Fuzzy Extension Principle and Fuzzy Arithmetic

Lecture 06

Extension Principle for Crisp Sets

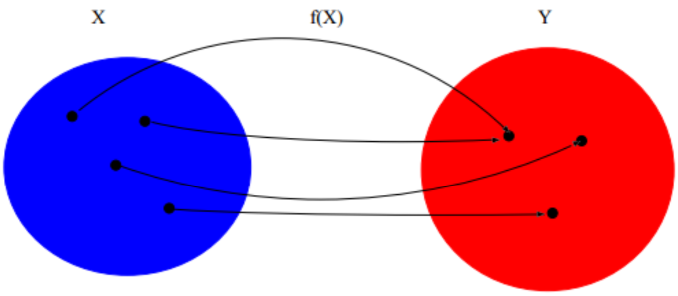
$$x \rightarrow \boxed{f(x)} \rightarrow y$$

Extention is mapping from x-to-y through f(·).

y = f(x)

f: X $\longrightarrow$ Y (From one universe to another universe)
x: input variable
y: input image of x under f(output)
y = f(x) $\Rightarrow$ $x = f^{-1}(y)$ (original image of y)

Crisp Mapping



Extension Principle for Crisp Sets

A mapping can also be expressed by a realtion R on the Cartesian space XxY with the characteristic function:

$$\chi_R(x,y) = \begin{cases} 1\ , & y = f(x) \\ 0\ , & y \neq f(x) \end{cases}$$

Extension Principle for Crisp Sets

Let A be a crisp set defined on X. The mapping y=f(x) will result in a set B defined on Y such that

$$B = f(A) = \{y \mid \forall x \in A,\ y = f(x)\},$$

and the characteristic function of B will be

$$\chi_B(y) = \chi_{f(A)}(y) = \bigvee_{y=f(x)} \chi_A(x)$$

Here, B is another crisp set.

Extension Principle for Crisp Sets

Example: Let X={ -2, -1, 0, 1, 2} and A={ 0, 1} defined on X. y= |4x|+2 mapping is applied to A, find B.

$$\left.\begin{array}{lll} x=-2 & \Rightarrow & y=10 \\ x=-1 & \Rightarrow & y=6 \\ x=0 & \Rightarrow & y=2 \\ x=1 & \Rightarrow & y=6 \\ x=2 & \Rightarrow & y=10 \end{array}\right\} \Rightarrow Y=\{2,6,10\}$$

Extension Principle for Crisp Sets

Method #1:
Directly appliying the formula:

$$\chi_B(y) = \bigvee_{y=f(x)} \chi_A(x)$$

$$\chi_A(0) = 1 \text{ , } \chi_A(1) = 1$$

$$\chi_A(-2) = \chi_A(-1) = \chi_A(2) = 0$$

Extension Principle for Crisp Sets

$$\left.\begin{aligned} &\chi_B(2) = \chi_A(0) = 1 \\ &\chi_B(6) = \left(\chi_A(-1) \vee \chi_A(1)\right) = \left(0 \vee 1\right) = 1 \\ &\chi_B(10) = \left(\chi_A(-2) \vee \chi_A(2)\right) = \left(0 \vee 0\right) = 0 \end{aligned}\right\} \Rightarrow B = \{2, 6\}$$

Extension Principle for Crisp Sets

Method #2:

Use relation matrix

$$\chi_R(x,y)=\begin{cases}1\ , & y=f(x)\\ 0\ , & y\neq f(x)\end{cases}$$

$$R=\begin{array}{c|ccc} & 2 & 6 & 10\\ \hline -2 & 0 & 0 & 1\\ -1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 2 & 0 & 0 & 1\end{array}$$

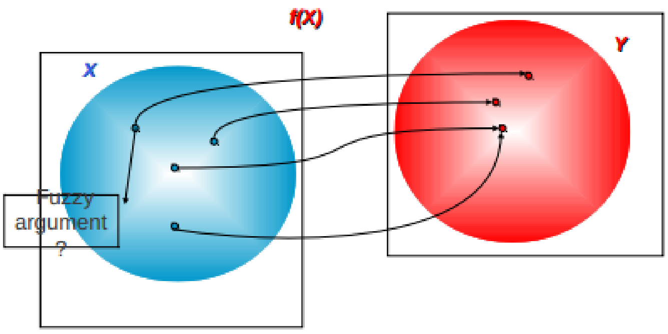
$$A=\begin{array}{ccccc} -2 & -1 & 0 & 1 & 2\\ \hline 0 & 0 & 1 & 1 & 0\end{array}$$

