NATURAL AND STEP RESPONSES OF RLC CIRCUITS

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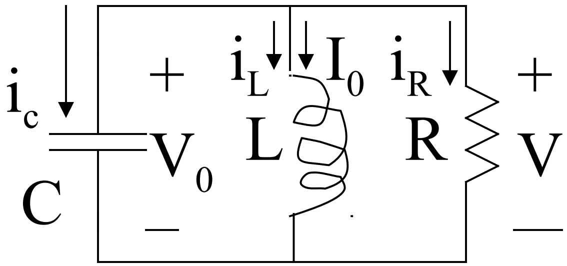
NATURAL RESPONSE OF A PARALLEL RALC CIRCUIT



$$\frac{V}{R} + \frac{1}{L}\int_0^t V d\tau + I_0 + C\frac{dV}{dt} = 0$$

$$\frac{1}{R}\frac{dV}{dt} + \frac{V}{L} + C\frac{d^2V}{dt^2} = 0$$

$$\frac{d^2V}{dt^2} + \frac{1}{RC}\frac{dV}{dt} + \frac{V}{LC} = 0$$

Ordinary, second-order differential equation with constant coefficients. Therefore, this circuit is called a second-order circuit.

GENERAL SOLUTION OF SECOND-ORDER DIFFERENTIAL EQUATIONS

Assume that the solution of the differential equation is of exponential form $V=Ae^{st}$ where A and s are unknown constants.

$$As^2e^{st}+\frac{As}{RC}e^{st}+\frac{Ae^{st}}{LC}=0$$

$$Ae^{st}(s^2+\frac{s}{RC}+\frac{1}{LC})=0$$

This equation can be satisfied for all values of t only if A=0 or the term in parentheses is zero.

A=0 cannot be used as a general solution because to do so implies that the voltage is zero for all time-a physical impossibility if energy is stored in either inductor or capacitor.

THE CHARACTERISTIC EQUATION

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$ is called the characteristic equation

The two roots of the characteristic equation are

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

If either s_1 or s_2 is substituted into Ae^{st}, the assumed solution satisfies the differential equation, regardless of the value of A

$$V_1 = A_1 e^{s_1 t}, \qquad V_2 = A_2 e^{s_2 t}$$

These two solutions as well as their summation $V=V_1+V_2$ satisfy the differential equation

$$V = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

FREQUENCIES

The behavior of V(t) depends on the values of s_1 and s_2.

Writing the roots in a notation as widely used in the literature

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2\mathrm{RC}} \qquad \omega_0^2 = \frac{1}{\sqrt{LC}}$$

The exponent of e must be dimensionless, so both s_1 and s_2 (and hence α and ω_0) must have the dimension of the reciprocal of time (frequency). s_1 and s_2 are referred to as complex frequencies, α is called the neper frequency, and ω_0 is called the resonant radian frequency. They have the unit radians per second (rad/s)

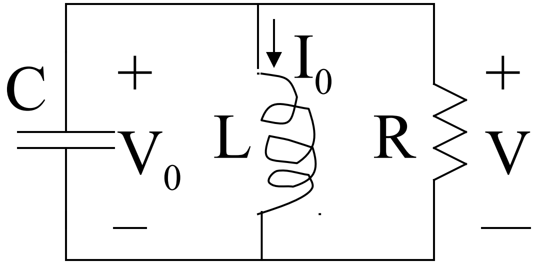
The nature of the roots s_1 and s_2 depends on the values of α and ω_0. There are three possible cases:

1) If $\omega_0^2<\alpha^2$, both roots will be real and distinct. The voltage response is said to be **overdamped**.

2) If $\omega_0^2>\alpha^2$, both roots will be complex and, in addition, will be conjugates of each other. The voltage response is said to be **underdamped**.

3) If $\omega_0^2=\alpha^2$, roots will be real and equal. The voltage response is said to be **critically damped**.

EXAMPLE



Find the roots of the characteristic equation if R=200 Ω, L=50 mH, and C=0.2 μF.

$$\alpha = \frac{1}{2RC} = \frac{10^6}{(400)(0.2)} = 1.25 \times 10^4 \text{ rad / s}$$

Overdamped

$$\omega_0^2 = \frac{1}{LC} = \frac{10^3(10^6)}{(50)(0.2)} = 10^8 \text{ rad}^2 / s^2$$

$$s_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8} = -5000 \text{ rad / s}$$

$$s_2 = -1.25 \times 10^4 - \sqrt{1.5625 \times 10^8 - 10^8} = -20000 \text{ rad / s}$$

Repeat the problem for R=312.5 Ω

$$\alpha = \frac{1}{2RC} = \frac{10^6}{(625)(0.2)} = 8000 \text{ rad / s}$$

Underdamped

$$\omega_0^2 = \frac{1}{LC} = \frac{10^3(10^6)}{(50)(0.2)} = 10^8 \text{rad}^2 / s^2$$

$$s_1 = -8000 + \sqrt{0.64 \times 10^8 - 10^8} = -8000 + j6000 \text{ rad / s}$$

$$s_2 = -8000 - \sqrt{0.64 \times 10^8 - 10^8} = -8000 - j6000 \text{ rad / s}$$

Find the value of R for a critically damped circuit.

