PROBABILITY FINAL EXAM

Dr. Salih FADIL **January 15, 2009**

#1) Event occurs according to Poisson process with rate $\lambda = 10$ per hour.

a) What is the probability that 8 events occur in 45 minutes?

b) What is the probability that the time between the 2^{nd} and 3^{rd} events bigger than six minutes?

c) What is the probability that the 6^{th} event occurs before the 50^{th} minute?

#2) The lifetimes interactive computer chips produced by a certain semiconductor manufacturer are normally distributed with parameters $\mu = 1.4 \times 10^6$ hours and $\sigma = 3 \times 10^5$ hours. What is the approximate probability that a batch of 100 chips will contain at least 20 whose lifetimes are less than 1.8×10^6 hours?

#3) Probability density function of a continuous random variable X is given as follows.

$$f(x) = \begin{cases} \dfrac{(0.02)^2}{\Gamma(2)} x\, e^{-0.02x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

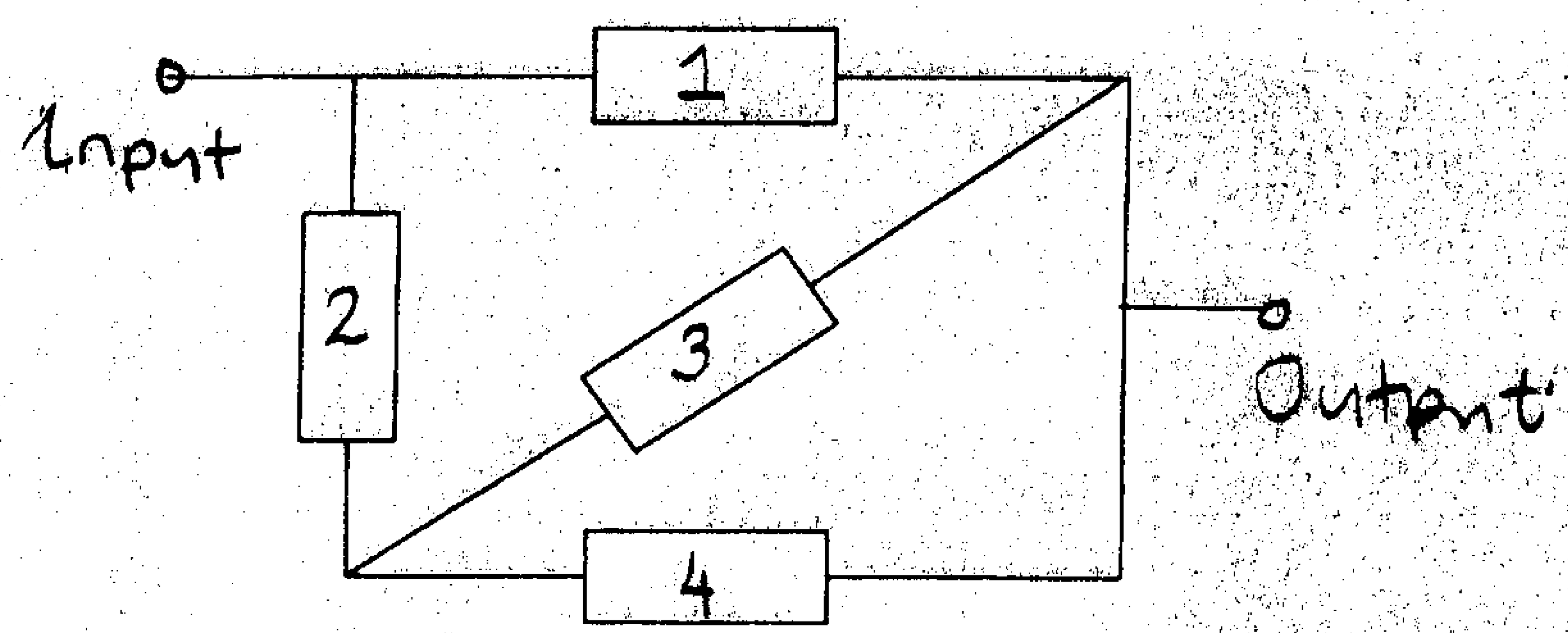
a) Calculate a bound for $P(|X - 100| < 80)$.

b) Calculate the exact probability.

c) Are the exact result and the bound conforming?