Then, B = A o R

$$B=\begin{array}{ccc} 2 & 6 & 10\\ \hline 1 & 1 & 0\end{array}$$

$$B=\{2,6\}$$

Fuzzy Mapping



Extension Principle for Fuzzy Sets

$\underset{\sim}{A}$: *fuzzy set defined on X*

$y = f(x)$: *functional transform or mapping*

$\underset{\sim}{B}$: *image of* $\underset{\sim}{A}$ *on X under f.*

$\underset{\sim}{B}$ *is a fuzzy set having universe of discourse Y.*

$$\underset{\sim}{B} = f\left(\underset{\sim}{A}\right)$$

$$\mu_{\underset{\sim}{B}}(y) = \bigvee_{f(x)=y} \mu_{\underset{\sim}{A}}(x)$$

Fuzzy Extension Principle

General definition: Suppose f is a mapping from an n-dimentional Cartesian product space $X_1 \times X_2 \times \ldots \times X_n$ to a one dimentional universe Y such that $y = f(x_1, x_2, \ldots, x_n)$ and suppose $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ are n fuzzy sets in $x_1, x_2, \ldots, x_n$ respectively. Then, the image of $\underset{\sim}{A}_1, \underset{\sim}{A}_2, \ldots, \underset{\sim}{A}_n$ under f is given as:

$$\boxed{\mu_B(y) = \max_{f(x_1, x_2, \ldots, x_n) = y} \left\{ \min(\mu_{\underset{\sim}{A}_1(x_1)}, \mu_{\underset{\sim}{A}_2(x_2)}, \ldots, \mu_{\underset{\sim}{A}_n(x_n)}) \right\}}$$

$\Downarrow$

Zadeh's extension principle

Fuzzy Extension Principle

Example:

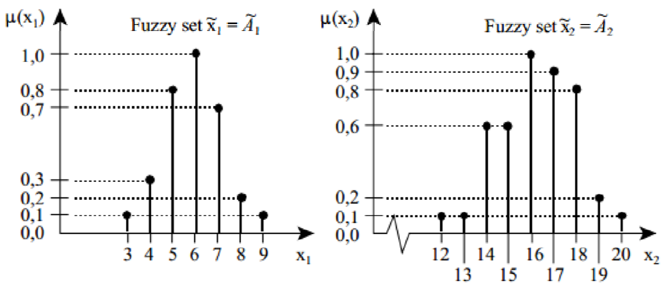
$$\underset{\sim}{A} = \left\{ \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.8}{0} + \frac{0.9}{1} + \frac{0.3}{2} \right\}, \quad f(x) = x^2 - 3$$

$$\underset{\sim}{B} = \frac{0.8}{-3} + \frac{(0.4 \vee 0.9)}{-2} + \frac{(0.1 \vee 0.3)}{1}$$

$$\underset{\sim}{B} = \left\{ \frac{0.8}{-3} + \frac{0.9}{-2} + \frac{0.3}{1} \right\}$$

Fuzzy Extension Principle

Example:



$$z = f(x_1; x_2) = 3 \cdot x_1 - x_2 + 5$$

Example: cont.

	(3; 0,1)	(4; 0,3)	(5; 0,8)	(6; 1,0)	(7; 0,7)	(8; 0,2)	(9; 0,1)
(12; 0,1)	(2; 0,1)	(5; 0,1)	(8; 0,1)	(11; 0,1)	(14; 0,1)	(17; 0,1)	(20; 0,1)
(13; 0,1)	(1; 0,1)	(4; 0,1)	(7; 0,1)	(10; 0,1)	(13; 0,1)	(16; 0,1)	(19; 0,1)
(14; 0,6)	(0; 0,1)	(3; 0,3)	(6; 0,6)	(9; 0,6)	(12; 0,6)	(15; 0,2)	(18; 0,1)
(15; 0,6)	(-1; 0,1)	(2; 0,3)	(5; 0,6)	(8; 0,6)	(11; 0,6)	(14; 0,2)	(17; 0,1)
(16; 1,0)	(-2; 0,1)	(1; 0,3)	(4; 0,8)	(7; 1,0)	(10; 0,7)	(13; 0,2)	(16; 0,1)
(17; 0,9)	(-3; 0,1)	(0; 0,3)	(3; 0,8)	(6; 0,9)	(9; 0,7)	(12; 0,2)	(15; 0,1)
(18; 0,8)	(-4; 0,1)	(-1; 0,3)	(2; 0,8)	(5; 0,8)	(8; 0,7)	(11; 0,2)	(14; 0,1)
(19; 0,2)	(-5; 0,1)	(-2; 0,2)	(1; 0,2)	(4; 0,2)	(7; 0,2)	(10; 0,2)	(13; 0,1)
(20; 0,1)	(-6; 0,1)	(-3; 0,1)	(0; 0,1)	(3; 0,1)	(6; 0,1)	(9; 0,1)	(12; 0,1)

