PROBABILITY FINAL EXAM
(Summer School-2008)

Dr. Salih FADIL **August 18, 2008**

#1) There are three boxes in an experiment. The number and color of the balls in each box are given as follows:

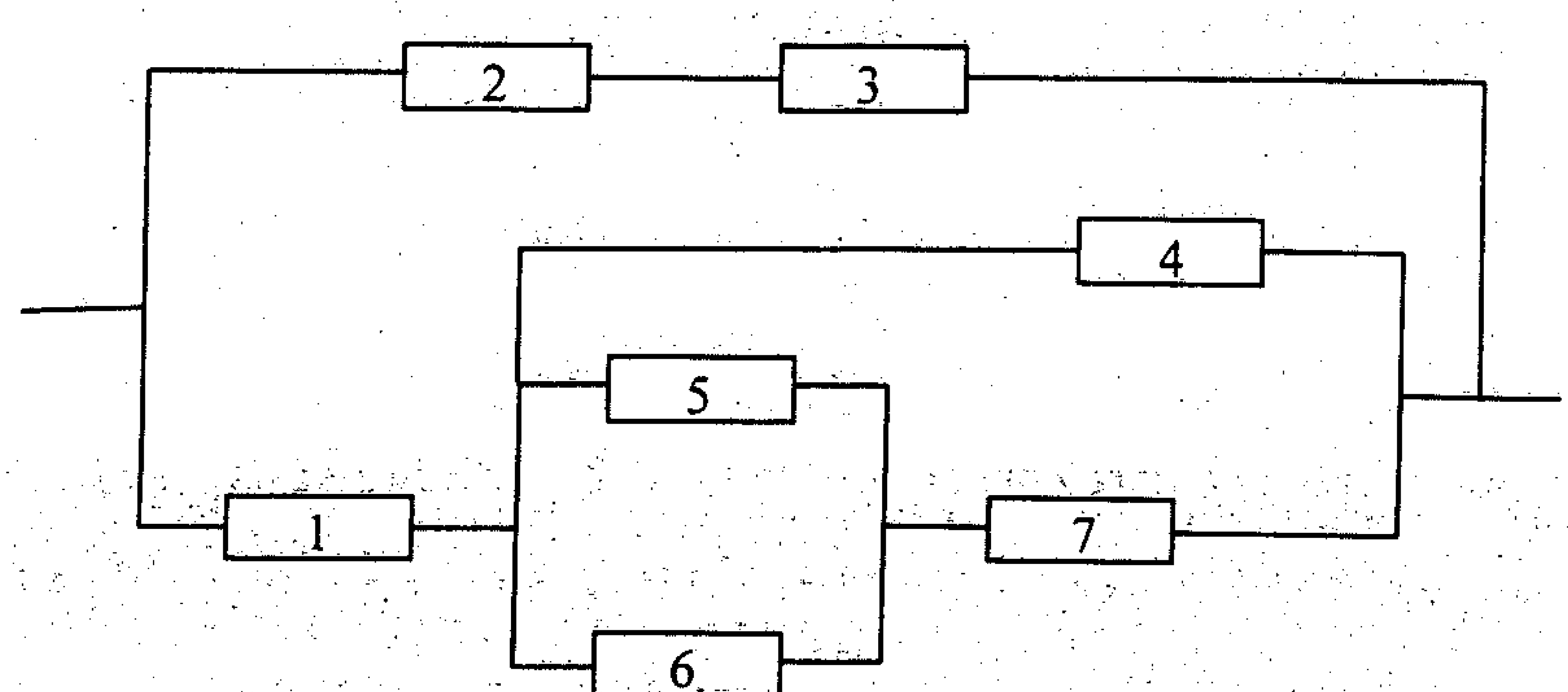
Box-1: 1 black, 2 white balls
Box-2: 3 black, 2 white balls
Box-3: 2 black, 3 red balls.