For critical damping, $\alpha^2=\omega_0^2$

$$\left(\frac{1}{2RC}\right)^2=\frac{1}{LC}=10^8$$

$$\frac{1}{RC}=10^4 \Rightarrow R=\frac{10^6}{(2\times 10^4)(0.2)}=250\Omega$$

THE OVERDAMPED RESPONSE

When the roots of the characteristic equation are real and distinct, the response is said to be overdamped in the form

$$V(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

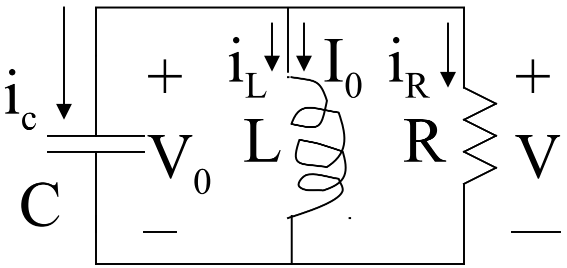
The constant A_1 and A_2 are to be determined by the initial conditions $V(0^+)$ and $dV(0^+)/dt$.

$$V(0^+) = A_1 + A_2, \qquad \frac{dV(0^+)}{dt} = s_1 A_1 + s_2 A_2$$

$$V(0^+) = V_0 \qquad \frac{dV(0^+)}{dt} = \frac{i_c(0^+)}{C}$$

$$i_c(0^+) = \frac{-V_0}{R} - I_0$$

EXAMPLE



Fin V(t), if R=200 Ω, L=50 mH, and C=0.2 μF. V_0=12V, I_0=30 mA.

$$i_L(0^+) = I_0 = 30mA$$

$$i_R(0^+) = \frac{V_c(0^+)}{R} = \frac{V_0}{R} = \frac{12}{200} = 60mA$$

$$i_c(0^+) = -i_L(0^+) - i_R(0^+) = -90mA$$

$$\frac{dV(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{-90\times10^{-3}}{0.2\times10^{-6}} = -450kV/s$$

From the previous example, we determined s_1=-5000 rad/s and s_2=-20000 rad/s. Then

