PROBABILITY SECOND MIDTERM EXAM

Dr. Salih FADIL — November 26, 2008

#1) pdf of a continous random varible X is given as follows.

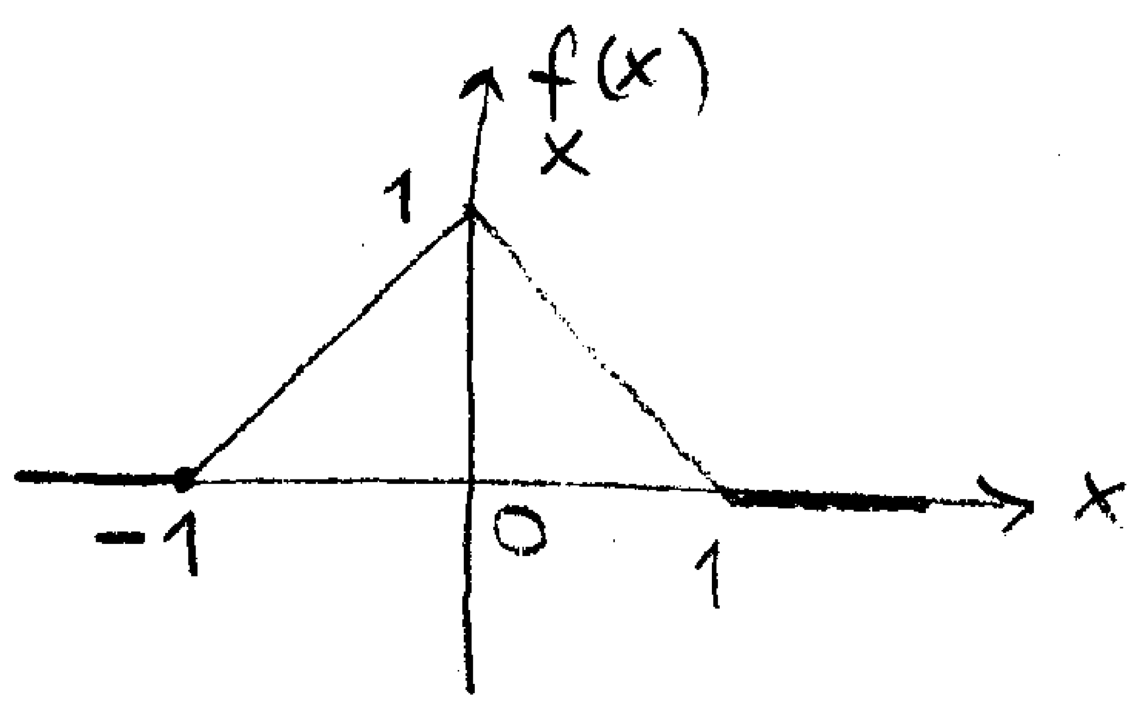
$$f(x) = \frac{1}{2} e^{-|x|} \qquad -\infty < x < +\infty$$

a) Determine $F(x)$, and give it as mathematical expression. Calculate $P\{-0.5 < X < 1.5\}$ by using $F(x)$

b) Determine MGF of random varible X.

c) Determine $V(X)$ by using the MGF of X found in part b.

#2) pdf of a continous random varible X is given as follows.