Fuzzy Extension Principle

Definition: A fuzzy vector is a vector containing fuzzy membership values.

$$\vec{\underset{\sim}{a}} = \{a_1,\dots,a_n\} = \left\{\mu_{\underset{\sim}{A}(x_1)},\dots,\mu_{\underset{\sim}{A}(x_n)}\right\} = \left\{\mu_{\underset{\sim}{A}(x_i)}\right\},\ i = 1,\dots,n$$

$$\vec{\underset{\sim}{b}} = \{b_1,\dots,b_m\} = \left\{\mu_{\underset{\sim}{B}(x_1)},\dots,\mu_{\underset{\sim}{B}(x_m)}\right\} = \left\{\mu_{\underset{\sim}{B}(x_j)}\right\},\ j = 1,\dots,n$$

$\underset{\sim}{B} = f\left(\underset{\sim}{A}\right)$ can be determined directly from $\underset{\sim}{B} = \underset{\sim}{A} \circ \underset{\sim}{R}$ using vector form:

$\vec{\underset{\sim}{b}} = \vec{\underset{\sim}{a}} \circ \underset{\sim}{R}$ where $\underset{\sim}{R}$ is an nxm fuzzy relation matrix

Composition: max-min

$$\underset{\sim}{B}_j = \max_i\left(\min\left(a_i, r_{ij}\right)\right). \quad \underset{\sim}{B}_j : j^{th} \textit{ element of the fuzzy image } \underset{\sim}{B}.$$

Fuzzy Extension Principle

Definition: If the input is a sigle element(a fuzzy singleton), the image of this singleton will be fuzzy and this case is termed as fuzzy transform.

$$\underset{\sim}{B}_i = f(x_i) \Rightarrow \quad \mu_{\underset{\sim}{B}_i(x_j)} = r_{ij}$$

$$\underset{\sim}{R} = \begin{array}{c|c} & \begin{array}{ccc} y_1 & y_j & y_m \end{array} \\ \begin{array}{c} x_1 \\ \\ x_i \\ \\ x_n \end{array} & \begin{bmatrix} r_{11} \ldots\ldots & r_{1j} \ldots\ldots & r_{1m} \\ \vdots & \vdots & \vdots \\ r_{i1} & r_{ij} & r_{im} \\ \vdots & \vdots & \vdots \\ r_{n1} \ldots\ldots & r_{nj} \ldots\ldots & r_{nm} \end{bmatrix} \end{array}$$

$$\vec{\underset{\sim}{b}_i} = \{r_{i1}, r_{i2}, \ldots, r_{ij}, \ldots, r_{im}\}$$

Fuzzy Extension Principle

Fuzzy transform of the singleton x_i is given by the i^{th} row of the fuzzy relation R.

Definition: For a function f that performs a one-to-one mapping (i.e.,maps one element in universe U to one element in universe V), the mapping is said to be one-to-one.

Exmample: $Let\ U = \{1,2,3\}\ and\ \underset{\sim}{A} = \left\{\frac{0.6}{1}, \frac{0.9}{2}, \frac{1}{3}\right\}\ defined\ on\ U$

$$mapping: v = f(u) = 2u - 1 \Rightarrow f\left(\underset{\sim}{A}\right) = ?$$

$$U = \{1,2,3\} \Rightarrow V = \{1,3,5\}$$

$$and\ f\left(\underset{\sim}{A}\right) = \left\{\frac{0.6}{1} + \frac{0.9}{3} + \frac{1}{5}\right\}$$

Mapping of more than one input variable:

Suppose U_1 and U_2 inputs are mapped to V through $V = f(u_1, u_2)$, If the mapping is one-to-one, the same membership grades results but if the mapping is not one-to-one , maximum membership grades maping to the same output variable is accepted.

$$\mu_{\underset{\sim}{A}}(u_1, u_2) = \max_{f(u_1, u_2) = V} \left[\min\left(\mu_1(u_1), \mu_2(u_1)\right)\right]$$

Fuzzy Arithmetic and Fuzzy Numbers:

Let $\underset{\sim}{I}$ and $\underset{\sim}{J}$ be two fuzzy numbers with $\underset{\sim}{I}$ defined on X and $\underset{\sim}{J}$ defined on Y, and let the symbol * denote a general arithmetic operation.

$$* = \{+, -, \times, \div\}$$

Fuzzy Extension Principle

An arithmetic operation between these two fuzzy numbers, denoted $\underset{\sim}{I} * \underset{\sim}{J}$ is a mapping to another universe, say Z, and accomplished by using the extention principle:

$$\mu_{\underset{\sim}{I} * \underset{\sim}{J}}(z) = \bigvee_{x * y = z} \left(\mu_{\underset{\sim}{I}}(x) \wedge \mu_{\underset{\sim}{J}}(y)\right)$$