$$V(t) = A_1 e^{-5000t} + A_2 e^{-20000t}$$

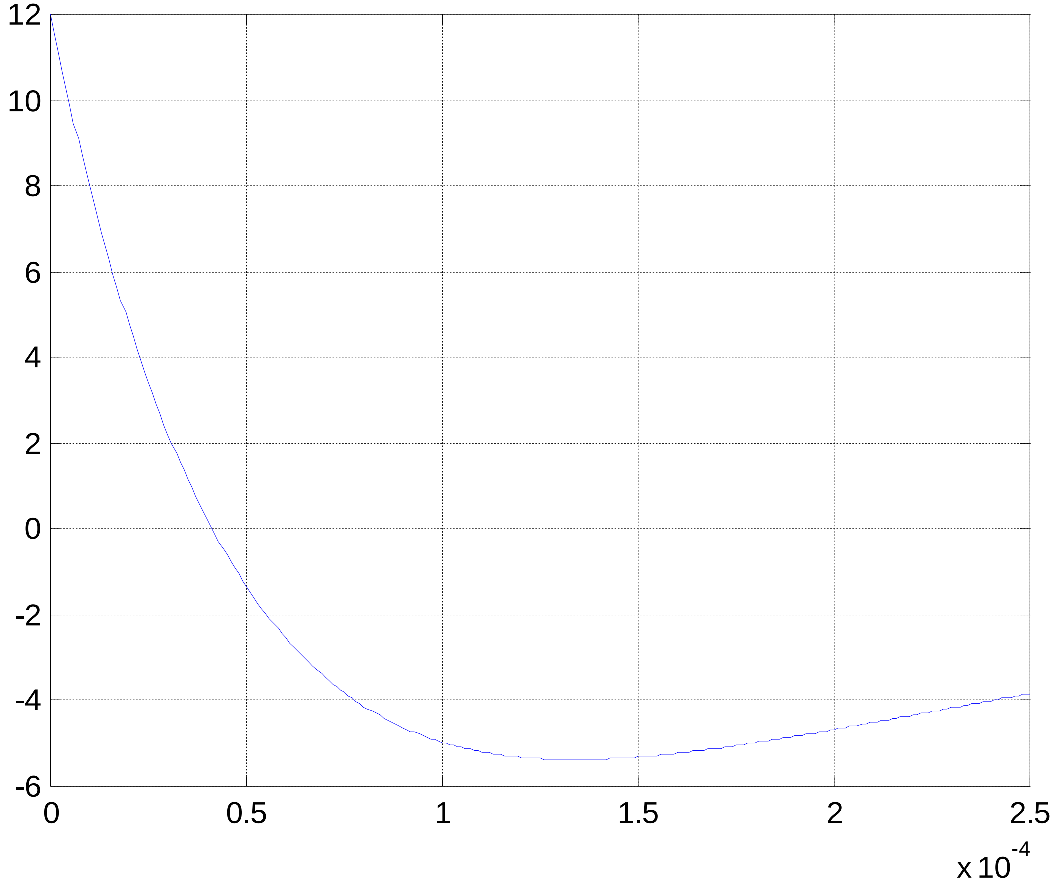
$$V(0^+) = A_1 + A_2 = 12$$

$$\frac{dV(t)}{dt} = -5000 A_1 e^{-5000t} - 20000 A_2 e^{-20000t}$$

$$\frac{dV(0^+)}{dt} = -5000 A_1 - 20000 A_2 = -450 \times 10^3$$

$$A_1 = -14V \quad A_2 = 26V$$

$$V(t) = (-14e^{-5000t} + 26e^{-20000t})V \quad t \geq 0$$


12
10
8
6
4
2
0
-2
-4
-6
0
0.5
1
1.5
2
2.5
x 10^-4

$$i_R(t) = \frac{V(t)}{200} = (-70e^{-5000t} + 130e^{-20000t})mA \quad t \geq 0$$

$$i_c(t) = C\frac{dV(t)}{dt} = 0.2 \times 10^{-6}(70000e^{-5000t} - 520000e^{-20000t})$$

$$= (14e^{-5000t} - 104e^{-20000t})mA \quad t \geq 0^+$$

$$i_L(t) = -i_R(t) - i_c(t)$$

$$= (56e^{-5000t} - 26e^{-20000t})mA \quad t \geq 0$$

THE UNDERDAMPED RESPONSE

When $\omega_0^2 > \alpha^2$, the roots of the characteristic equation are complex, and the response is underdamped.

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

ω_d is called the **damped radian frequency**.

$$
\begin{aligned}
V(t) &= A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t} \\
&= A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t} \\
&= e^{-\alpha t}(A_1 \cos\omega_d t + jA_1 \sin\omega_d t + A_2 \cos\omega_d t - jA_2 \sin\omega_d t) \\
&= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t]
\end{aligned}
$$

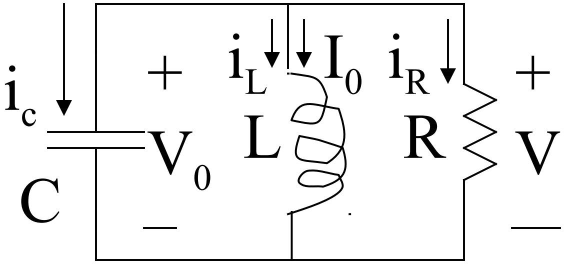
A_1 and A_2 are complex conjugates. Therefore, their sum is a real number and their difference is imaginary. Then $j(A_1 - A_2)$ is also a real number. Denoting $B_1 = A_1 + A_2$, and $B_2 = j(A_1 - A_2)$

$$
V(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)
$$

DAMPING FACTOR

The trigonometric functions indicate that the response is oscillatory; that is, the voltage alternates between positive and negative values. The rate at which the voltage oscillates is fixed with ω_d. The rate at which the amplitude decreases is determined by α. Because α determines how quickly the amplitude decreases, it is called as the **damping factor**. If there is no damping, $\alpha=0$ and the frequency of oscillations is ω_0. When there is a dissipative element, R, in the circuit, α is not zero and the frequency of oscillations is, ω_d, less than ω_0. Thus, when α is not zero, the frequency of oscillation is said to be damped.

EXAMPLE



Find V(t) if R=20 kΩ, L=8H, C=0.125μF, V_0=0, and I_0=-12.25 mA

$$\alpha = \frac{1}{2RC} = \frac{10^6}{2(20)10^3(0.125)} = 200 \text{ rad / s}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{10^6}{8(0.125)}} = 10^3 \text{ rad / s}$$

$$\omega_0^2 > \alpha^2 \Rightarrow \omega_d = \sqrt{\omega_0^2 - \alpha^2} = 979.8 \text{ rad / s}$$

$$s_1 = -\alpha + j\omega_d = -200 + j979.8 \text{ rad / s}$$

$$s_2 = -\alpha - j\omega_d = -200 - j979.8 \text{ rad / s}$$

$V(0^+)=V_0=0$, then $i_R(0^+)=V(0^+)/R=0$.

$i_c(0^+)=-i_L(0^+)=12.25$ mA

$$\frac{dV(0^+)}{dt}=\frac{12.25\times10^{-3}}{0.125\times10^{-6}}=98000\ \mathrm{V/s}$$

$$B_1=0,\quad B_2=\frac{98000}{\omega_d}\approx 100V$$

$$V(t)=100e^{-200t}\sin 979.8t\ \ V\quad t\geq 0$$

THE CRITICALLY DAMPED RESPONSE

The second-order circuit is critically damped when $\alpha^2=\omega_0^2$. The two roots of the characteristic equation are equal

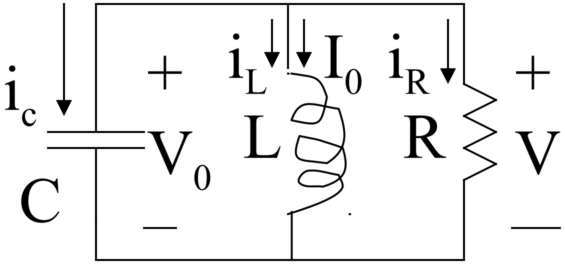
$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