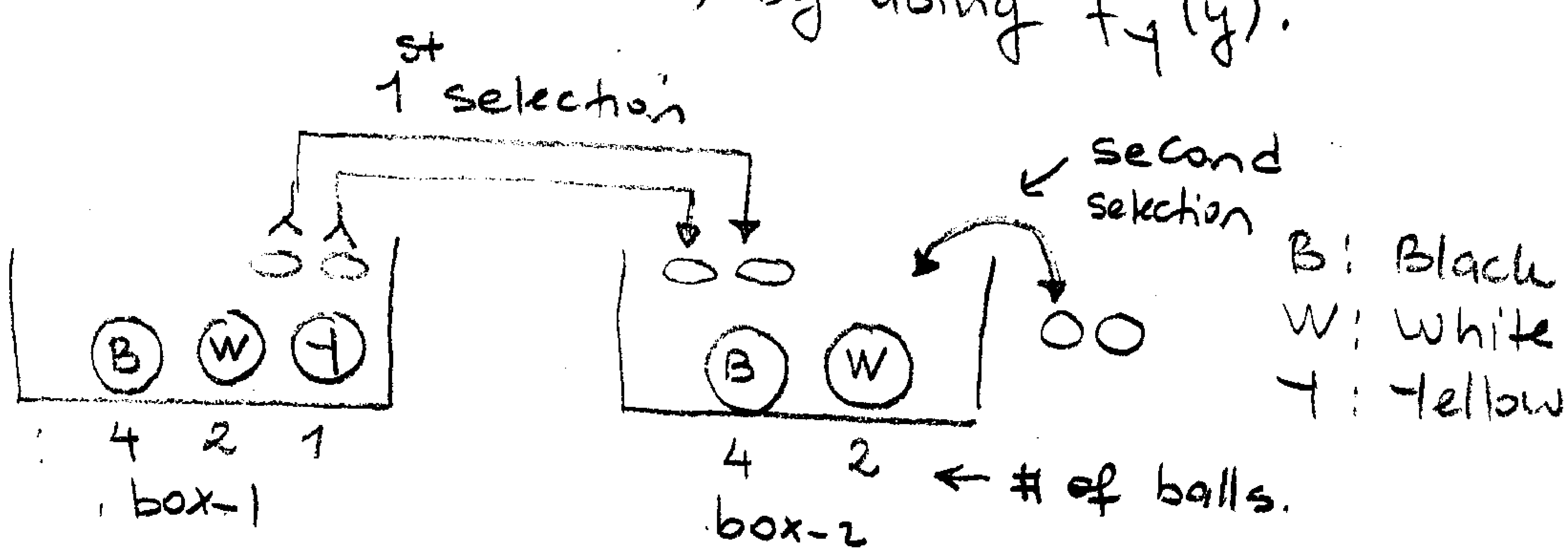


A new random varible is defined as $Y = X^2$. Determine $E(Y)$ value

a) directly (i.e by using $f_X(x)$)

b) by using $f_Y(y)$.

#3)



A person selects 2 balls randomly from box-1 and put them into box-2. Later on, the person selects 2 balls randomly from box-2 with replacement. If X is defined the number of selected black balls in the second selection, find $p(x)$.

PROBABILITY SECOND MIDTERM EXAM SOLUTION MANUAL

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#1) a) $F(x) = \int_{-\infty}^{x} \frac{1}{2} e^{x} dx = \frac{1}{2} e^{x} \Big|_{-\infty}^{x} = \frac{1}{2}(e^{x} - 0)$ for $x \le 0$

$$F(x) = \int_{-\infty}^{0} \frac{1}{2} e^{x} dx + \int_{0}^{x} \frac{1}{2} e^{-x} dx = \frac{1}{2} e^{x} \Big|_{-\infty}^{0} - \frac{1}{2} e^{-x} \Big|_{0}^{x}$$

$$F(x) = \frac{1}{2}(1-0) - \frac{1}{2}(e^{-x} - 1) = 1 - \frac{1}{2} e^{-x} \qquad x \geq 0$$

$$F(x) = \begin{cases} \frac{1}{2} e^{x} & x \le 0 \\ 1 - \frac{1}{2} e^{-x} & x \geq 0 \end{cases}$$

$$P(-0{,}5 < X < 1{,}5) = F(1{,}5) - F(-0{,}5) = \left(1 - \frac{1}{2} e^{-1{,}5}\right) - \frac{1}{2} e^{-0.5}$$

$$= 1 - \frac{1}{2}\left(e^{-1{,}5} + e^{-0{,}5}\right) = 0{,}585169$$

b) $M_X(t) = E(e^{X t}) = \int_{-\infty}^{0} \frac{1}{2} e^{xt} e^{x} dx + \int_{0}^{\infty} \frac{1}{2} e^{+xt} e^{-x} dx$

$$= \frac{1}{2} \int_{-\infty}^{0} e^{x(t+1)} dx + \frac{1}{2} \int_{0}^{\infty} e^{x(t-1)} dx = \frac{1}{2} \left\{ \frac{1}{t+1} e^{x(t+1)} \Big|_{-\infty}^{0} + \frac{1}{(t-1)} e^{x(t-1)} \Big|_{0}^{\infty} \right.$$

$$= \frac{1}{2}\left[\left\{ \underbrace{\frac{1}{(t+1)} (1-0)}_{t+1>0,\ t>-1} \right\} + \left\{ \frac{1}{(t-1)} \underbrace{(0-1)}_{t-1<0,\ t<1} \right\} \right] = \frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1} \right)$$

