdots + a_{1n}\phi_n \\
\frac{d\phi_2}{dt} &= a_{21}\phi_1 + a_{22}\phi_2 + \cdots + a_{2n}\phi_n \\
&\vdots \\
\frac{d\phi_n}{dt} &= a_{n1}\phi_1 + a_{n2}\phi_2 + \cdots + a_{nn}\phi_n
\end{aligned}$$

for all t such that $a \leq t \leq b$.

Theorem

Any linear combination of the solutions of the homogeneous linear system (88) is itself a solution of the system (88).

Definition

There exist sets of n linearly independent solutions of the homogeneous linear system (88). Every solution of system (88) can be written as a linear combination of any n linearly independent solutions of (88).

Definition

Let

$$\Phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{bmatrix}; \quad \Phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{n2} \end{bmatrix}; \cdots ; \Phi_n = \begin{bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{nn} \end{bmatrix}$$

be n linearly independent solutions of the homogeneous linear system (88). Let $c_1, c_2, \ldots, c_n$ be n arbitrary constants. Then the solution

$$X = c_1\Phi_1(t) + c_2\Phi_2(t) + \cdots + c_n\Phi_n(t)$$

is called a general solution of the system (88).

Definition

Consider the n vector functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ defined respectively, by

$$\Phi_1 = \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{n1} \end{bmatrix}; \quad \Phi_2 = \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{n2} \end{bmatrix}; \cdots ; \Phi_n = \begin{bmatrix} \phi_{1n} \\ \phi_{2n} \\ \vdots \\ \phi_{nn} \end{bmatrix}$$

The $n \times n$ determinant

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

is called Wronskian of the n vector functions $\Phi_1, \Phi_2, \ldots, \Phi_n$. We will denote its value at t by $W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t))$.

Theorem

n solutions $\Phi_1, \Phi_2, \dots, \Phi_n$ of the homogeneous linear system (88) are linearly independent on an interval $a \leq t \leq b$ if and only if $W(\Phi_1(t), \Phi_2(t), \dots, \Phi_n(t)) \neq 0$ for all $t \in [a, b]$.

Theorem

Let $\Phi_1, \Phi_2, \ldots, \Phi_n$ be n solutions of the homogeneous linear differential equation (88) on an interval $a \leq t \leq b$. Then either $W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) = 0$ for all $t \in [a, b]$ or $W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)) = 0$ for no $t \in [a, b]$.

Define $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then we assume the solutions of (88) have the form $X = ve^{\lambda t}$. Recalling

$$\frac{dX}{dt} = AX \tag{89}$$

substitute $X = ve^{\lambda t}$ into (89) to obtain

$$\begin{aligned} \lambda ve^{\lambda t} &= Ave^{\lambda t} \\ \to Av &= \lambda v \\ \to Av &= \lambda Iv \\ (A - \lambda I)v &= 0 \end{aligned}$$

$$(A-\lambda I)v = 0 \tag{90}$$

which is an algebraic equation in the explicit form

$$\begin{aligned}
(a_{11}-\lambda)v_1 + a_{12}v_2 + \cdots + a_{1n}v_n &= 0 \\
a_{21}v_1 + (a_{22}-\lambda)v_2 + \cdots + a_{2n}v_n &= 0 \\
&\vdots \\
a_{n1}v_1 + a_{n2}v_2 + \cdots + (a_{nn}-\lambda)v_n &= 0
\end{aligned}$$

This can be written in a matrix notation as follows:

$$\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This equation holds only for certain λ and $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ pairs.

This equation set has a non trivial solution if and only if

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0,$$

or in matrix notation $|A - \lambda I| = 0$. This is called characteristic equation for system (89). The λ values satisfying the characteristic equation are called characteristic values of (89). Solutions of (90) corresponding to characteristic values are called characteristic vectors of (89). Recall that

$$\frac{dX}{dt} = AX \qquad (89)$$

$$(A - \lambda I)v = 0 \qquad (90)$$

Case of n distinct characteristic values

Suppose that each of the n characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the $n \times n$ square coefficient matrix A of the vector differential equation is distinct and let $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$ be a set of n respective corresponding characteristic vectors of A. Then the n distinct vector functions $x_1, x_2, \ldots, x_n$ defined respectively by

$$x_1(t) = v^{(1)} e^{\lambda_1 t},\ x_2(t) = v^{(2)} e^{\lambda_2 t},\ \ldots, x_n(t) = v^{(n)} e^{\lambda_n t}$$

are solutions of the vector differential equation (89) on every real interval $[a, b]$. This can be verified by direct substitution.