For a critically damped circuit, the solution takes the following form

$$V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

D_1 and D_2 are constant which must be determined using the initial conditions $V(0^+)$ and $dV(0^+)/dt$

EXAMPLE



Determine the value of R for a critically damped response when, L=8H, C=0.125μF, V_0=0, and I_0=-12.25 mA

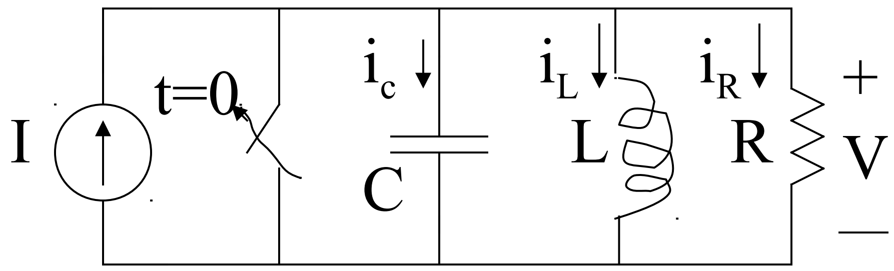
From the previous example $\omega_0^2=10^6$. Then

$$\alpha = 10^3 = \frac{1}{2RC} \Rightarrow R = 4000\Omega$$

Again, from the previous example $V(0^+)=0$ and $dV(0^+)/dt=98000$ V/s
Then $D_1=0$ and $D_2=98000$ V/s.

$$V(t) = 98000te^{-1000t}V \quad t \geq 0$$

THE STEP RESPONSE OF A PARALLEL RLC CIRCUIT



$$i_L + i_R + i_c = I$$

$$i_L + \frac{V}{R} + C\frac{dV}{dt} = I$$

$$V = L\frac{di_L}{dt} \Rightarrow \frac{dV}{dt} = L\frac{d^2 i_L}{dt^2}$$

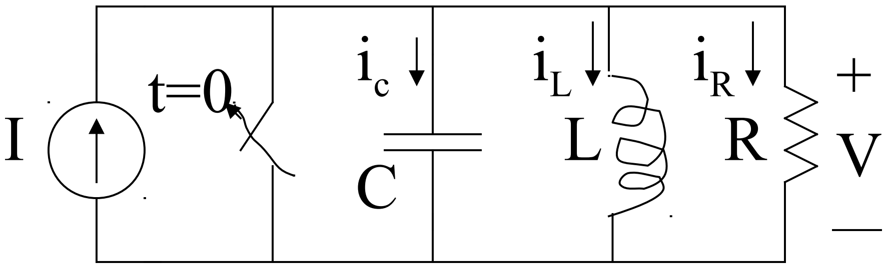
$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

The equation describing the step response of a second-order circuit is a second-order differential equation with constant coefficients and with a constant forcing function. The solution of this differential equation equals the forced response which is in the same form of the forcing function (constant for a step input) plus a response function identical in form to the natural response. Thus, the solution for the inductor current is in the form

$$i_L = I_f + \begin{Bmatrix} \text{function of the same form} \\ \text{as the natural response} \end{Bmatrix}$$

EXAMPLE



C=25nF, L=25mH, R=400Ω
The initial energy in the circuit is zero. I=24 mA.
Find $i_L(t)$

Since there is no initial energy in the circuit, $i_L(0^+)=0$ and $V(0^+)=0$

$V(0^+)=L[di_L(0^+)/dt]=0$, then $di_L(0^+)/dt=0$

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16\times 10^8$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{2(400)(25)} = 5\times 10^4 \text{ rad / s}$$

$$\alpha^2 = 25\times 10^8 > \omega_0^2$$

Overdamped circuit

$$s_1 = -5\times10^4 + 3\times10^4 = -20000 \text{ rad / s}$$

$$s_2 = -5\times10^4 - 3\times10^4 = -80000 \text{ rad / s}$$

As t$\longrightarrow\infty$, circuit reaches dc steady state where inductor is short and capacitor is open. All of the input current flows through the inductor. Then I_f=24 mA.

$$i_L(t) = 24\times10^{-3} + A_1e^{-20000t} + A_2e^{-80000t}\,A$$

$$i_L(0^+) = 24\times10^{-3} + A_1 + A_2$$

$$\frac{di_L}{dt} = -20000A_1e^{-20000t} - 80000A_2e^{-80000t}$$

$$\frac{di_L(0^+)}{dt} = -20000A_1 - 80000A_2 = 0$$

$$A_1 = -32mA \qquad A_2 = 8mA$$

$$i_L(t) = (24 - 32e^{-20000t} + 8e^{-80000t})mA \quad t \geq 0$$

EXAMPLE: If the resistor in the circuit is increased to 625Ω, find $i_L(t)$ in the circuit.