#4) Consider the system given in the figure. The system works if there is at least one path consisting of working components between its input and output. Assume that all components in the system functions independently and working probability of each device is $p = 0.8$. Calculate the probability that the system works.

Hint: Try to find the system's working probability conditioning on device 3.



GOOD LUCK.....☺

PROBABILITY FINAL EXAM SOLUTION MANUAL

Dr. Salih FADIL

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#1) $P(X(t)=r) = \frac{(\lambda t)^r}{r!} e^{-\lambda t}$ $\quad \lambda = 10\ hr^{-1} = \frac{1}{6}\ min^{-1}$

a) $P(X(45)=8) = \frac{(45/6)^8}{8!} e^{-45/6} = 0{,}13732$

b) Since interarrival times are exponentially distributed,

$f_T = (\frac{1}{6}) e^{-t/6}$ $\quad t \geq 0$ and t is in minute.

$$P(T>6) = 1 - P(T \leq 6) = 1 - \int_0^6 (\tfrac{1}{6}) e^{-t/6} dt = 1 + e^{-t/6} \Big|_0^6$$

$$= 1 + (e^{-1} - 1) = \frac{1}{e} = 0{,}36788$$

c) $P(W_6 < 50) = P(X(50) \geq 6 = 1 - P(X(50) < 6)$

$$= 1 - \{ P(X(50) = 5 + P(X(50) = 4 + \dots + P(X(50) = 0 \}$$

$$= 1 - e^{-50/6} \{ 1 + \frac{(50/6)^1}{1!} + \frac{(50/6)^2}{2!} + \frac{(50/6)^3}{3!} + \frac{(50/6)^4}{4!} + \frac{(50/6)^5}{5!} \}$$

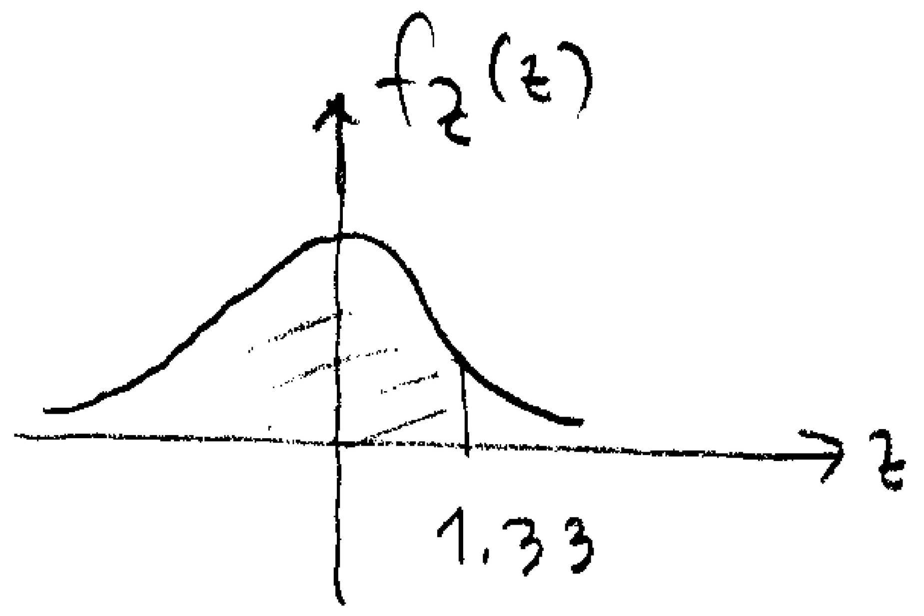
$= 0{,}83742$

#2) T = life time of the chips in hour

$T \sim N(\mu = 1{,}4 \times 10^6,\ \sigma = 3 \times 10^5)$

$$P(T < 1{,}8\ 10^6) = P(\underbrace{\frac{T-\mu}{\sigma}}_{Z} < \frac{1{,}8\ 10^6 - 1{,}4\ 10^6}{3 \times 10^5})$$

$$P\left(Z < \frac{0.4\ 10^6}{3\ 10^5}\right) = P\left(Z < \frac{4}{3}\right) = P(Z < 1.333)$$



$$= 0.5 + P(0 < Z < 1.33) = 0.5 + 0.4082$$
$$= 0.9082$$

N = # of chips whose life is less than $1.8\ 10^6$ h, then N is binomially distributed with $p = 0.9082$, $n = 100$