Now consider the Wronskian of the n solutions $x_1, x_2, \ldots, x_n$:

$$\begin{vmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} & \cdots & v_{1n}e^{\lambda_n t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} & \cdots & v_{2n}e^{\lambda_n t} \\ \vdots & & & \vdots \\ v_{n1}e^{\lambda_1 t} & v_{n2}e^{\lambda_2 t} & \cdots & v_{nn}e^{\lambda_n t} \end{vmatrix}$$

$$= e^{(\lambda_1+\lambda_2+\cdots+\lambda_n)t} \begin{vmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & & & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{vmatrix} \neq 0$$

Since exponential functions never result in zero, and from linear algebra eigenvectors corresponding to distinct eigenvalues are linearly independent which makes the determinant above nonzero. The n solutions $x_1, x_2, \ldots, x_n$ are linearly independent.

Theorem

Consider the homogeneous linear system

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

That is, the vector differential equation

$$\frac{dX}{dt} = AX$$

with obvious definitions. Suppose each of the n characteristic values $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A is distinct; and let $v^{(1)}, v^{(2)}, \ldots, v^{(n)}$ be a set of respective corresponding characteristic vectors of A.

Continued from the previous page

Then on every real interval, the n vector functions defined by

$$v^{(1)}e^{\lambda_1 t}, v^{(2)}e^{\lambda_2 t}, \ldots, v^{(n)}e^{\lambda_n t}$$

form a linearly independent set of solutions of (88), that is (89), and

$$X = c_1 v^{(1)}e^{\lambda_1 t} + c_2 v^{(2)}e^{\lambda_2 t} + \ldots + c_n v^{(n)}e^{\lambda_n t},$$

where $c_1, c_2, \ldots, c_n$ are n arbitrary constants, is a general solution of (88).

Example

Consider

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

Continued from the previous page

$$\rightarrow |A-\lambda I| = \begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = \lambda^3 - 10\lambda^2 + 31\lambda - 30$$

Characteristic values are obtained by equating characteristic expression above to zero:

$$\lambda_1 = 2, \ \lambda_2 = 3, \ \lambda_3 = 5$$

Let us find characteristic vectors for each characteristic value. To find a characteristic vector for $\lambda_1 = 2$, we need to solve

$$(A-\lambda_1 I)v = 0 \text{ or } \begin{bmatrix} 7-\lambda_1 & -1 & 6 \\ -10 & 4-\lambda_1 & -12 \\ -2 & 1 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Continued from the previous page

$$\begin{bmatrix} 7-\lambda_1 & -1 & 6 \\ -10 & 4-\lambda_1 & -12 \\ -2 & 1 & -1-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7-2 & -1 & 6 \\ -10 & 4-2 & -12 \\ -2 & 1 & -1-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & 6 \\ -10 & 2 & -12 \\ -2 & 1 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{\rightarrow}_{GaussianElim.} \quad v^{(1)} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Next find a characteristic vector for $\lambda_2 = 3$

Continued from the previous page

$$\begin{bmatrix} 7-\lambda_2 & -1 & 6 \\ -10 & 4-\lambda_2 & -12 \\ -2 & 1 & -1-\lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7-3 & -1 & 6 \\ -10 & 4-3 & -12 \\ -2 & 1 & -1-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 6 \\ -10 & 1 & -12 \\ -2 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{\rightarrow}_{Gaussian Elim.} \quad v^{(2)} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

Next find a characteristic vector for $\lambda_3 = 5$

Continued from the previous page

$$\begin{bmatrix} 7-\lambda_3 & -1 & 6 \\ -10 & 4-\lambda_3 & -12 \\ -2 & 1 & -1-\lambda_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7-5 & -1 & 6 \\ -10 & 4-5 & -12 \\ -2 & 1 & -1-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ -10 & -1 & -12 \\ -2 & 1 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \underbrace{\rightarrow}_{GaussianElim.} v^{(3)} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix}$$

Continued from the previous page

For $\lambda = \lambda_1 = 2 \to v^{(1)} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

For $\lambda = \lambda_2 = 3 \to v^{(2)} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

For $\lambda = \lambda_3 = 5 \to v^{(3)} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix}$ We have distinct characteristic values and corresponding characteristic vectors. For a general solution, we use them in the solution formula:

$$X = c_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t}$$

Case of repeated characteristic values

We again consider the vector differential equation

$$\frac{dX}{dt} = AX$$

where A is an $n \times n$ real constant matrix. We suppose that A has a real characteristic value λ_1 of multiplicity m, where $1 < m \leq n$, and that all the other characteristic values $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n$ (if there are any) are distinct.

Example

Let 6×6 matrix A have the characteristic equation $(\lambda - 7)^4(\lambda - 2)(\lambda - 5) = 0$. Here $\lambda_1 = 7$ repeated 4 times; $\lambda_5 = 2$ and $\lambda_6 = 5$ are distinct. Linear algebra says we obtain 4 or less linearly independent characteristic vectors for $\lambda_1 = 7$, depending on the matrix A.