Since L and C remain fixed, resonant frequency has the same value. But neper frequency decreases to 3.2×10^4 rad/s. With these values circuit is underdamped with complex conjugate roots.

$$s_1 = -3.2\times10^4 + j2.4\times10^4 \text{ rad / s}$$

$$s_2 = -3.2\times10^4 - j2.4\times10^4 \text{ rad / s}$$

$$i_L(t) = I_f + B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

$$I_f = 24\times10^{-3} A$$

$$i_L(0^+) = 24\times10^{-3} + B_1 = 0$$

$$\frac{di_L}{dt} = -\alpha e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)$$

$$+\,\omega_d e^{-\alpha t}(-B_1\sin\omega_d t + B_2\cos\omega_d t)$$

$$\frac{di_L(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = 0$$

$$B_1 = -24mA \quad B_2 = -32mA$$

$$i_L(t) = (24 - e^{-32000t}(24\cos 24000t + 32\sin 24000t))mA \quad t \geq 0$$

EXAMPLE: Find $i_L(t)$ if R=500Ω

The resonant frequency remains the same, but neper frequency becomes $4\text{x}10^4$ rad/s. These values correspond to critical damping. Roots of the characteristic equation are real and equal at s=-40000

$$i_L(t) = I_f + D_1 t e^{-40000t} + D_2 e^{-40000t} A$$

$$I_f = 24 \times 10^{-3} A$$

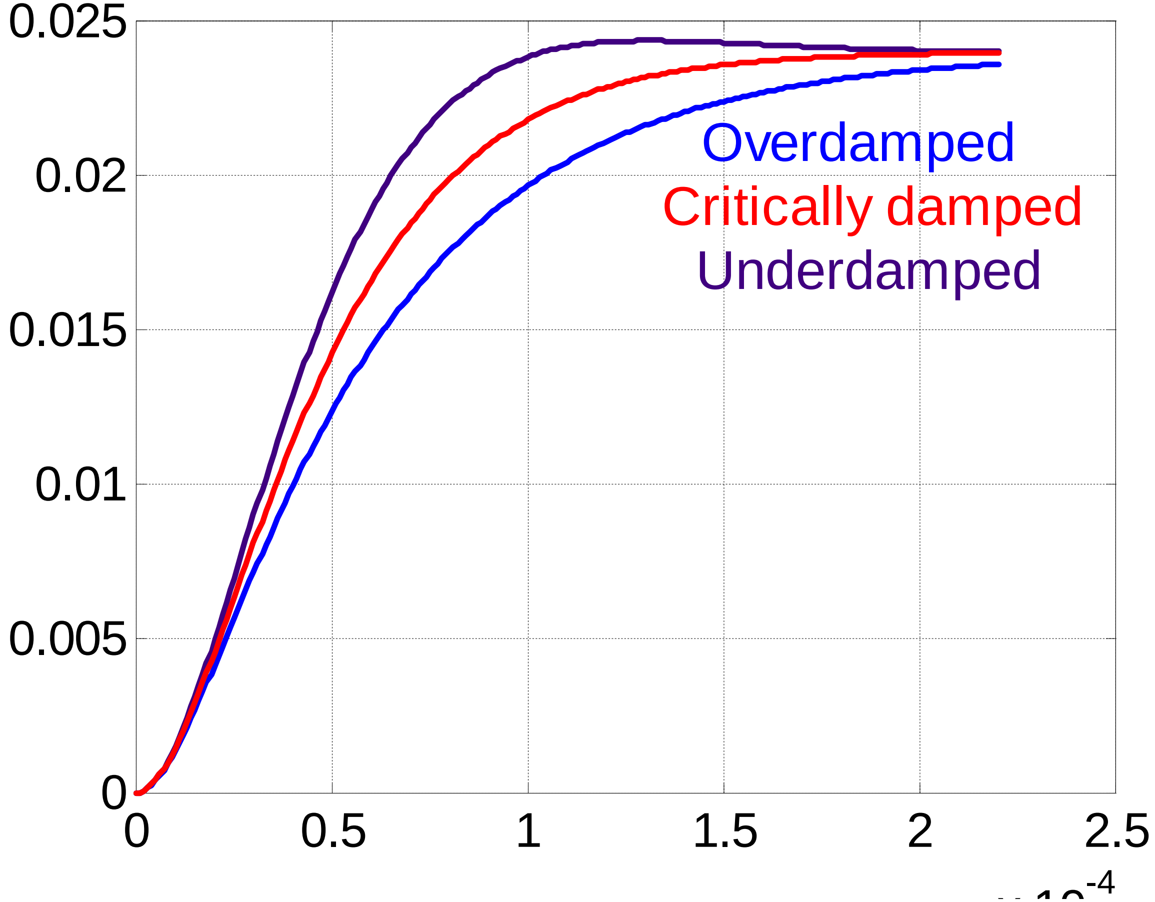
$$i_L(0^+) = 24 \times 10^{-3} + D_2 = 0$$