$$P(N \geq 20) = 1 - P(N \leq 19) \qquad 1-p = 0.0918$$

$$= 1 - \{ p(0) + p(1) + p(2) + \dots + 19 \}$$

$$= 1 - \left\{ \binom{100}{0} p^0 (1-p)^{100} + \binom{100}{1} p^1 (1-p)^{99} + \binom{100}{2} p^2 (1-p)^{98} \right.$$

$$+ \binom{100}{3} p^3 (1-p)^{97} + \binom{100}{4} p^4 (1-p)^{96} + \underbrace{\binom{100}{5} p^5 (1-p)^{95}}_{1.373\ 10^{-91}} + \binom{100}{6} p^6 (1-p)^{94}$$

$$\left. + \dots + \binom{100}{19} p^{19} (1-p)^{81} \right\} \cong 1.$$

$2.076\ 10^{-20}$

↑ the maximum one

#3)
$$f(x) = \begin{cases} \frac{(0.02)^2}{\Gamma(2)} x\, e^{-(0.02)x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

X is gamma distributed $\alpha = 2$, $\beta = 0.02$, $\Gamma(2) = 1! = 1$.

$$E(X) = \frac{\alpha}{\beta} = \frac{2}{0.02} = \frac{200}{2} = 100$$

$$\sigma = \frac{\sqrt{\alpha}}{\beta} = \frac{\sqrt{2}}{0.02} = 70.7106$$

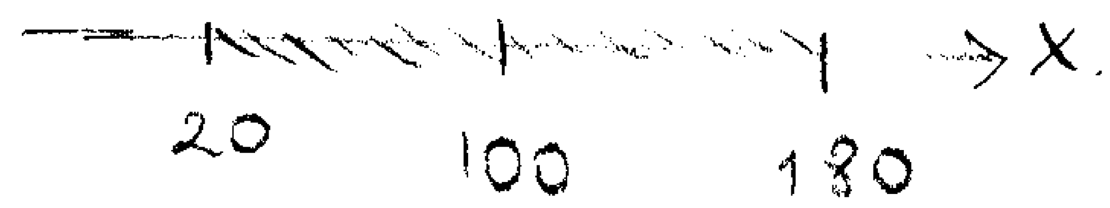
a) $P\{|X-100|<80\}$ $\qquad P\{|X-\mu|<\sigma t\} > 1-\frac{1}{t^2} \quad t>1$

$\mu = 100$

$80 = 70.7106 \cdot t \qquad t = \frac{80}{70.7106} = 1.13137.$

$$P\{|X-100|<80\} > 1-\frac{1}{(1.13137)^2}$$

$$> 0.21875$$



b) $P\{|X-100|<80\} = P\{20<X<180\} = \int_{20}^{180} (0.02)^2 x \, e^{-0.02x} dx$

$$= (0.02)\left\{ -x\, e^{-0.02x}\Big|_{20}^{180} + \underbrace{\int_{20}^{180} e^{-0.02x} dx}_{-\frac{1}{0.02} e^{-0.02}} \right\}$$

