PROBABILITY FIRST EXAM
(Summer School-2008)

Dr. Salih FADIL **July 21, 2008**

#1) A customer visiting the suit department of a certain store will purchase a suit with probability 0.22, a shirt with probability 0.30, and a tie with probability 0.28. The customer will purchase both a suit and a shirt with probability 0.11, both a suit and a tie with probability 0.14 and both a shirt and a tie with probability 0.10. A customer will purchase all three items with probability 0.06. What is the probability that a customer purchases

a) none of these item
b) exactly one of these items

#2) A and B are involved in a duel. The rules of the duel are that they are to pick their guns and shoot at each other simultaneously. If one or both are hit, the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each of A will hit B with the probability p_A and each shots of B will hit A with the probability p_B. What is

a) the probability that A is not hit;
b) the probability that both duelist are hit;
c) the probability that the duel ends after the n^{th} round of shots;
d) the conditional probability that the duel ends after the n^{th} round of shots given that A is not hit.

#3) Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let random variable X denote the highest ranking achieved by a woman (for instance, $X=1$ if the top-ranked person is female).

a) Find the probability function of random variable X.
b) Find the cumulative distribution function of random variable X.
c) Find $P(X \geq 3)$.

#4) Suppose that we have three cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black. Three cards are mixed up in a hat and one card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is black?

Note!

* Time: 90 min
* Answer Q1, Q4 and either Q2 or Q3.

PROBABILITY FIRST EXAM SOLUTION MANUAL
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#1) a) A = customer buys a suit
B = " " a shirt
C = " " a tie

$$P\left\{\begin{matrix}\text{customer buys}\\ \underline{\text{none}}\text{ of these}\\ \text{items}\end{matrix}\right\} = 1 - P\left\{\begin{matrix}\text{customer buys}\\ \underline{\text{at least}}\text{ one of}\\ \text{these items}\end{matrix}\right\} = 1 - P\{A \cup B \cup C\}$$

$$P\{A \cup B \cup C\} = P(A) + P(B) + P(C) - \{P(A\cap B) + P(A \cap C) + P(B \cap C\} + P(A \cap B \cap C)$$

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$$= 0.22 + 0.30 + 0.28 - \{0.11 + 0.14 + 0.1\} + 0.06 = 0.51$$

$$P\left\{\begin{matrix}\text{customer buys}\\ \text{none of these items}\end{matrix}\right\} = 1 - 0.51 = \underline{0.49}$$

b) $$P\left\{\begin{matrix}\text{customer buys}\\ \text{exactly one of}\\ \text{these items}\end{matrix}\right\} = P\left\{\begin{matrix}\text{customer buys}\\ \underline{\text{at least}}\text{ one}\\ \text{of these items}\end{matrix}\right\} - P\left\{\begin{matrix}\text{customer buys}\\ \underline{\text{at least}}\text{ two}\\ \text{of these items}\end{matrix}\right\}$$

$$P\left\{\begin{matrix}\text{customer buys}\\ \text{at least two of}\\ \text{these items}\end{matrix}\right\} = P\{(A\cap B) \cup (A \cap C) \cup (B \cap C)\}$$

$$= P(A\cap B) + P(A\cap C) + P(B \cap C) - 3.\{P(A\cap B\cap C\} + P(A\cap B \cap C)$$

$$= 0.11 + 0.14 + 0.1 - 2 \times 0.06 = 0.23$$

$$P\left\{\begin{matrix}\text{customer buys}\\ \text{exactly one of}\\ \text{these items}\end{matrix}\right\} = 0.51 - 0.23 = \underline{0.28}$$

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2) shots are independent

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probability that A hits B = P_A

" " B hits A = P_B

probability that A misses B = $1-P_A = q_A$

" " B misses A = $1-P_B = q_B$

a) Consider the final round of the duel.