$$\frac{di_L}{dt} = -\alpha e^{-\alpha t}(D_1 t + D_2) + D_1 e^{-\alpha t}$$

$$\frac{di_L(0^+)}{dt} = -\alpha D_2 + D_1 = 0$$

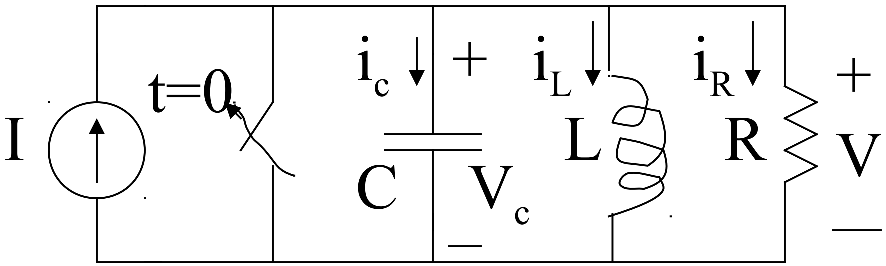
$$D_1 = -960000 mA/s \quad D_2 = -24 mA$$

$$i_L(t) = (24 - 960000 t e^{-40000t} - 24 e^{-40000t}) mA \quad t \geq 0$$



0.025
0.02
0.015
0.01
0.005
0
0
0.5
1
1.5
2
2.5
Overdamped
Critically damped
Underdamped

EXAMPLE



C=25nF, L=25mH, R=500Ω The initial current in the inductor is 29 mA, the initial voltage across the capacitor is 50 V. I=24 mA. Find $i_L(t)$ and V(t)

From the previous example, we know that this circuit is critically damped with $s_1=s_2=-40000$ rad/s

$$i_L(0^+) = i_L(0^-) = 29mA$$

$$V_c(0^+) = V_c(0^-) = 50V$$

$$V_L(0^+) = V_c(0^+) = L\frac{di_L(0^+)}{dt} \Rightarrow \frac{di_L(0^+)}{dt} = \frac{50}{25\times 10^{-3}} = 2000\,A/s$$

$$i_L(t) = I_f + D_1 t e^{-40000t} + D_2 e^{-40000t}\,A$$

$$I_f = 24\times 10^{-3}\,A$$

$$i_L(0^+) = 24\times 10^{-3} + D_2 = 29\times 10^{-3} \Rightarrow D_2 = 5\times 10^{-3}\,A$$

$$\frac{di_L(0^+)}{dt} = D_1 - \alpha D_2 = 2000 \Rightarrow D_1 = 2200\,A/s$$

$$i_L(t) = (24 + 2.2\times 10^{6} t e^{-40000t} + 5e^{-40000t})mA \quad t \geq 0$$

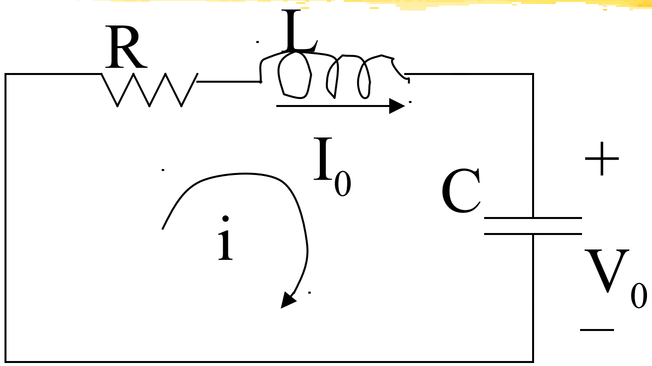
$$V(t) = L\frac{di_L}{dt}$$

$$= (25\times 10^{-3})[(2.2\times 10^{6})(-40000)te^{-40000t}$$

$$+ 2.2\times 10^{6}e^{-40000t} + 5(-40000)e^{-40000t}]\times 10^{-3}$$

$$= -2.2\times 10^{6}te^{-40000t} + 50e^{-40000t}V$$

THE NATURAL AND STEP RESPONSE OF A SERIES RLC CIRCUIT



$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau + V_0 = 0$$

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

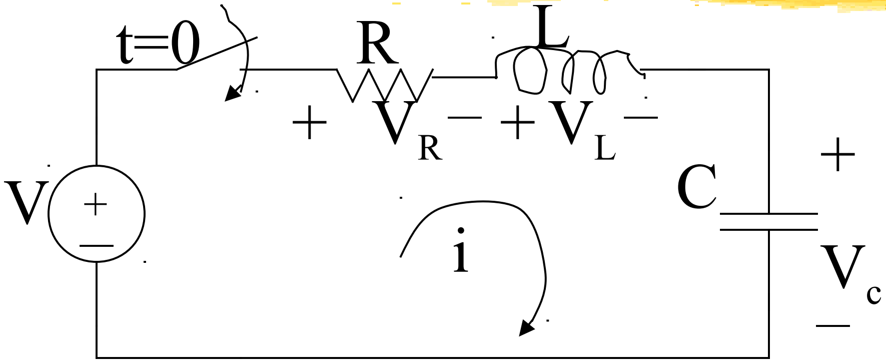
$$\alpha = \frac{R}{2L} \text{ rad/s} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

These equations are in the same form that of the equations for the parallel RLC circuit. Therefore, the solution will be overdamped, critically damped, or underdamped depending on relative magnitudes of the resonant frequency and neper frequency.