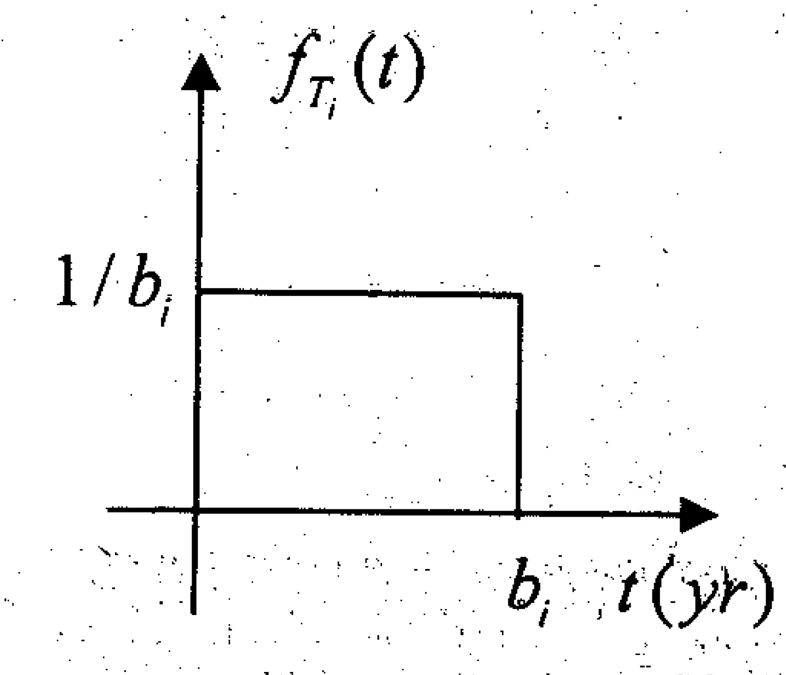
A person selects randomly two balls from box-1 and put them into box-3. Later on, the person selects one ball randomly from box-2 and put it into box-3. Lastly, the same person continues to select balls with replacement from box-3 until 2 black balls are selected. What is the probability that at least three trials in the last selection will be required?

#2) The hardness (Rockwell hardness) of a metal specimen is determined by impressing the surface of the specimen with a hardened point, and then measuring the depth of penetration. The hardness of a certain alloy is normally distributed with mean of 70 units and standard deviation of 3 units.

a) If a specimen is acceptable only if its hardness is between 66 and 74 units, what is the probability that a randomly chosen specimen is acceptable?
b) If the acceptable range is $70 \pm c$, for what value of c would 95% of all specimens be acceptable?

#3) Consider the system shown in the following figure. The life time pdf of each component $f_{T_i}(t)$, $i = 1, \cdots, 7$ is also shown in this figure.





$b_i = 1 + i0.5, \quad i = 1,2,3,4,5,6,7$

Calculate the probability that the system will be working after one year later. Assume that components are working independently.

PROBABILITY FINAL EXAM SOLUTION MANUAL
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#1)

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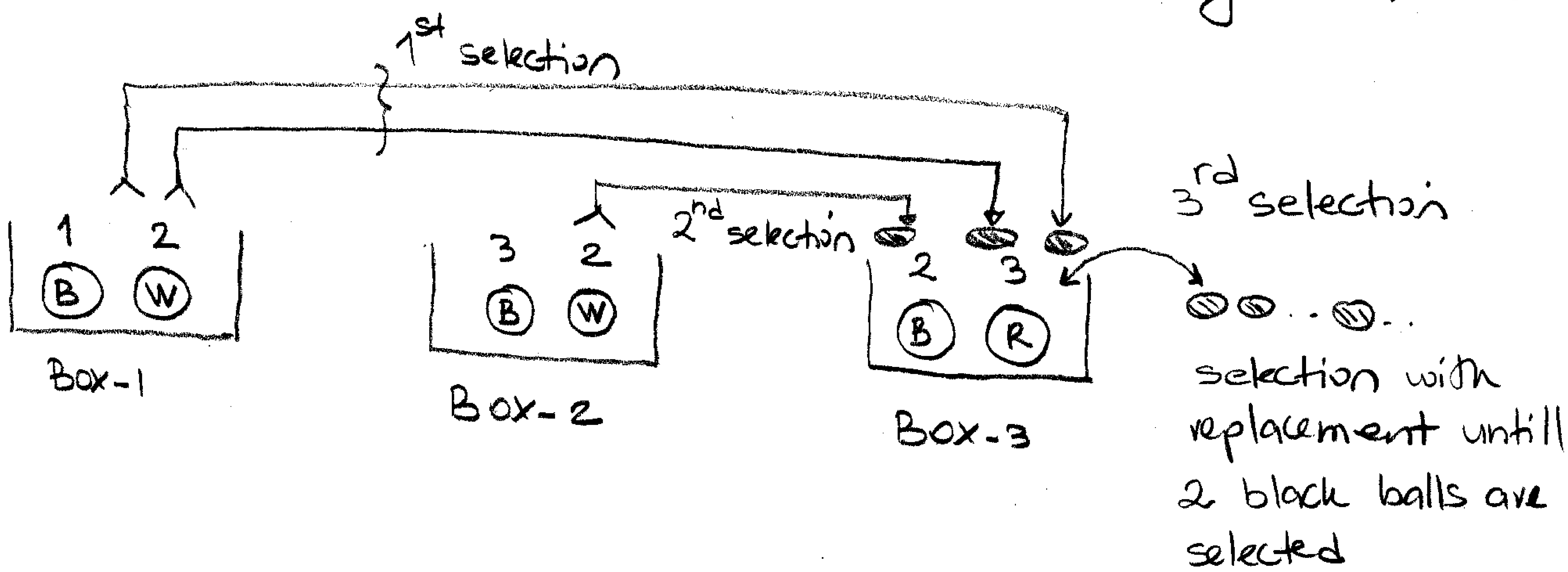


