PROBABILITY SECOND EXAM
(Summer School-2008)

Dr. Salih FADIL **August 04, 2008**

#1) The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a) Compute the expected life time of such a tube.
b) Find a bound for the $P(2-3\sqrt{2} \leq X \leq 2+3\sqrt{2})$
c) Calculate the exact probability value given in part b.

#2) Diagonal value of a square is not known exactly. It is given as a random variable X that is distributed between 0 and 1 uniformly.

a) Take the area of the square as random variable Y and calculate $f_Y(y)$.
b) Calculate the expected value of the area of the square by using $f_Y(y)$.
c) Calculate the expected value of the area of the square directly.

#3) Suppose that a normal die is rolled twice. Random variable X is defined as the numbers of dots obtained in the first roll minus that of in the second roll. Calculate the probability function of X. Calculate also $P(X < 4)$.

PROBABILITY SECOND EXAM
(Summer School - 2008)

Dr. Salih FADIL

August 04, 2008

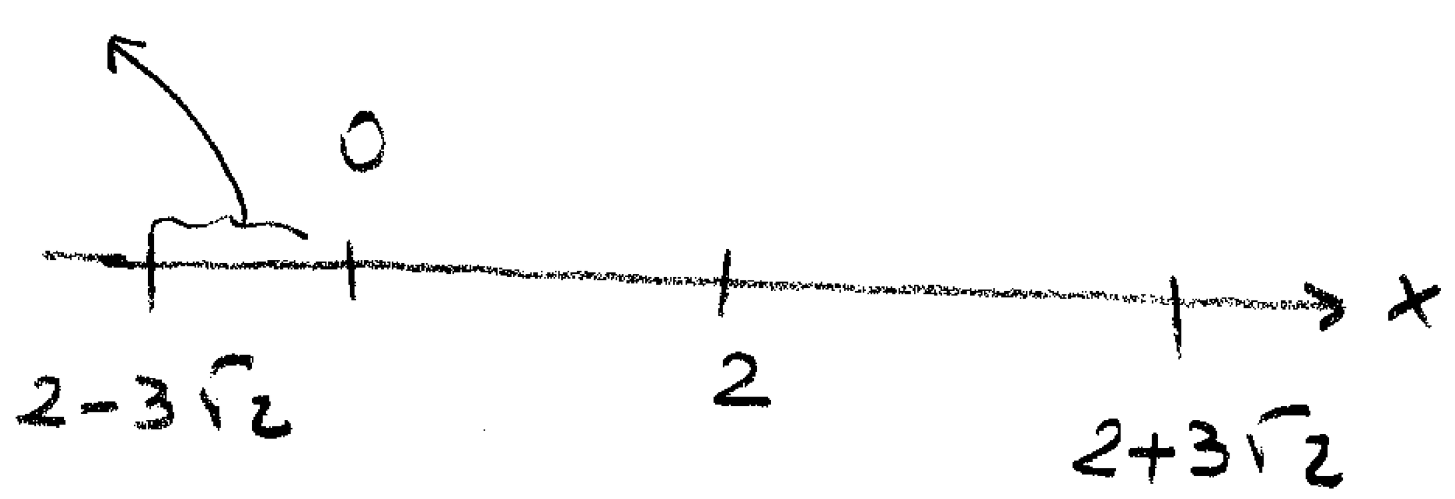
a) $E(X) = \int_0^\infty x \, x \, e^{-x} \, dx = \cancelto{0}{-x^2 e^{-x}} \Big|_0^\infty + 2 \int_0^\infty x \, e^{-x} \, dx$

$u = x^2 \rightarrow du = 2x \, dx$ | $u = x \rightarrow du = dx$

$dv = e^{-x} dx \rightarrow v = -e^{-x}$ | $dv = e^{-x} dx \rightarrow v = -e^{-x}$

$E(X) = 2 \left\{ \cancelto{0}{-x e^{-x}} \Big|_0^\infty + \int_0^\infty e^{-x} dx \right\} = 2 \left\{ -e^{-x} \Big|_0^\infty \right\} = 2(-0 + 1) = 2$

b)



$P\{2 - 3\sqrt{2} \le X \le 2 + 3\sqrt{2}\} =$

$P\{|X - 2| \le 3\sqrt{2}\} = ?$

$\sigma^2 = E(X^2) - \mu^2$

$E(X^2) = \int_0^\infty x^2 \, x \, e^{-x} \, dx = \cancelto{0}{x^3 e^{-x}} \Big|_0^\infty + 3 \underbrace{\int_0^\infty x^2 e^{-x} dx}_{2} = 3 \times 2 = 6$