We know that the repeated characteristic value λ_1 of multiplicity m has p linearly independent characteristic vectors, where $1 \leq p \leq m$. Now we consider two subcases (1) $p = m$ and (2) $p < m$.

Case 1 If $p = m$ then we will have totally n linearly independent characteristic vectors for the matrix A. In this case the general solution has the form that is the same as the one for all distinct characteristic values. The next example illustrates this:

Example

Consider

$$\frac{dX}{dt} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} X$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

$$\rightarrow |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = \lambda^3 - 5\lambda^2 + 8\lambda - 4$$

Characteristic values are obtained by equating characteristic equation to zero:

$$\underbrace{\lambda_1 = 1}_{\text{distinct}},\ \underbrace{\lambda_2 = 2,\ \lambda_3 = 2}_{\text{repeated}}$$

Continued from the previous page

Evaluate $Av = \lambda v$ at the characteristic values:

At $\lambda = 1$

$$v^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

is a characteristic vector.

At $\lambda = 2$

$$v^{(2)} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad v^{(3)} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are characteristic vectors. $v^{(2)}$ and $v^{(3)}$ are linearly independent.

General solution is

$$X(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Continued from the previous page

Case (2), $p < m$: In this case there are less than m linearly independent characteristic vectors $v^{(1)}$ corresponding to the characteristic value λ_1 of multiplicity m. Hence there are less than m linearly independent solutions of system (88) of the form $v^{(1)}e^{\lambda_1 t}$ corresponding to λ_1. Thus there is not a full set of n linearly independent solutions of (88) of basic exponential form $v^{(k)}e^{\lambda_k t}$.

Clearly we must seek linearly independent solutions of another form.

Continued from the previous page

Let λ be a characteristic value of multiplicity $m = 2$. Suppose $p = 1 < m$, so that there is only one type of characteristic vector v and hence only one type of solution of the basic exponential form $ve^{\lambda t}$ corresponding to λ. We need two linearly independent solutions in order to write the general solution. The second solution is of the form

$$(vt + w)e^{\lambda t}$$

together with $ve^{\lambda t}$ form a linearly independent set of two solutions. Let us substitute this in the differential equation

$$\frac{dX}{dt} = AX$$

Continued from the previous page

$$(vt + w)\lambda e^{\lambda t} + ve^{\lambda t} = A(vt + w)e^{\lambda t}$$

Dividing throughout by $e^{\lambda t}$ and rearranging, this can be written as

$$(\lambda v - Av)t + (\lambda w + v - Aw) = 0$$

This implies

$$(A - \lambda I)v = \underline{0}$$

$$\lambda w + v - Aw = 0$$

We already know the v satisfying the first equation. From the second equation we want to find w:

$$(A - \lambda I)w = v$$

Upon finding w, the general solution will be

$$X(t) = c_1 ve^{\lambda t} + c_2(vt + w)e^{\lambda t}$$

Continued from the previous page

Now let λ be a characteristic value of multiplicity $m = 3$, and suppose $p < m$. Here there are two possibilities: $p = 1$ and $p = 2$. If $p = 1$, there is only one type of characteristic vector v and hence only one type of solution of the form

$$ve^{\lambda t}$$

corresponding to λ. Then a second solution corresponding to λ is of the form

$$(vt + w)e^{\lambda t}$$

Substitute this in the d.e. $\frac{dX}{dt} = AX$

Continued from the previous page

$$(vt + w)\lambda e^{\lambda t} + ve^{\lambda t} = A(vt + w)e^{\lambda t}$$

Dividing throughout by $e^{\lambda t}$ and rearranging, this can be written as

$$(\lambda v - Av)t + (\lambda w + v - Aw) = 0$$

This implies

$$(A - \lambda I)v = \underline{0}$$

$$\lambda w + v - Aw = 0$$

We already know the v satisfying the first equation. From the second equation we want to find w:

$$(A - \lambda I)w = v$$

Upon finding w, an already found part of the general solution will be

$$X(t) = c_1 ve^{\lambda t} + c_2(vt + w)e^{\lambda t}$$

Continued from the previous page

In this case the third solution corresponding to λ is of the form

$$(v\frac{t^2}{2} + wt + z)e^{\lambda t}$$

Upon substituting this in the d.e. $\frac{dX}{dt} = AX$ we observe that z satisfies

$$(A - \lambda I)z = w$$

z obtained from this is used in the third solution. These three solutions obtained are linearly independent. The general solution will be

$$X(t) = c_1 v e^{\lambda t} + c_2(vt + w)e^{\lambda t} + c_3(v\frac{t^2}{2} + wt + z)e^{\lambda t}$$

Continued from the previous page

If $p = 2$, there are two linearly independent characteristic vectors $v^{(1)}$ and $v^{(2)}$ corresponding to λ and hence there are two linearly independent solutions of the form

$$v^{(1)} e^{\lambda t} \text{ and } v^{(2)} e^{\lambda t}$$

Then a third solution corresponding to λ is of the form

$$(vt + w)e^{\lambda t}$$

where v satisfies

$$(A - \lambda I)v = \underline{0} \tag{91}$$

and w satisfies

$$(A - \lambda I)w = v \tag{92}$$

The v in (91) is defined by $k_1 v^{(1)} + k_2 v^{(2)}$. We need to determine k_1 and k_2 which satisfy (92).