1st and 2nd selection : Hypergeometric distribution

3rd selection : Negative Binomial distribution

X = # of trials (selections) to select 2 black balls in the third selection

$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X=2)$

<u>After the 2nd selection:</u>

B_0 : # of black balls does not change

B_{+1} : # of black balls increases by one

B_{+2} : # of black balls increases by two.

$B_j^i \triangleq$ selecting j black balls in the i^{th} selection

$i = 1 \rightarrow j = 0, 1$

$i = 2 \rightarrow j = 0, 1$

$$P(B_0^1) = \frac{\binom{1}{0}\binom{2}{2}}{\binom{3}{2}} = \frac{1}{\frac{3!}{2!\,2!}} = 1/3$$

$$P(B_0^2) = \frac{2}{5}$$

$$P(B_1^1) = \frac{\binom{1}{1}\binom{2}{1}}{\binom{3}{2}} = \frac{1 \times 2}{3} = 2/3$$

$$+ \overline{\quad 1 \quad}$$

$$P(B_1^2) = \frac{3}{5}$$

$$+ \overline{\quad 1 \quad}$$

$$P(B_2^1) = 0$$

$$P(B_0) = P(B_0^1)\cdot P(B_0^2) = \frac{1}{3}\times\frac{2}{5} = \frac{2}{15}$$

$$P(B_{+1}) = P(B_0^1)\,P(B_1^2) + P(B_1^1)\,P(B_0^2) = \frac{1}{3}\times\frac{3}{5} + \frac{2}{3}\times\frac{2}{5} = \frac{7}{15}$$

$$P(B_{+2}) = P(B_1^1)\,P(B_1^2) = \frac{2}{3}\times\frac{3}{5} = \frac{6}{15}$$

$$\left\{\frac{2}{15}+\frac{7}{15}+\frac{6}{15} = \frac{15}{15} = 1\right\}$$

$$P(X=x) = p(x) = \binom{x-1}{r-1} p^r q^{x-r} \qquad x = r, r+1, r+2, \dots$$

In our problem: $r=2$ $\quad P(X=2) = \binom{2-1}{2-1} p^2 q^0 = p^2$

$$P(X=2) = P(X=2/B_0)\,P(B_0) + P(X=2/B_{+1})\,P(B_{+1}) + P(X=2/B_{+2})\,P(B_{+2})$$

$$P(X=2) = \left(\frac{2}{8}\right)^2\left(\frac{2}{15}\right) + \left(\frac{3}{8}\right)^2\left(\frac{7}{15}\right) + \left(\frac{4}{8}\right)^2\left(\frac{6}{15}\right) = \frac{8+63+96}{960}$$

$$P(X=2) = \frac{167}{960}$$

$$\boxed{P(X\geq 3) = 1 - \frac{167}{960} = \frac{960-167}{960} = \frac{793}{960} = 0.8260416}$$