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$P\{\text{A is not hit}\} = P\{\text{A is not hit} / \text{at least one is hit}\}$

$P\{\text{A is not hit and B is hit}\} / P\{\text{at least one is hit}\}$

$P\{\text{at least one is hit}\} = 1 - P\{\text{hit}\} = 1 - q_A q_B$

$$P\{\text{A is not hit}\} = \frac{q_B P_A}{1 - q_A q_B}$$

b) $P\{\text{both A and B are hit}\} = P\{\text{both A and B are hit} / \text{at least one is hit}\}$

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$= P\{\text{both A and B are hit}\} / P\{\text{at least one is hit}\}$

$$P\{\text{both A and B are hit}\} = \frac{P_A P_B}{1 - q_A q_B}$$

c) $P\left\{\begin{array}{l}\text{duel ends after}\\ \text{the } n^{th} \text{ round}\\ \text{of shots}\end{array}\right\} = P\left\{\begin{array}{l}\text{both A and B}\\ \text{miss their}\\ \text{shots in the}\\ n-1 \text{ rounds}\end{array} \cap \begin{array}{l}\text{at least}\\ \text{one is hit}\\ \text{in the } n\text{th round}\end{array}\right\}$

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$= P\left\{\begin{array}{l}\text{both A and B}\\ \text{miss their}\\ \text{shots in the}\\ n-1 \text{ rounds}\end{array}\right\} \cdot P\left\{\begin{array}{l}\text{at least one is}\\ \text{hit in the } n\text{th}\\ \text{round}\end{array}\right\}$

$$= (q_A . q_B)^{n-1} (1 - q_A q_B) = (q_A q_B)^{n-1} - (q_A q_B)^n$$

$$P\left\{\begin{matrix}\text{duel ends after}\\ \text{the } n^{th} \text{ round}\end{matrix} \Big/ A \text{ is unhit}\right\} = \frac{P\left\{\begin{matrix}\text{it ends after}\\ n^{th} \text{ round}\end{matrix} \cap A \text{ is unhit}\right\}}{P\{A \text{ is unhit}\}}$$

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$$= \frac{(q_A q_B)^{n-1} \cdot (\cancel{p_A q_B})}{\dfrac{\cancel{q_B p_A}}{(1-q_A q_B)}} = (q_A q_B)^{n-1}(1 - q_A q_B)$$

$$= (q_A q_B)^{n-1} - (q_A q_B)^n \quad \text{the same as part c. !}$$

$$= P\left\{\begin{matrix}\text{duel end after}\\ \text{the } n^{th} \text{ round}\end{matrix}\right\}$$

Similarly (this part is NOT asked)

- $$P\left\{\begin{matrix}\text{duel ends after}\\ \text{the } n^{th} \text{ round}\end{matrix} \Big/ B \text{ is unhit}\right\} = \frac{(q_A q_B)^{n-1} (\cancel{q_A p_B})}{\dfrac{\cancel{q_A p_B}}{1 - q_A q_B}}$$

$$= (q_A q_B)^{n-1} - (q_A q_B)^n \quad \text{the same.}$$

$$P\left\{\begin{matrix}\text{duel ends after}\\ \text{the } n^{th} \text{ round}\end{matrix} \Big/ \begin{matrix}\text{both A}\\ \text{and B are hit}\end{matrix}\right\} = \frac{(q_A q_B)^{n-1} (\cancel{p_A p_B})}{\dfrac{\cancel{p_A p_B}}{1 - q_A q_B}}$$

- $$= (q_A q_B)^{n-1} - (q_A q_B)$$

Ending of the duel after n^{th} round does not depend on the events; A is unhit, B is unhit, both A and B are hit

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#3) X = the highest ranking achieved by a women.

a)

$$A = \{1, 2, 3, 4, 5, 6\} \leftarrow \begin{cases} 5 & \text{women} \\ 5 & \text{men} \end{cases}$$