Example

Consider

$$\frac{dX}{dt} = \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} X$$

or in vector-matrix notation

$$\frac{dX}{dt} = AX$$

$$\rightarrow |A - \lambda I| = \begin{vmatrix} 4-\lambda & 3 & 1 \\ -4 & -4-\lambda & -2 \\ 8 & 12 & 6-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 + 12\lambda - 8$$

Characteristic values are obtained by equating characteristic equation to zero:

$$\lambda_1 = \lambda_2 = \lambda_3 = 2$$

Continued from the previous page

Evaluate $Av = \lambda v$ at the characteristic value. We obtain two linearly independent characteristic vectors:

$$\rightarrow v^{(1)} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad v^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

For the third solution we solve

$$(A - \lambda I)w = v$$

with

$$v = k_1 v^{(1)} + k_2 v^{(2)} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix}$$

Continued from the previous page

$$(A-2I)w = v \to \begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix}$$

Notice that rows on the lefthand side of the equality are proportional. For consistency we must have $k_2 = -2k_1$. Select

$k_1 = 1,\ k_2 = -2 \to v = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ Solving for w we obtain

$w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus the general solution is $X(t) =$

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) e^{2t}$$

Definition

Sturm-Liouville BVP is boundary value problem which consists of
(a) A second order homogeneous linear d.e. of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0 \tag{93}$$

where p, q, and r are real functions such that p has a continuous derivative, q and r are continuous, and $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \leq x \leq b$; and λ is a parameter independent of x; and
(b) two supplementary conditions

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0 \\ B_1 y(b) + B_2 y'(b) &= 0 \end{aligned} \tag{94}$$

where A_1, A_2, B_1 and B_2 are real constants such that A_1 and A_2 are not both zero, and B_1 and B_2 are not both zero.

Example

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

$$y(0) = 0, \quad y(\pi) = 0$$

is a Sturm Liouville problem. The differential equation may be written as

$$\frac{d}{dx}\left[\underbrace{1}_{p(x)} \cdot \frac{dy}{dx}\right] + \left[\underbrace{0}_{q(x)} + \lambda \cdot \underbrace{1}_{r(x)}\right] y = 0$$

$$y(\underbrace{0}_{a}) = 0$$

$$y(\underbrace{\pi}_{b}) = 0$$

This verifies the claim.

Example

The boundary value problem

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \left[2x^2 + \lambda x^3\right] y = 0$$

$$3y(1) + 4y'(1) = 0$$
$$5y(2) - 3y'(2) = 0$$

is a Sturm-Liouville problem.

Example

Find nontrivial solutions of the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

$$y(0) = 0, \quad y(\pi) = 0$$

Solution

We consider three cases $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

Case 1: $\lambda = 0$ reduces the the problem to

$$\frac{d^2y}{dx^2} = 0$$

The general solution is

$$y = c_1 + c_2 x$$

The first condition $y(0) = 0$ yields $c_1 = 0$. The second condition $y(\pi) = c_1 + c_2\pi = 0$ yields $c_2 = 0$.

Continued from the previous page

Thus, when $\lambda = 0$ the only solution is the trivial solution.
Case 2: For the d.e

$$\frac{d^2y}{dx^2} + \lambda y = 0, \; y(0) = 0, \quad y(\pi) = 0$$

when $\lambda < 0$, the characteristic equation is

$$m^2 + \lambda = 0$$

Its roots $\pm\sqrt{-\lambda}$ are real and unequal. The corresponding general solution is

$$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

where $\alpha = \sqrt{-\lambda}$. Apply the conditions $y(0) = 0$ and $y(\pi) = 0$:

$$c_1 + c_2 = 0, \;\; c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} = 0$$

Continued from the previous page

Solve the equations arising from applying the condition:

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi} &= 0 \end{aligned}$$

The only solution is $c_1 = c_2 = 0$
$\therefore$ When $\lambda < 0$ we have only the trivial solution.