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(overdamped)}$$

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t \ \text{(underdamped)}$$

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} \quad \text{(critically damped)}$$



$$V = Ri + L\frac{di}{dt} + V_c$$

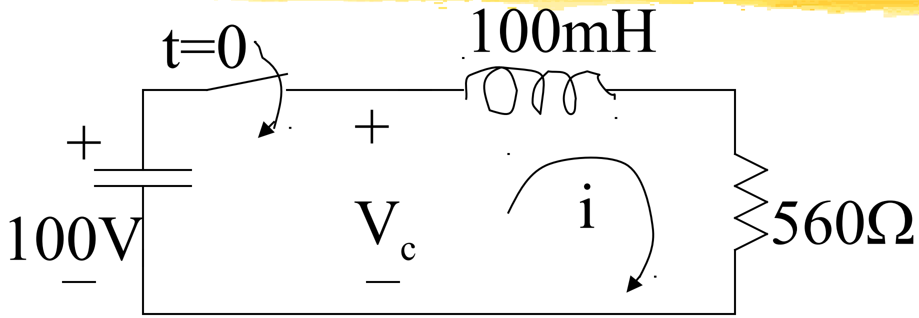
$$i = C\frac{dV_c}{dt} \Rightarrow \frac{di}{dt} = C\frac{d^2V_c}{dt^2}$$

$$\frac{d^2V_c}{dt^2} + \frac{R}{L}\frac{dV_c}{dt} + \frac{V_c}{LC} = \frac{V}{LC}$$

$$V_c(t) = V_f + A_1e^{s_1t} + A_2e^{s_2t} \text{ (overdamped)}$$

$$V_c(t) = V_f + B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t \text{ (underdamped)}$$

$$V_c(t) = V_f + D_1te^{-\alpha t} + D_2e^{-\alpha t} \text{ (critically damped)}$$



The capacitor is initially charged to 100V. At t=0, switch closes. Find i(t) and V_c(t) for t≥0

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{100(0.1)} = 10^8$$

$$\alpha = \frac{R}{2L} = \frac{560}{2(100)} \times 10^3 = 2800 \text{ rad / s}$$

$$\alpha^2 = 7.84 \times 10^6 < \omega_0^2 \Rightarrow \text{Underdamped}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 9600 \text{ rad / s}$$

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

$$i(0^+) = 0 = B_1$$

$$-V_c(0^+) + Ri(0^+) + L\frac{di(0^+)}{dt} = 0$$

$$\frac{di(0^+)}{dt} = \frac{V_c(0^+)}{L} = \frac{100}{100} \times 10^3 = 1000\, A/s$$

$$\frac{di}{dt} = -2800 B_2 e^{-2800t} \sin 9600t + 9600 B_2 e^{-2800t} \cos 9600t$$

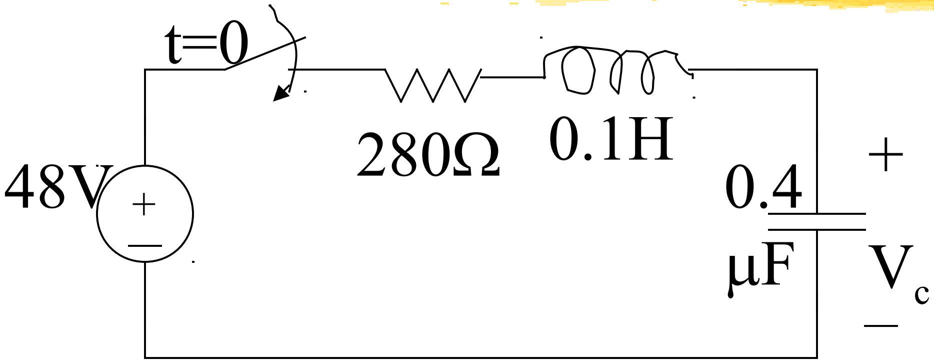
$$\frac{di(0^+)}{dt} = 9600 B_2 = 1000 \Rightarrow B_2 = \frac{1000}{9600} A$$

$$i(t) = \frac{1}{9.6} e^{-2800t} \sin 9600t \text{ A} \quad t \geq 0$$

$$V_c(t) = iR + L\frac{di}{dt}$$

$$V_c(t) = (100\cos 9600t + 29.17\sin 9600t)e^{-2800t} V \quad t \geq 0$$

EXAMPLE



There is no initial stored energy in the circuit at t=0. Find $V_c(t)$ for $t \geq 0$.

$$s_1 = -\frac{280}{2(0.1)} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$

$$= -1400 + j4800 \text{ rad / s}$$

$$s_2 = -1400 - j4800 \text{ rad / s}$$

$$V_c(t) = V_{cf} + B_1 e^{-1400t} \cos 4800t + B_2 e^{-1400t} \sin 4800t, \quad t \geq 0$$