$u = x^3 \rightarrow du = 3x^2$

$dv = e^{-x} dx \rightarrow v = -e^{-x}$

$\sigma^2 = 6 - 4 = 2$ $\qquad \underline{\underline{\sigma = \sqrt{2}}}$

$P\{|X - \mu| \le \sigma t\} \ge \left(1 - \frac{1}{t^2}\right)$ $\qquad \sigma t = 3\cancel{\sqrt{2}} = \cancel{\sqrt{2}}\, t$ $\qquad t = 3$

$|t| > 1$ o.k

$P\{|X - 2| \le 3\sqrt{2}\} \ge \left(1 - \frac{1}{9}\right) = 8/9$

$$\boxed{P\{|X - 2| \le 3\sqrt{2}\} > 8/9 = 0.8888}$$

c)

$$P\{2-3\sqrt{2} \leq X \leq 2+3\sqrt{2}\} = P\{0 \leq x \leq 2+3\sqrt{2}\}$$

$$= \int_0^{2+3\sqrt{2}} x\, e^{-x}\, dx = -x e^{-x} \Big|_0^{2+3\sqrt{2}} + \int_0^{2+3\sqrt{2}} e^{-x} dx = -x e^{-x} \Big|_0^{2+3\sqrt{2}} - e^{-x} \Big|_0^{2+3\sqrt{2}}$$

$u = x \rightarrow du = dx$

$dv = e^{-x} dx \rightarrow v = -e^{-x}$

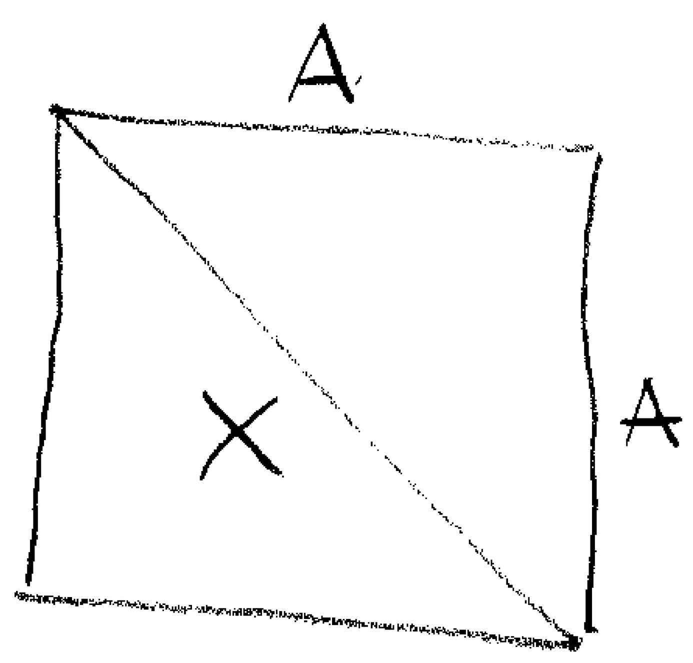
$$P\{0 \leq X \leq 2+3\sqrt{2}\} = -(2+3\sqrt{2})\, e^{-(2+3\sqrt{2})} + 0 - e^{-(2+3\sqrt{2})} + 1$$