Example:

$$\underset{\sim}{A} = \underset{\sim}{2} = \text{"}approximately\ 2\text{"} = \left\{\frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3}\right\}$$

$$\underset{\sim}{B} = \underset{\sim}{6} = \text{"}approximately\ 6\text{"} = \left\{\frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7}\right\}$$

Fuzzy Extension Principle

Let's map the product of $\underset{\sim}{2}$ and $\underset{\sim}{6}$ to a fuzzy number $\underset{\sim}{12} = "approximately\ 12"$

Extension principle:

$$\underset{\sim}{2}\, x\, \underset{\sim}{6} = \left(\frac{0.6}{1} + \frac{1}{2} + \frac{0.8}{3}\right) x \left(\frac{0.8}{5} + \frac{1}{6} + \frac{0.7}{7}\right)$$

$$= \left\{\frac{\min(0.6, 0.8)}{5} + \frac{\min(0.6, 1)}{6} + ... + \frac{\min(0.8, 0.7)}{21}\right\}$$

$$= \left\{\frac{0.6}{5} + \frac{0.6}{6} + \frac{0.6}{7} + \frac{0.8}{10} + \frac{1}{12} + \frac{0.7}{14} + \frac{0.8}{15} + \frac{0.8}{18} + \frac{0.7}{21}\right\}$$

Fuzzy Extension Principle

$$\underset{\sim}{I} = \left\{\frac{0.2}{0} + \frac{1}{1} + \frac{0.2}{2}\right\} \quad Let's\ find\ \underset{\sim}{I} + \underset{\sim}{I}\ using\ the\ extention\ principible:$$

$$\underset{\sim}{1} + \underset{\sim}{1} = \underset{\sim}{2} = \left(\frac{0.2}{0} + \frac{1}{1} + \frac{0.2}{2}\right) + \left(\frac{0.2}{0} + \frac{1}{1} + \frac{0.2}{2}\right)$$

$$= \left\{\frac{\min(0.2, 0.2)}{0} + \frac{\max\left[\min(0.2, 0.1), \min(1, 0.2)\right]}{1} + \ldots + \frac{\min(0.2, 0.2)}{4}\right\}$$

$$= \left\{\frac{0.2}{0} + \frac{0.2}{1} + \frac{1}{2} + \frac{0.2}{3} + \frac{0.2}{4}\right\}$$

Fuzzy Extension Principle

Example:

$$\underset{\sim}{I} = \left\{ \frac{0.3}{1} + \frac{1}{2} + \frac{0.7}{3} \right\}$$

$$\underset{\sim}{J} = \left\{ \frac{0.2}{4} + \frac{1}{5} + \frac{0.6}{6} \right\}$$

$$\underset{\sim}{I} + \underset{\sim}{J} = \left(\frac{0.3}{1} + \frac{1}{2} + \frac{0.7}{3} \right) + \left(\frac{0.2}{4} + \frac{1}{5} + \frac{0.6}{6} \right)$$

$$\underset{\sim}{I} + \underset{\sim}{J} = \left\{ \frac{\min(0.3, 0.2)}{5} + \frac{\max(\min(0.3,1), \min(1,0.2))}{6} + \frac{\max(\min(0.3,0.6), \min(1,1), \min(0.7,0.2))}{7} \right.$$

$$\left. + \frac{\max(\min(1,0.6), \min(0.7,1))}{8} + \frac{\min(0.7,0.6)}{9} \right\} = \left\{ \frac{0.2}{5} + \frac{0.3}{6} + \frac{1}{7} + \frac{0.7}{8} + \frac{0.6}{9} \right\}$$

Fuzzy Extension Principle

Example: We have two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, each defined on its own universe as follows:

$$\underset{\sim}{A} = \left\{ \frac{0.2}{1} + \frac{1}{2} + \frac{0.7}{4} \right\} \quad \text{and} \quad \underset{\sim}{B} = \left\{ \frac{0.5}{1} + \frac{1}{2} \right\}$$

We wish to determine the membership values for the algebraic product mapping

$$f(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{A} \times \underset{\sim}{B} \text{ (arithmetic product)}$$

$$= \left\{ \frac{\min(0.2, 0.5)}{1} + \frac{\max[\min(0.2, 1), \min(0.5, 1)]}{2} \right.$$

$$\left. + \frac{\max[\min(0.7, 0.5), \min(1, 1)]}{4} + \frac{\min(0.7, 1)}{8} \right\}$$

$$= \left\{ \frac{0.2}{1} + \frac{0.5}{2} + \frac{1}{4} + \frac{0.7}{8} \right\}$$