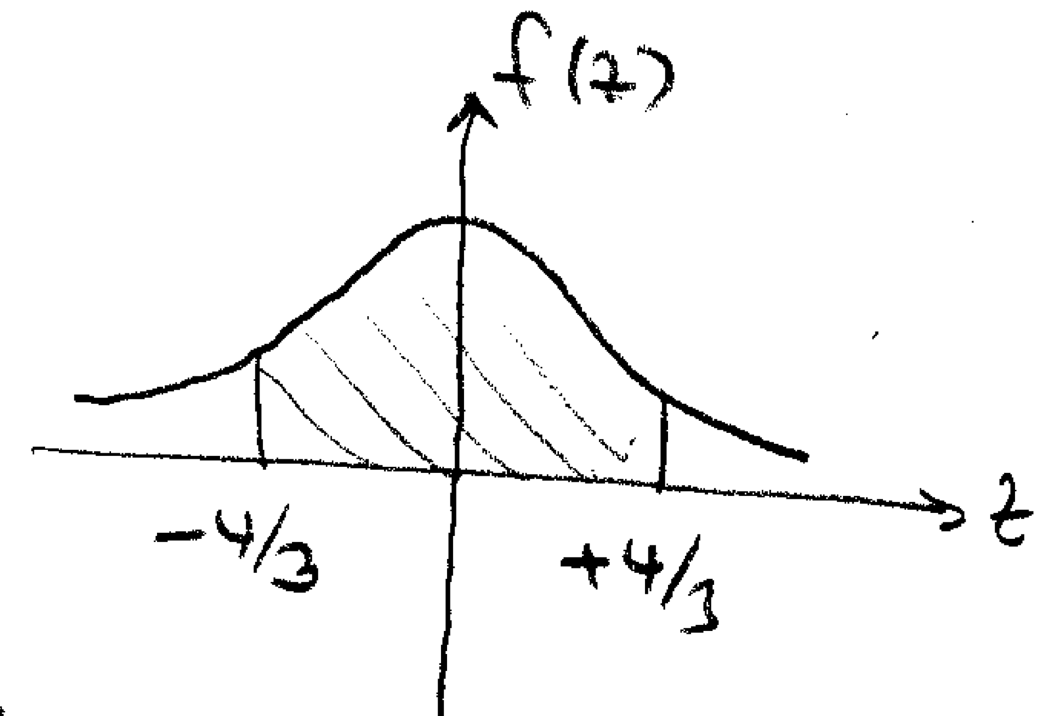
#2) X = hardness of the alloy $\qquad X \sim N(\mu=70, \sigma=3)$

a) $P(66 < X < 74) = P\left(\frac{66-70}{3} < \frac{X-70}{3} < \frac{74-70}{3}\right)$

$= P(-4/3 < Z < \frac{4}{3}) = 2P(0 < Z < 4/3) = 2P(0 < Z < 1.33\overline{3})$

$\underset{\uparrow \text{Table}}{=} 2\times 0{,}4082 = 0{,}8164$

$$\boxed{P(66 < X < 74) = 0{,}8164}$$



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b) $P(\overbrace{\mu - k\sigma}^{70-c} < X < \overbrace{\mu + \sigma k}^{70+c}) = P(-k < Z < +k) = 2\,P(0 < Z < k)$