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X	1	2	3	4	5	6
$p(X)$	$\frac{126}{252}$	$\frac{70}{252}$	$\frac{35}{252}$	$\frac{15}{252}$	$\frac{5}{252}$	$\frac{1}{252}$

$$\sum_{X \in A} p(X) = 1 \checkmark$$

$$\overset{1}{\underset{\underset{W}{\uparrow}}{\underline{5}}} \times \underbrace{\overset{2}{\underline{9}} \times \overset{3}{\underline{8}} \times \overset{4}{\underline{7}} \times \overset{5}{\underline{6}} \times \overset{6}{\underline{5}} \times \overset{7}{\underline{4}} \times \overset{8}{\underline{3}} \times \overset{9}{\underline{2}}}_{\text{does not matter W or M}} \times \overset{10}{\underline{1}} = 5 \times 9!$$

$\leftarrow$ rank

\# of cases where top-ranked person is a woman

$$p(1) = \frac{5 \times 9!}{10!} = \frac{5}{10} = \frac{1}{2} \Rightarrow p(1) = \frac{126}{252}$$

$$\overset{1}{\underset{\underset{M}{\uparrow}}{\underline{5}}} \times \overset{2}{\underset{\underset{W}{\uparrow}}{\underline{5}}} \times \underbrace{\overset{3}{\underline{8}} \times \overset{4}{\underline{7}} \times \overset{5}{\underline{6}} \times \overset{6}{\underline{5}} \times \overset{7}{\underline{4}} \times \overset{8}{\underline{3}} \times \overset{9}{\underline{2}} \times \overset{10}{\underline{1}}}_{\text{does not matter}} = 25 \times 8!$$

$$p(2) = \frac{25 \times 8!}{10!} = \frac{\cancel{25}^{5}}{9 \times \cancel{10}_{2}} = \frac{5}{18} \Rightarrow p(2) = \frac{70}{252}$$

$$\overset{1}{\underset{M}{\underline{5}}} \times \overset{2}{\underset{M}{\underline{4}}} \times \overset{3}{\underset{W}{\underline{5}}} \times \underbrace{\overset{4}{\underline{7}} \times \overset{5}{\underline{6}} \times \overset{6}{\underline{5}} \times \overset{7}{\underline{4}} \times \overset{8}{\underline{3}} \times \overset{9}{\underline{2}} \times \overset{10}{\underline{1}}}_{\text{does not matter}} = 100 \times 7!$$

$$p(3) = \frac{100 \times 7!}{10!} = \frac{\cancel{100}}{8 \times 9 \times \cancel{10}} = \frac{10}{72} \Rightarrow p(3) = \frac{35}{252}$$

$$\overset{1}{\underset{M}{\underline{5}}} \times \overset{2}{\underset{M}{\underline{4}}} \times \overset{3}{\underset{M}{\underline{3}}} \times \overset{4}{\underset{W}{\underline{5}}} \times \underbrace{\overset{5}{\underline{6}} \times \overset{6}{\underline{5}} \times \overset{7}{\underline{4}} \times \overset{8}{\underline{3}} \times \overset{9}{\underline{2}} \times \overset{10}{\underline{1}}}_{\text{does not matter}} = 300 \times 6!$$

$$p(4) = \frac{300 \times 6!}{10!} = \frac{30\cancel{0}}{7 \times 8 \times 9 \times 1\cancel{0}} = \frac{30}{504} = \frac{10}{168} \Rightarrow p(4) = \frac{15}{252}$$