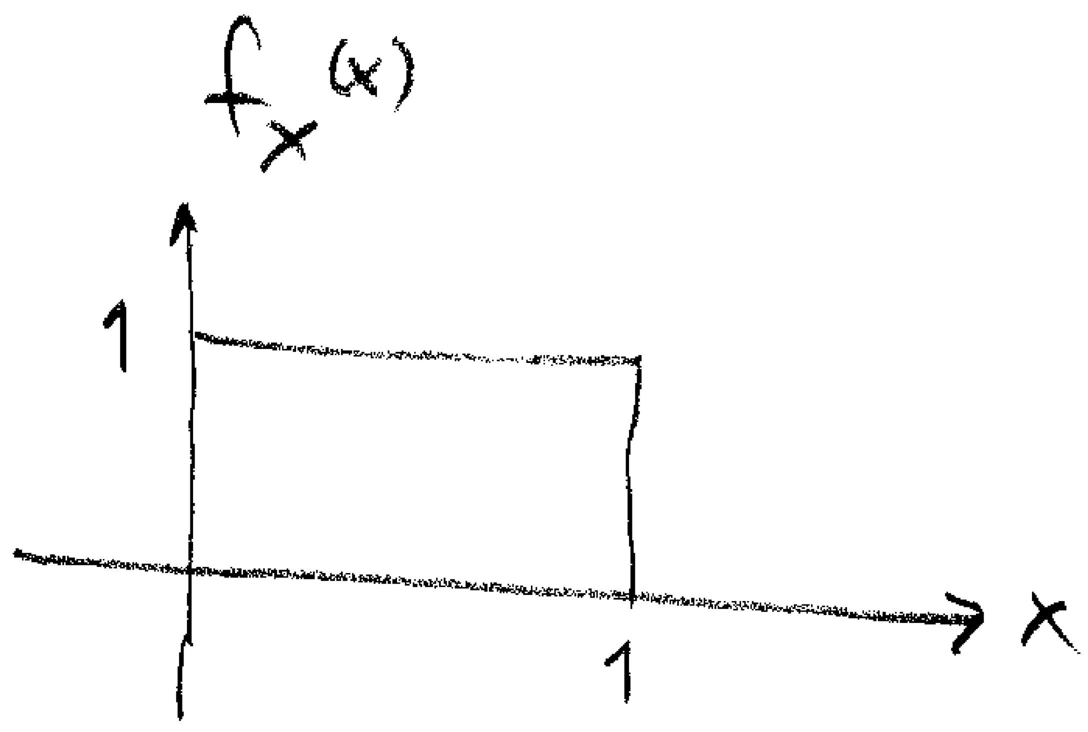
$$= -(3+3\sqrt{2})\, e^{-(2+3\sqrt{2})} + 1 = 0{,}985915139 > 0{,}8888$$

correct!

#2)

a)





$X = \sqrt{2}\, A.$

$Y = A^2 = \frac{1}{2} X^2$

$y = \frac{1}{2} x^2 \qquad x_1 = \sqrt{2y} \qquad \cancel{x_2 = -\sqrt{2y}}$ impossible

$$f_Y(y) = \frac{1}{\sqrt{2y}}$$

$u(x) = \frac{x^2}{2}$

$u'(x) = x.$

$0 \leq \sqrt{2y} \leq 1, \quad 0 \leq y \leq 1/2$

As a result

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2y}} & 0 \leq y \leq 1/2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\frac{1}{\sqrt{2}} \int_0^{1/2} y^{-1/2} dy = \frac{1}{\sqrt{2}} \frac{y^{1/2}}{1/2} \Big|_0^{1/2}$$

$$= \frac{2}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} - 0\right) = 1,$$

O.K.

b) $$E(Y) = \int_0^{1/2} y \cdot \frac{1}{\sqrt{2}\sqrt{y}}\, dy = \frac{1}{\sqrt{2}} \int_0^{1/2} \sqrt{y}\, dy = \frac{1}{\sqrt{2}} \left. \frac{y^{3/2}}{3/2} \right|_0^{1/2}$$

$$E(Y) = \frac{1}{\sqrt{2}}\, \frac{2}{3}\, \frac{1}{\sqrt{2}\, 2} = \frac{1}{\sqrt{2}}\, \frac{2}{3}\, \frac{1}{2\sqrt{2}} = \frac{1}{6}\, (\text{unit})^2$$

c) $$E\left(\frac{X^2}{2}\right) = \frac{1}{2} E(X^2) = \frac{1}{2} \int_0^1 x^2\, 1 \, dx = \frac{1}{2} \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{6}\, (\text{unit})^2$$

the same!

order is reversed!

#3)

a)

-5	-4	-3	-2	-1	0	1	2	3	4	5
(1,6)	(1,5)	(1,4)	(1,3)	(1,2)	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
	(2,6)	(2,5)	(2,4)	(2,3)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)	
		(3,6)	(3,5)	(3,4)	(3,3)	(4,3)	(5,3)	(6,3)		
			(4,6)	(4,5)	(4,4)	(5,4)	(6,4)			
				(5,6)	(5,5)	(6,5)				
					(6,6)					
1	2	3	4	5	6	5	4	3	2	1

1st die, 2nd die

The last row is labelled "# of outcomes".

Total number of outcomes = 36

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b) $$P(X<4) = 1 - P(X \geq 4) = 1 - \left(\frac{2}{36} + \frac{1}{36}\right) = 1 - \frac{3}{36} = \frac{33}{36}$$