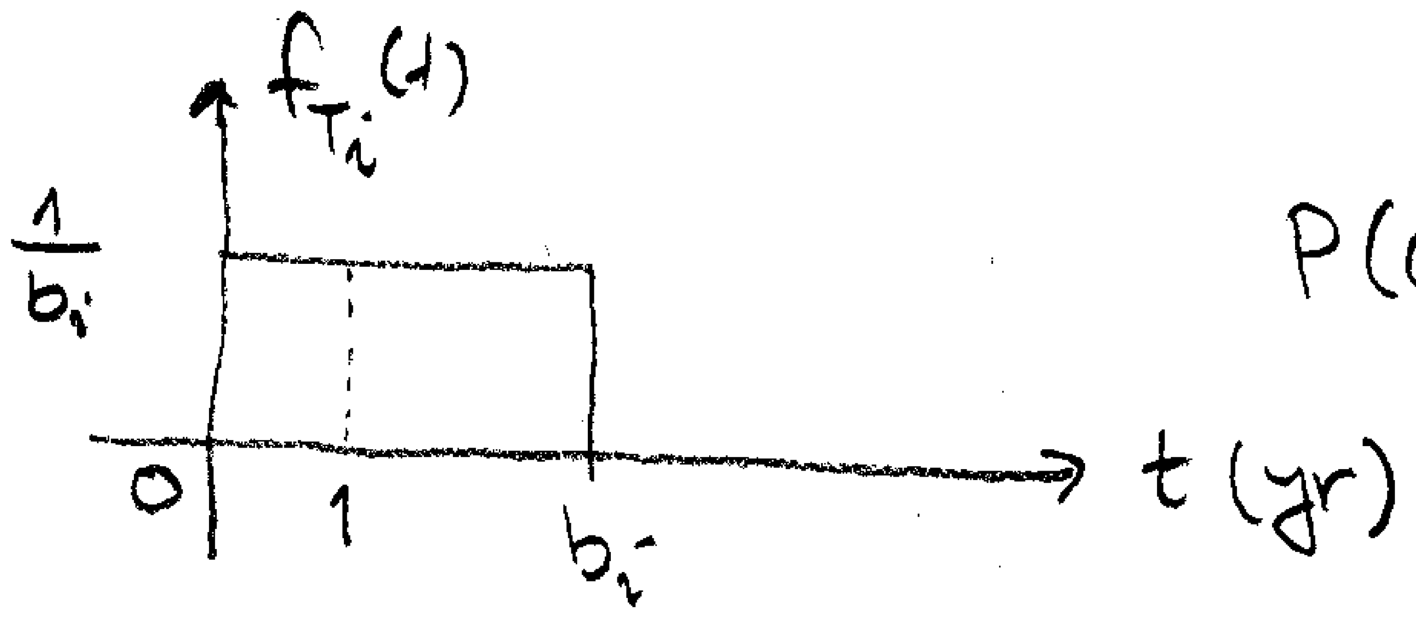
$$2\,P(0 < Z < k) = 0{,}95 \quad \rightarrow \quad P(0 < Z < k) = 0{,}475$$

From the table, we get $k = 1{,}96$.

$\sigma k = c \qquad c = 3 \times 1{,}96$

$$\boxed{c = 5{,}88}$$

#3) $C_i \triangleq$ ith component works ~~after~~ one year



$$P(C_i) = \int_1^{b_i} \frac{1}{b_i}\cdot dt = \frac{1}{b_i}(b_i - 1) = 1 - \frac{1}{b_i}$$

$P(C_1) = 1 - \frac{1}{1.5} = \frac{0{,}5}{1{,}5} = \frac{1}{3}$; $P(C_2) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(C_3) = 1 - \frac{1}{2{,}5} = \frac{1{,}5}{2{,}5} = \frac{3}{5}$; $P(C_4) = 1 - \frac{1}{3} = \frac{2}{3}$

$P(C_5) = 1 - \frac{1}{3{,}5} = \frac{2{,}5}{3{,}5} = \frac{5}{7}$; $P(C_6) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(C_7) = 1 - \frac{1}{4{,}5} = \frac{3{,}5}{4{,}5} = \frac{7}{9}$

$S \triangleq$ the system works after one year

$$P(\overbrace{C_5 \cup C_6}^{A}) = P(C_5) + P(C_6) - P(C_5 \cap C_6) \overset{\text{independency}}{=} P(C_5) + P(C_6) - P(C_5)\,P(C_6)$$

$$= \frac{5}{7} + \frac{3}{4} - \frac{15}{28} = 0{,}928571$$

$$P(\overbrace{A \cap C_7}^{B}) = P(A)\cdot P(C_7) = 0{,}928571 \times \frac{7}{9} = 0{,}722222$$

$$P(\overbrace{B \cup C_4}^{D}) = P(B) + P(C_4) - P(B)\,P(C_4)$$

$$= 0{,}722222 + \frac{2}{3} - \frac{2 \times 0{,}722222}{3} = 0{,}9074074$$

$$P(\overbrace{D \cap C_1}^{E}) = P(D)\cdot P(C_1) = 0{,}9074074 \times \frac{1}{3} = 0{,}302469.$$

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$P(\overbrace{C_2 \cap C_3}^{F}) = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10} = 0,3$

$P(S) = P(E \cup F) = 0,302469 + 0,3 - 0,302469 \times 0,3 = 0,511728$

$\boxed{P(S) = 0,511728}$

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