$$= \frac{\cancel{t} - 1 - \cancel{t} - 1}{t^2 - 1} \cdot \frac{1}{\cancel{2}}$$

$$\boxed{M_X(t) = \frac{1}{1-t^2}, \qquad -1 < t < 1}$$

c) $V(X) = E(X^2) - (E(X))^2$

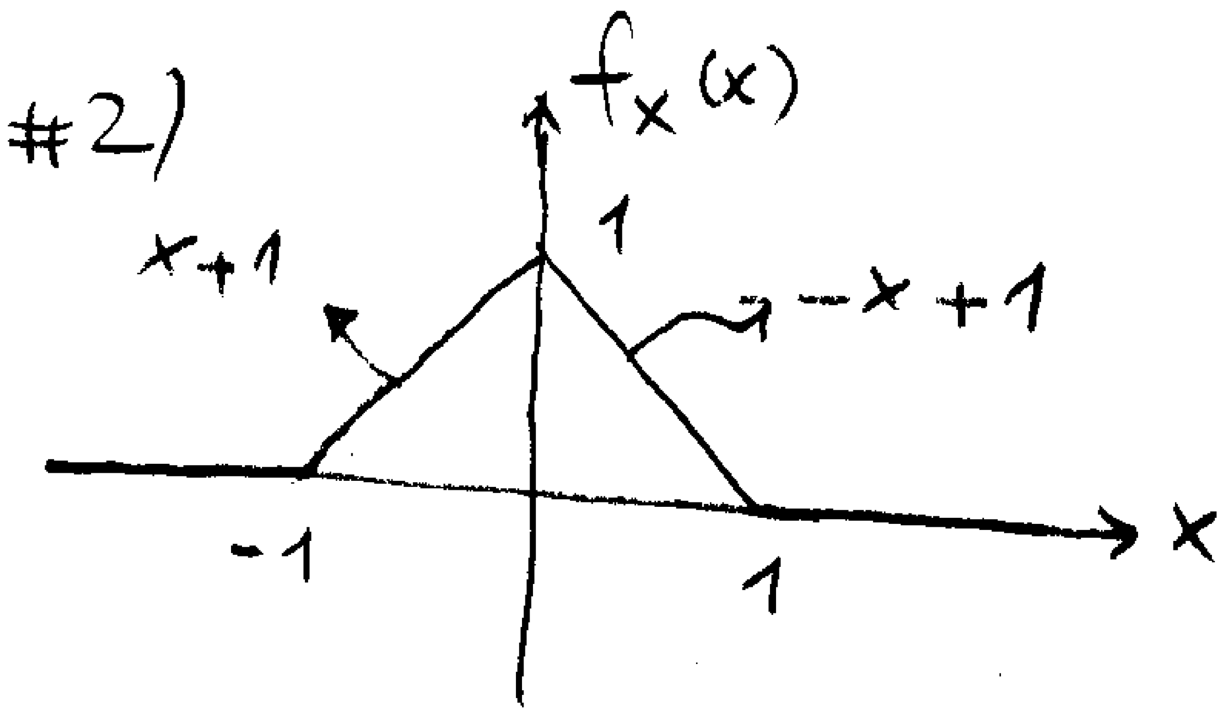
$E(X) = \mu = M_X'(0) = (-1)(1-t^2)^{-2}(-2t)\Big|_{t=0} = 0$ as expected

$E(X^2) = \mu_2' = M_X''(0) = \frac{d}{dt}\left\{(2t)(1-t^2)^{-2}\right\} = \left\{ 2(1-t^2)^{-2} - 2(2t)(1-t^2)^{-3}(-2t\right.$

$= \left\{ 2(1-t^2)^{-2} + 8t^2(1-t^2)^{-3} \right\}_{t=0} = 2$

$\boxed{V(X) = E(X^2) = 2}$

#2)



$$f_X(x) = \begin{cases} x+1 & -1 < x \leq 0 \\ -x+1 & 0 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

a) $Y = X^2$

$E(Y) = \int_{-1}^{0} x^2 (x+1)\,dx + \int_{0}^{1} x^2(-x+1)\,dx$

$= \int_{-1}^{0} (x^3 + x^2)\,dx + \int_{0}^{1} (-x^3 + x^2)\,dx = \left(\frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_{-1}^{0} + \left(-\frac{x^4}{4} + \frac{x^3}{3}\right.$

$= \left[0 - \left(\frac{1}{4} - \frac{1}{3}\right)\right] + \left[\left(-\frac{1}{4} + \frac{1}{3}\right) - 0\right] = \left(-\frac{1}{4} + \frac{1}{3}\right)2$