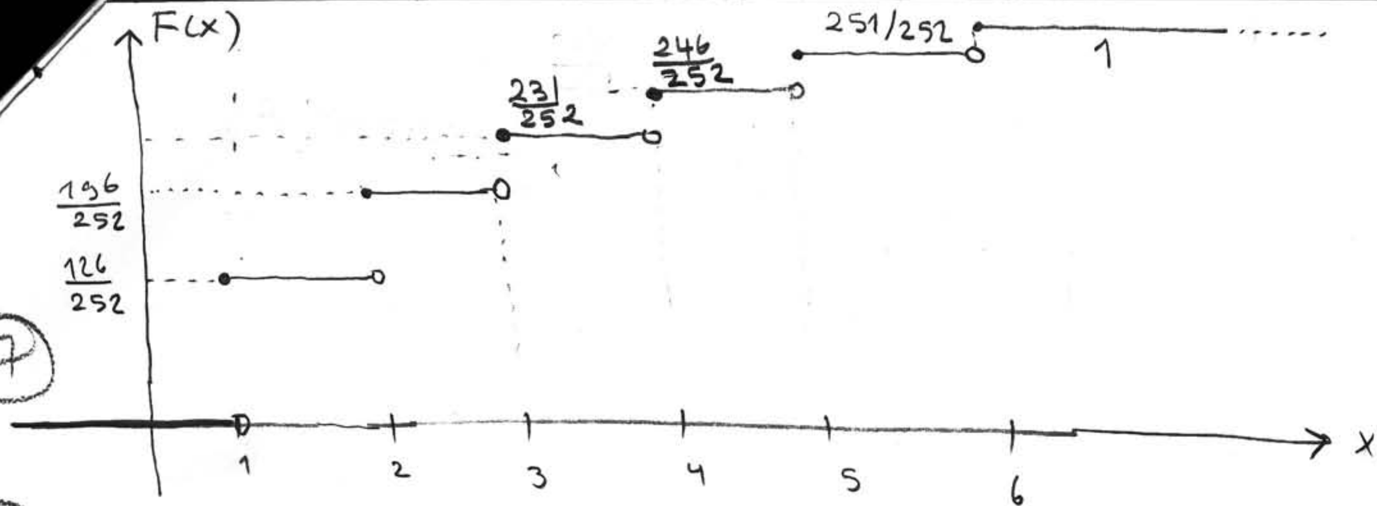
$$\overset{1}{\underset{M}{\underline{5}}} \times \overset{2}{\underset{M}{\underline{4}}} \times \overset{3}{\underset{M}{\underline{3}}} \times \overset{4}{\underset{M}{\underline{2}}} \times \overset{5}{\underset{W}{\underline{5}}} \times \underbrace{\overset{6}{\underline{5}} \times \overset{7}{\underline{4}} \times \overset{8}{\underline{3}} \times \overset{9}{\underline{2}} \times \overset{10}{\underline{1}}}_{\text{does not matter}} = 600 \times 5!$$

$$p(5) = \frac{600 \times 5!}{10!} = \frac{\cancel{600}^{10}}{\cancel{6} \times 7 \times 8 \times 9 \times \cancel{10}} = \frac{10}{504} = \frac{5}{252}$$

$$\overset{1}{\underset{M}{\underline{5}}} \times \overset{2}{\underset{M}{\underline{4}}} \times \overset{3}{\underset{M}{\underline{3}}} \times \overset{4}{\underset{M}{\underline{2}}} \times \overset{5}{\underset{M}{\underline{1}}} \times \overset{6}{\underset{W}{\underline{5}}} \times \underbrace{\overset{7}{\underline{4}} \times \overset{8}{\underline{3}} \times \overset{9}{\underline{2}} \times \overset{10}{\underline{1}}}_{\text{does not matter}} = \frac{5!\,5!}{10!}$$

$$= \frac{\cancel{120}\,\cancel{120}^{2}}{\cancel{6} \times 7 \times 8 \times 9 \times \cancel{10}} = \frac{1}{252}$$

$$p(6) = \frac{1}{252}$$



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c) $P(X \geq 3) = 1 - P(X \leq 2) = 1 - \frac{196}{252} = \frac{56}{252} = 0.22\overline{2}$

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#4) RR = chosen card is all red
BB = " " is all black
RB = " " is red-black

R = the event that the upturned side of the chosen card is red

$$P(RB/R) = \frac{P(R)\,P(R/RB)}{P(R/RR)\,P(RR) + P(R/BB)\,P(BB) + P(R/RB)\,P(RB)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{2}}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{1/6}{\frac{1}{3} + \frac{1}{6}}$$

$$= \frac{1}{\cancel{6}} \cdot \frac{\cancel{6}}{3} = 1/3 = P\left\{\begin{matrix}\text{selecting}\\ \text{RB card}\end{matrix}\right\}$$

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