Continued from the previous page

Case 3: $\lambda > 0$ implies that the characteristic equation has the roots $\pm i\sqrt{\lambda}$.This leads to the general solution

$$y = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$$

Now apply the condition $y(0) = 0$:

$$c_1 \sin 0 + c_2 \cos 0 = 0$$

This results in $c_2 = 0$. The other condition $y(\pi) = 0$ yields:

$$c_1 \sin \sqrt{\lambda}\pi + c_2 \cos \sqrt{\lambda}\pi = 0$$

Because $c_2 = 0$, this reduces to

$$c_1 \sin \sqrt{\lambda}\pi = 0$$

If we let $c_1 = 0$, then we get a trivial solution. This is not desired. Therefore we make $\sin \sqrt{\lambda}\pi = 0$

The general solution corresponding to $\lambda > 0$ from the previous slide:

$$y = c_1 \sin\sqrt{\lambda}x + \underbrace{\cancel{c_2 \cos\sqrt{\lambda}x}}_{\to 0}$$

Continued from the previous page

$\sin\sqrt{\lambda}\pi = 0$ is satisfied if $\sqrt{\lambda} = n$, or equivalently, $\lambda = n^2$. In other words, λ must be a member of the infinite sequence

$$1, 4, 9, 16, \ldots$$

$\therefore$ For $\lambda = n^2$ $(n = 1, 2, 3, \ldots)$ we have nontrivial solutions

$$y = c_n \sin nx$$

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0 \qquad \text{(cf. 93)}$$

$$\begin{array}{rcl} A_1 y(a) + A_2 y'(a) & = & 0 \\ B_1 y(b) + B_2 y'(b) & = & 0 \end{array} \qquad \text{(cf. 94)}$$

Definition

Consider the Sturm-Liouville equation (93) and the supplementary conditions (94). The values of the parameter λ in (93) for which there exists nontrivial solutions of the problem are called the **characteristic values** of the problem. The corresponding nontrivial solutions themselves are called the **characteristic functions** of the problem.

Example

Find the characteristic values and characteristic functions of

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\frac{\lambda}{x}y=0$$

$$y'(1)=0,\quad y'(e^{2\pi})=0$$

where we assume that the parameter λ is nonnegative.

Solution: We consider separately the cases $\lambda=0$ and $\lambda>0$.

Case 1: $\lambda=0$ reduces the problem to

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right]=0$$

Integrate twice for the general solution:

$$x\frac{dy}{dx}=C\rightarrow\frac{dy}{dx}=\frac{C}{x}\rightarrow y=C\ln|x|+C_0$$

where C and C_0 are arbitrary constants.

Continued from the previous page

Apply the supplementary conditions to this general solution:

$$y = C \ln|x| + C_0, \;\; y'(1) = 0, \;\;\; y'(e^{2\pi}) = 0$$

$$y'(1) = \frac{C}{1} = 0 \to C = 0, \;\&\; y'(e^{2\pi}) = \frac{C}{e^{2\pi}} = 0 \to C = 0$$

Thus C becomes 0. There is no condition imposed on C_0. Solution becomes

$$y = C_0$$

Thus $\lambda = 0$ is a characteristic value and the corresponding characteristic functions are $y = C_0$, where C_0 is an arbitrary nonzero constant.

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Case 2: $\lambda > 0$:

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] + \frac{\lambda}{x}y = 0$$

$$1 \cdot \frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{\lambda}{x}y = 0$$

$$x \cdot \frac{dy}{dx} + x^2\frac{d^2y}{dx^2} + \lambda y = 0$$

For $x \neq 0$, this is equivalent to the Cauchy Euler equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0$$

Letting $x = e^t$, the solution is found to be

$$y = c_1 \sin\sqrt{\lambda}t + c_2 \cos\sqrt{\lambda}t.$$

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$$y = c_1 \sin \sqrt{\lambda} t + c_2 \cos \sqrt{\lambda} t.$$

Back to the x gives

$$y = c_1 \sin(\sqrt{\lambda} \ln x) + c_2 \cos(\sqrt{\lambda} \ln x).$$

Apply the supplementary conditions $y'(1) = 0, \quad y'(e^{2\pi}) = 0$ to the general solution. Let us apply the first condition first:

$$\frac{dy}{dx} = \frac{c_1\sqrt{\lambda}}{x} \cos(\sqrt{\lambda} \ln x) - \frac{c_2\sqrt{\lambda}}{x} \sin(\sqrt{\lambda} \ln x)$$

$$y'(1) = 0 \to \frac{c_1\sqrt{\lambda}}{1} \cos(\sqrt{\lambda} \ln 1) - \frac{c_2\sqrt{\lambda}}{1} \sin(\sqrt{\lambda} \ln 1) = 0$$

$$\to c_1\sqrt{\lambda} = 0 \to c_1 = 0$$

Continued from the previous page

Now apply the second supplementary conditions $y'(e^{2\pi}) = 0$ to the general solution. This leads to