$u = x \rightarrow du = dx$

$dv = 0.02\, e^{-0.02x} \rightarrow v = -e^{-0.02x}$

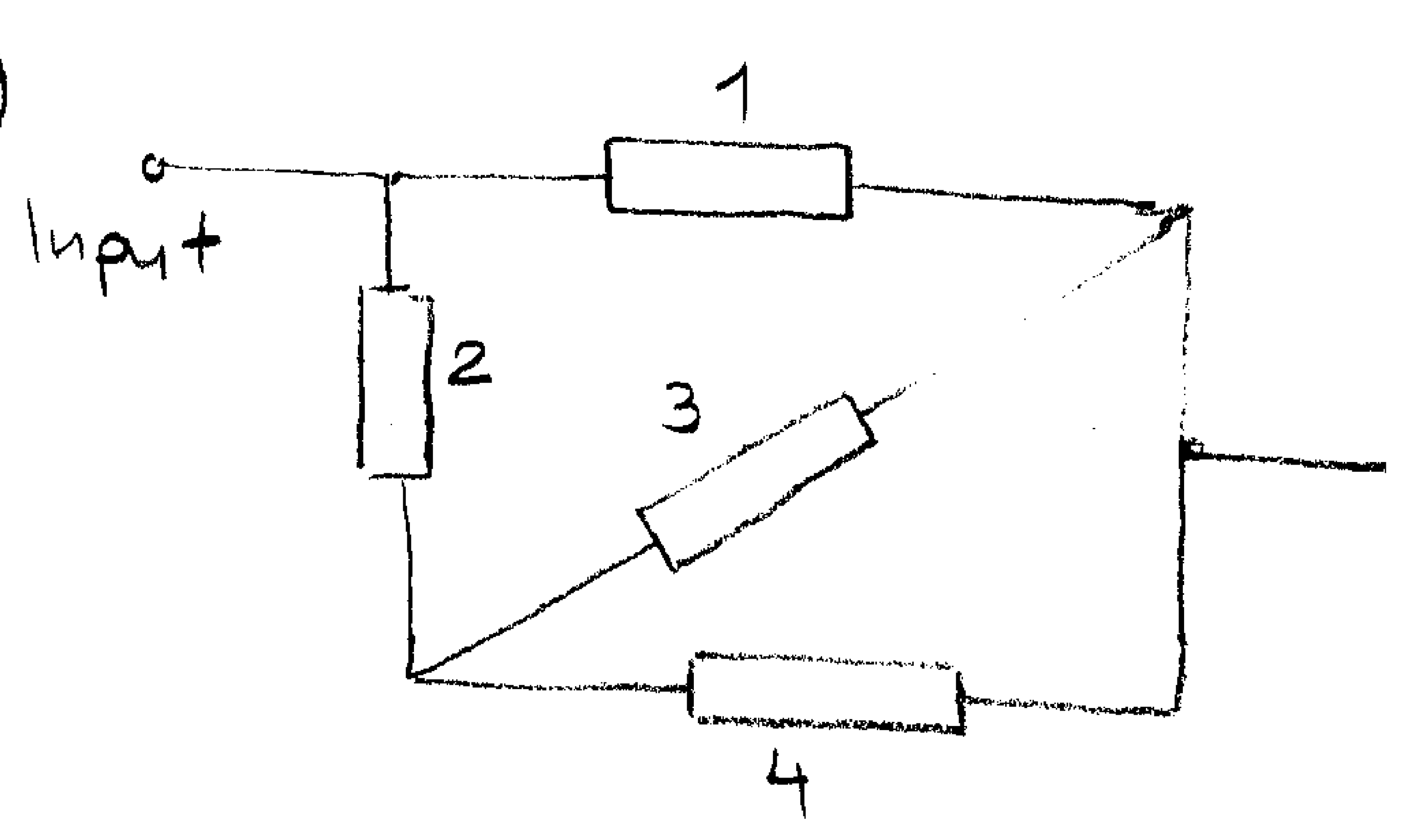
$$= (0.02)\left\{ -180\, e^{-3.6} + 20\, e^{-0.4} + \left(-\frac{1}{0.02}\left(e^{-3.6} - e^{-0.4}\right)\right)\right\}$$

$$= (0.02)\left\{ -180\, e^{-3.6} + 20\, e^{-0.4} - 50\, e^{-3.6} + 50\, e^{-0.4}\right\}$$

$$= (0.02)\left\{ -230\, e^{-3.6} + 70\, e^{-0.4}\right\} = 0.81276 > 0.21875$$

c) Yes they are conforming $P\{20<X<180\} = 0.81276 > 0.21875$

#4)