$= \frac{-3+4}{12}\,2 = \frac{1}{6}$

b) $y = x^2$ $\qquad x_1 = \sqrt{y}$ $\qquad u(x)$

$\qquad\qquad x_2 = -\sqrt{y}$ $\qquad u'(x) = 2x$

$f_Y(y) = \frac{f_X(x_1)}{|u'(x_1)|} + \frac{f_X(x_2)}{|u'(x_2)|} = \frac{-\sqrt{y}+1}{2\sqrt{y}} + \frac{-\sqrt{y}+1}{2\sqrt{y}}$

$$f_Y(y) = \frac{\cancel{2}(1-\sqrt{y})}{\cancel{2}\sqrt{y}} = \left(\frac{1}{\sqrt{y}} - 1\right)$$

$$\begin{cases} -1 < -\sqrt{y} \leq 0 \\ 0 \leq y < 1 \end{cases}$$

$$0 \leq \sqrt{y} < 1$$

$$f_Y(y) = \begin{cases} \left(\frac{1}{\sqrt{y}} - 1\right) & 0 \leq y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(Y) = \int_0^1 y\left(\frac{1}{\sqrt{y}} - 1\right) dy = \int_0^1 (\sqrt{y} - y)\, dy = \frac{y^{3/2}}{3/2}\Big|_0^1 - \frac{y^2}{2}\Big|_0^1$$

$$= \underset{(2)}{\frac{2}{3}} - \underset{(3)}{\frac{1}{2}} = \frac{4-3}{6} = \frac{1}{6}$$ the same result is obtained.

#3) X_1 = # of black balls selected in the first selection (hypergeomet

$$P(X_1=0) = \frac{\binom{4}{0}\binom{3}{2}}{\binom{7}{2}} = \frac{3}{21} \qquad P(X_1=1) = \frac{\binom{4}{1}\binom{3}{1}}{\binom{7}{2}} = \frac{12}{21}$$

$$P(X_1=2) = \frac{\binom{4}{2}\binom{3}{0}}{\binom{7}{2}} = \frac{6}{21}$$

X = # of black balls selected in the second selection (Binomial)

By using total probability rule

$$P(X=0) = \underbrace{P(X=0/X_1=0)}_{p=\frac{4}{8}} P(X_1=0) + \underbrace{P(X=0/X_1=1)}_{p=\frac{5}{8}} P(X_1=1) + \underbrace{P(X=0/X_1=2)}_{p=\frac{6}{8}} P(X_1=2)$$

$$= \frac{3}{21}\left\{\binom{2}{0}\left(\frac{4}{8}\right)^0\left(\frac{4}{8}\right)^2\right\} + \frac{12}{21}\left\{\binom{2}{0}\left(\frac{5}{8}\right)^0\left(\frac{3}{8}\right)^2\right\} + \frac{6}{21}\left\{\binom{2}{0}\left(\frac{6}{8}\right)^0\left(\frac{2}{8}\right)^2\right\}$$

$$= \frac{3}{21}\left(\frac{4}{8}\right)^2 + \frac{12}{21}\left(\frac{3}{8}\right)^2 + \frac{6}{21}\left(\frac{2}{8}\right)^2$$

$$= 0.1339285$$

$$P(X=1) = P(X=1/X_1=0)\,P(X_1=0) + P(X=1/X_1=1)\,P(X_1=1)$$
$$+ P(X=1/X_1=2)\,P(X_1=2)$$
$$= \left(\frac{3}{21}\right)\left\{ \overset{2}{\binom{2}{1}} \left(\frac{4}{8}\right)^1 \left(\frac{4}{8}\right)^1 \right\} + \left(\frac{12}{21}\right) \left\{ \binom{2}{1} \left(\frac{5}{8}\right)^1 \left(\frac{3}{8}\right)^1 \right\}$$
$$+ \frac{6}{21} \left\{ \binom{2}{1} \left(\frac{6}{8}\right)^1 \left(\frac{2}{8}\right)^1 \right\}$$

$$= 2 \left\{ \left(\frac{3}{21}\right) \left(\frac{4}{8}\right)^2 + \left(\frac{12}{21}\right) \frac{15}{64} + \frac{6}{21}\ \frac{12}{64} \right\} = 0,4464285$$

$$P(X=2) = \left(\frac{3}{21}\right)\left\{ \overset{1}{\binom{2}{2}} \left(\frac{4}{8}\right)^2 \right\} + \left(\frac{12}{21}\right) \left\{ \binom{2}{2} \left(\frac{5}{8}\right)^2 \right\} + \left(\frac{6}{21}\right) \left\{ \binom{2}{2} \left(\frac{6}{8}\right)^2 \right\}$$

$$= 0,4196428$$

x	0	1	2
$p(x)$	0,1339285	0,4464285	0,4196428

$$\sum_{x=0}^{2} p(x) = 0,999999857$$ Correct?