$$c_2\sqrt{\lambda}e^{-2\pi}\sin(2\pi\sqrt{\lambda}) = 0$$

Nontrivial solutions will require $\lambda = \frac{n^2}{4}$, $(n = 1, 2, 3, \ldots)$ Thus, corresponding to the characteristic values $\lambda = \frac{n^2}{4}$, $(n = 1, 2, 3, \ldots)$, with $x > 0$, the characteristic functions are

$$y = C_n \cos\left(\frac{n \ln x}{2}\right)$$

Theorem

Hypothesis Consider the Sturm Liouville problem consisting of

1. the differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[q(x)+\lambda r(x)]\,y=0$$

where p, q, and r are real functions such that p has continuous derivative, q and r are continuous, $p(x)>0$ and $r(x)>0$ for all x on a real interval $a\leq x\leq b$, and λ is a parameter independent of x; and

2. the conditions

$$\begin{aligned} A_1y(a)+A_2y'(a) &= 0 \\ B_1y(b)+B_2y'(b) &= 0 \end{aligned}$$

where A_1, A_2, B_1, and B_2 are real constants such that A_1 and A_2 are not both zero, and B_1 and B_2 are not both zero.

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Conclusions:

1. There exists an infinite number of characteristic values λ_n of the given problem. These characteristic values can be arranged in a monotonic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

and such that $\lambda_n \to +\infty$ as $n \to +\infty$.

2.Corresponding to each characteristic value λ_n there exists a one parameter family of characteristic functions ϕ_n. Each of these characteristic functions is defined on $a \leq x \leq b$, and any two characteristic functions corresponding to the same characteristic value are nonzero constant multiples of each other.

3. Each characteristic function ϕ_n corresponding to the characteristic value λ_n $(n = 1, 2, 3, \dots)$ has exactly $(n-1)$ zeros in the open interval $a < x < b$.

Example

Consider the Sturm Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

$$y(0) = 0, \quad y(\pi) = 0$$

It has already been solved and was found that it has infinitely many characteristic values, therefore, the 1st conclusion is valid. Validity of the 2nd conclusion may be verified for a characteristic function For instance, for $\lambda = 9$ corresponding solutions are $c \sin 3x$ where c is arbitrary. Looking at some of the solutions $5 \sin 3x, \ 12 \sin 3x, \ -2.2 \sin 3x, \ldots$, one observes that for the same characteristic value, corresponding characteristic functions are multiple of each other.

Continued from the previous page

Conclusion 3 suggests the characteristic function $c_n \sin nx$ corresponding to $\lambda = n^2$ has exactly $n-1$ zeros in the open interval $0 < x < \pi$. We know that $\sin nx = 0$ if and only if $nx = k\pi$, where k is an integer. Thus the zeros of $c_n \sin nx$ are given by

$$x = \frac{k\pi}{n}, \quad (k = 0, \pm 1, \pm 2, \dots) \tag{95}$$

The zeros of (95) which lie in the open interval $0 < x < \pi$ are the ones corresponding to $k = 1, 2, \dots, n-1$. Totally, there are $n-1$ zeros in the interval.

Orthogonality of Characteristic Functions

Definition

Two functions f and g are called **orthogonal** with respect to the **weight function** r on the interval $a \leq x \leq b$ if and only if

$$\int_a^b f(x)g(x)r(x)dx = 0$$

Example

The functions $\sin x$ and $\sin 2x$ are orthogonal with respect to the weight function having the constant value 1 on the interval $0 \leq x \leq \pi$:

$$\int_0^\pi (\sin x)(\sin 2x)(1)dx = \left.\frac{2\sin^3 x}{3}\right|_0^\pi = 0$$

Definition

Let $\{\phi_n\}$, $n = 1, 2, 3, \ldots$, be an infinite set of functions defined on the interval $a \leq x \leq b$. The **set** $\{\phi_n\}$ is called **orthogonal system** with respect to the weight function r on $a \leq x \leq b$ if every two distinct functions of the set are orthogonal with respect to r on $a \leq x \leq b$. That is, the set $\{\phi_n\}$ is orthogonal with respect to r on $a \leq x \leq b$ if

$$\int_a^b \phi_m(x)\phi_n(x)r(x)dx = 0, \quad \text{for } m \neq n$$

Example

Consider the infinite set of functions $\{\sin x, \sin 2x, \sin 3x, \ldots\}$ on the interval $0 \leq x \leq \pi$. Let the weight function be 1. Then this set is orthogonal wrt this weight function:

$$\int_0^{\pi} (\sin mx)(\sin nx)(1)dx = \left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n}\right]_0^{\pi} = 0$$

Theorem

Hypothesis Consider the Sturm Liouville problem consisting of

1. the differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]\, y = 0$$

where p, q, and r are real functions such that p has continuous derivative, q and r are continuous, $p(x) > 0$ and $r(x) > 0$ for all x on a real interval $a \leq x \leq b$, and λ is a parameter independent of x; and

2. the conditions

$$\begin{aligned} A_1 y(a) + A_2 y'(a) &= 0 \\ B_1 y(b) + B_2 y'(b) &= 0 \end{aligned}$$

where A_1, A_2, B_1, and B_2 are real constants such that A_1 and A_2 are not both zero, and B_1 and B_2 are not both zero.