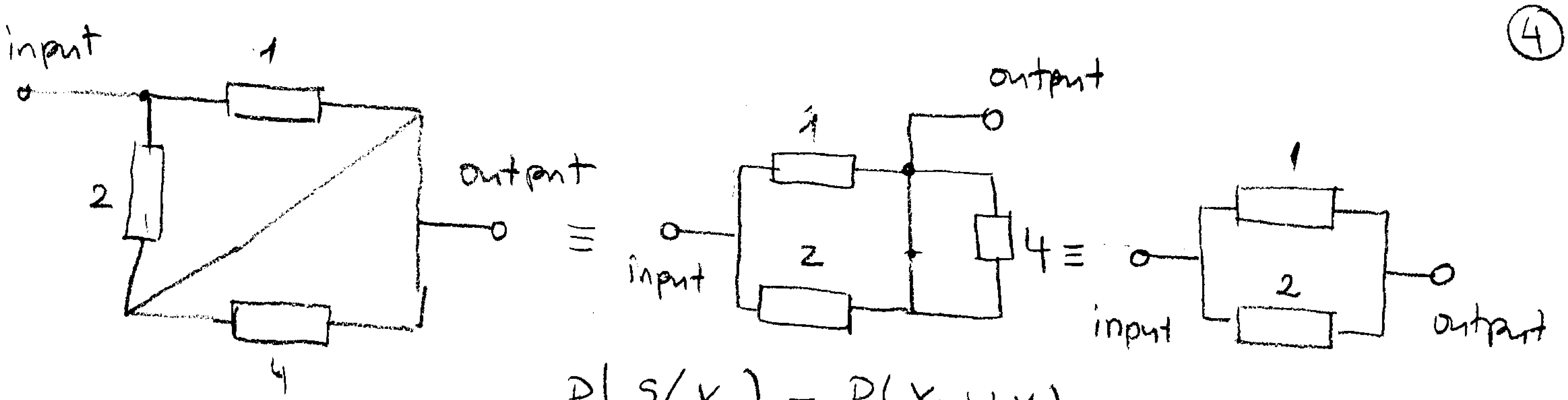


$X_i \triangleq$ component-i is working

$\overline{X_i} \triangleq$ " is failed

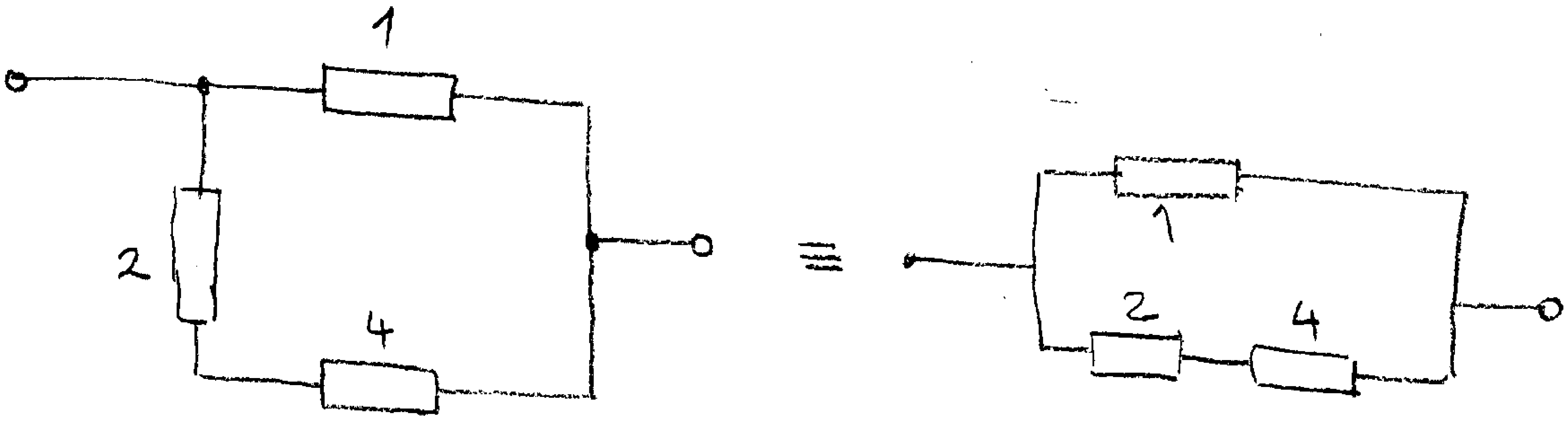
total probability theorem.

$$P(S) = \underbrace{P(S/X_3)}_{A}\, P(X_3) + \underbrace{P(S/\overline{X_3})}_{B}\, P(\overline{X_3})$$



A

$$P(s/X_3) = P(X_1 \cup X_2)$$
$$= 2p - p^2$$



B

$$P(s/\overline{X}_3) = P\{X_1 \cup (X_2 \cap X_4)\}$$
$$= P(X_1) + P(X_2)\,P(X_4) - P(X_1)\,P(X_2)\,P(X_4)$$
$$= p + p^2 - p^3$$

$$P(s) = (2p - p^2)\,p + (p + p^2 - p^3)\,(1 - p)$$
$$= 2p^2 - p^3 + p + p^2 - p^3 - p^2 - p^3 + p^4$$

$$P(s) = p^4 - 3p^3 + 2p^2 + p \Big|_{p=0,8} = 0,9536$$

components are functioning independently.