Continued from the previous page

Let λ_m and λ_n be two distinct characteristic values of this problem. Let ϕ_m be a characteristic function for λ_m and ϕ_n be a characteristic function for λ_n.

Conclusion The characteristic functions ϕ_m and ϕ_n are orthogonal with respect to the weight function r on the interval $a \leq x \leq b$.

Example

Consider the Sturm Liouville problem

$\frac{d^2y}{dx^2} + \lambda y = 0, \;\; y(0) = 0, \quad y(\pi) = 0$

where $r = 1$. Corresponding to each characteristic value $\lambda = n^2$ $(n = 1, 2, \ldots)$ we have characteristic functions $c_n \sin nx$ $(n = 1, 2, \ldots)$. Define $\phi_n(x) = \sin nx, \;\; n = 1, 2, \ldots$ The set $\{\phi_n\}, \;\; n = 1, 2, \ldots,$ is an orthogonal system because

$$\int_0^\pi (\sin mx)(\sin nx)(1)dx = 0, \;\; \text{for } m = 1, 2, \ldots, \;\; n = 1, 2, \ldots, \;\; m \neq n$$

Polynomial Division

Theorem

Let the polynomials $p(x)$ and $d(x) \neq 0$ be given. Then there exist unique $q(x)$ and $r(x)$ polynomials such that

$$p(x) = d(x)q(x) + r(x).$$

Furthermore, $\deg d(x) \geq 1$ implies $\deg r(x) < \deg d(x)$; and $\deg d(x) = 0$ implies $r(x) = 0$.

Proof We prove it by induction. We prove that it holds true firstly for $\deg p(x) = 0$. Then we show that if it is true for $\deg p(x) = k$, then it will be true for $\deg p(x) = k + 1$.

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" ... there exist unique $q(x)$ and $r(x)$ polynomials ..."
" ... $\deg d(x) \geq 1$ implies $\deg r(x) < \deg d(x)$; and $\deg d(x) = 0$ implies $r(x) = 0$" [Conclusion of the theorem].

Case of $\deg p(x) = 0$
In this case $p(x) = c$. Let us denote $\deg d(x)$ by m. There are two subcases: $m = 0$ and $m > 0$. Consider the $m = 0$ case first. In this case $d(x) = k$. Then $p(x) = k\frac{c}{k} + 0$. $\therefore$ The theorem holds.
In the $m > 0$ case choose $q(x) = 0$ and $r(x) = c$. This yields $p(x) = d(x) \cdot 0 + c$. $\therefore$ The theorem holds.

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Case of $\deg p(x) = k+1$

Let us assume that the theorem holds for $\deg p(x) = k$, we will show that this implies it holds true for $\deg p(x) = k+1$.

Let $p(x)$ have the form:

$$p(x) = a_{k+1}x^{k+1} + a_k x^k + \cdots + a_1 x + a_0$$

where $a_{k+1} \neq 0$. Now we have two cases: $m = 0$ case and $m > 0$ case.

Subcase of $m = 0$

$m = 0$ implies $d(x) = c$. In this case choose $q(x) = \frac{1}{c}p(x)$ and $r(x) = 0$. This yields

$p(x) = c\frac{1}{c}p(x) + 0$

" ... there exist unique $q(x)$ and $r(x)$ polynomials ..."

" ... $\deg d(x) \geq 1$ implies $\deg r(x) < \deg d(x)$; and $\deg d(x) = 0$ implies $r(x) = 0$" [Conclusion of the theorem].

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Subcase of $m > 0$

Let $d(x) = d_m x^m + \cdots + d_1 x + d_0$ where $d_m \neq 0$. Notice that $\frac{a_{k+1}}{d_m} \neq 0$. Choose $p_1(x) = p(x) - \frac{a_{k+1}}{d_m} x^{k+1-m} d(x)$. This annihilates the x^{k+1} term. The degree of $p_1(x)$ is k or lower. By the hypothesis, there exist q_1 and r_1 satisfying the theorem for $p_1(x)$. More explicitly,

$$p_1(x) = d(x)q_1(x) + r_1(x) = p(x) - \frac{a_{k+1}}{d_m} x^{k+1-m} d(x)$$

$$\to p(x) = \frac{a_{k+1}}{d_m} x^{k+1-m} d(x) + d(x)q_1(x) + r_1(x)$$

$$\to p(x) = d(x) \underbrace{\left[\frac{a_{k+1}}{d_m} x^{k+1-m} + q_1(x) \right]}_{q(x)} + \underbrace{r_1(x)}_{r(x)}$$

For $k+1$st degree polynomial $p(x)$ we have shown existence of q and r satisfying the theorem.

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Now let us show that for a given p, corresponding q and r are unique. Let

$$p(x) = d(x)q_1(x) + r_1(x) = d(x)q_2(x) + r_2(x)$$

$$\rightarrow d(x)\left[q_1(x) - q_2(x)\right] = r_2(x) - r_1(x)$$

Now there two cases: $m = 0$ and $m > 0$.

Case of $m = 0$:

$m = 0$ implies $r_1(x) = r_2(x) = 0 \rightarrow d(x)\left[q_1(x) - q_2(x)\right] = 0$

Since $d(x) \neq 0$, we have $[q_1(x) - q_2(x)] = 0$, this implies $q_1(x) = q_2(x)$.

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Case of $m > 0$:
Suppose $[q_1(x) - q_2(x)] \neq 0$ and calculate the degrees of the polynomials on both sides. Degree of LHS is m or larger. On the RHS we have two polynomials where each one has degree lower than m. Difference of them also has degree lower than m. The degrees of LHS and RHS are equal. This is a contradiction. The contradiction is caused by the supposition $[q_1(x) - q_2(x)] \neq 0$. Correcting this we have $q_1(x) = q_2(x)$. Also, the correction yields $r_2(x) - r_1(x) = 0$, which implies $r_2(x) = r_1(x)$.

Numerical Solutions by ode23.m Consider the first order differential equation

$$\frac{dy}{dx} + \frac{2x+1}{x} y = e^{-2x}, \quad y(1) = 2. \tag{96}$$

We want to find a solution in the interval $[1, 5]$. Form two m-files: Let their names be mymain.m and myequation.m.
mymain.m:

```
[t,x]=ode23('myequation',[1,5],2);
plot(t,x,'o')
```

myequation.m:

```
function ydot=myequation(x,y)
ydot=-((2*x+1)/x)*y +exp(-2*x);
```

The graphics is concatenation of o characters due to the 'o' option in the plot command.
Save mymain.m and myequation.m files in the work folder of MATLAB.
In the workplace of MATLAB, type mymain and press enter key.
The graphics obtained is depicted below:

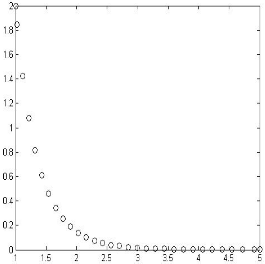

Figure: Numerical solution of the 1st order differential equation

The differential equation (96) is linear and its analytical solution is $y(x) = \frac{x}{2}e^{-2x} + \frac{14.27}{x}e^{-2x}$. For the purpose of comparison with the numerical solution we can plot this over the previous graphics by using the following codes in the workplace of MATLAB (Figure 8):

```
hold on
x=1:0.1:5
y=exp(-2*x).*x/2+14.27*exp(-2*x)./x
plot(x,y)
```

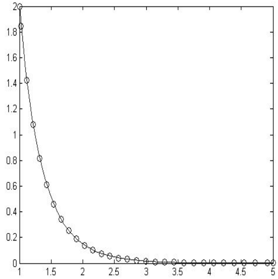

Figure: Analytical solution of the 1st order differential equation

Now let us modify the files mymain.m and myequation.m to solve the following second order differential equation in the interval $[0, 5]$

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = \sin(t), \quad y(0) = 3; \; \dot{y}(0) = 9. \tag{97}$$

This can be written in the normal form as:

$$\left.\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -4x_1 - 5x_2 + \sin(t) \end{array}\right\} \; x_1(0) = 3; \; x_2(0) = 9. \tag{98}$$

Corresponding m-files are shown below:
mymain.m:

```
[t,x]=ode23('myequation',[0,5],[3,9]);
plot(t,x(:,1),'o',t,x(:,2),'o')
```

myequation.m:

```
function xdot=myequation(t,x)
xdot=[x(2); -4*x(1)-5*x(2)+sin(t)];
```

The codes above yields the following graphics:

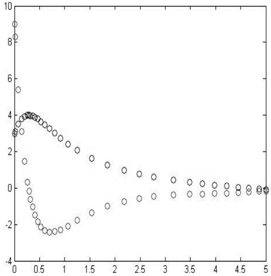

Figure: Numerical solution of the 2nd order differential equation