As t $\longrightarrow \infty$, circuit reaches dc steady state (inductor short, capacitor open). Thus, V_{cf}=48V

$$V_c(0^+) = 0, \quad \frac{dV_c(0^+)}{dt} = \frac{i_c(0^+)}{C} = \frac{i_L(0^+)}{C} = 0$$

$$V_c(0^+) = 48 + B_1 = 0 \Rightarrow B_1 = -48V$$

$$\frac{dV_c}{dt} = -1400e^{-1400t}(B_1 \cos 4800t + B_2 \sin 4800t)$$

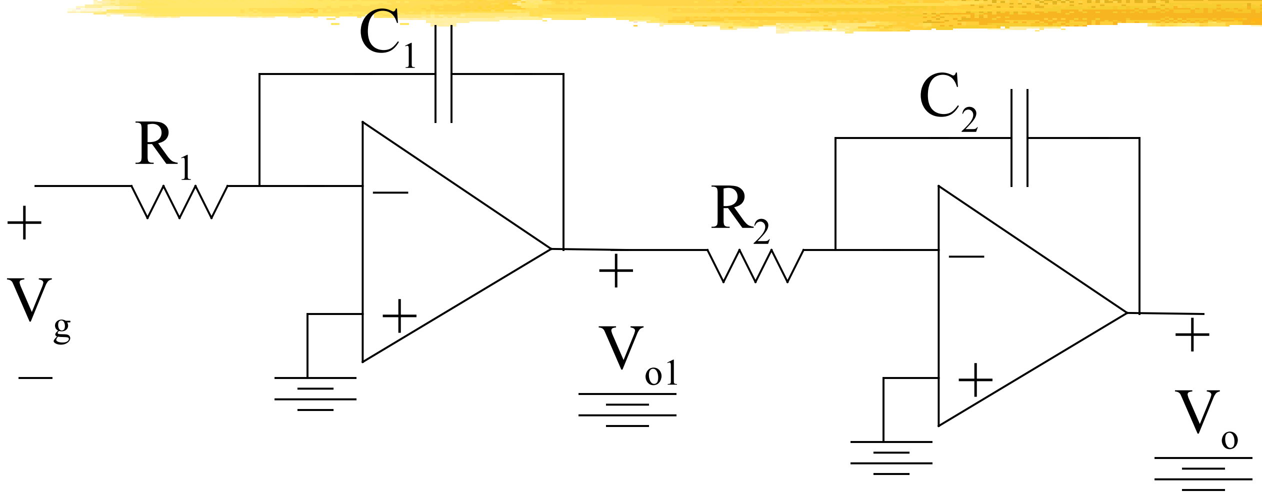
$$+ 4800e^{-1400t}(B_2 \cos 4800t - B_1 4800t)$$

$$\frac{dV_c(0^+)}{dt} = 4800B_2 - 1400B_1 = 0$$

$$B_2 = -14V$$

$$V_c(t) = (48 - 48e^{-1400t}\cos 4800t - 14e^{-1400t}\sin 4800t)V, \quad t \geq 0$$

A CIRCUIT WITH TWO INTEGRATING AMPLIFIERS



Assuming ideal opamps, find the relation between V_o and V_g

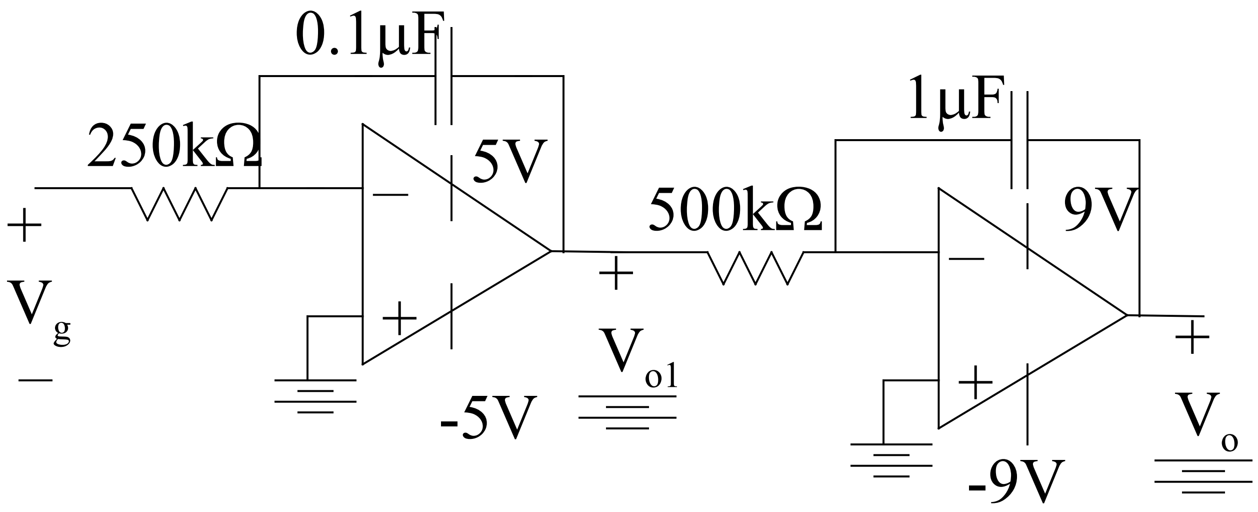
$$\frac{0-V_g}{R_1}+C_1\frac{d}{dt}(0-V_{o1})=0 \Rightarrow \frac{dV_{o1}}{dt}=-\frac{1}{R_1C_1}V_g$$

$$\frac{0 - V_{o1}}{R_2} + C_2 \frac{d}{dt}(0 - V_o) = 0 \Rightarrow \frac{dV_o}{dt} = -\frac{1}{R_2 C_2} V_{o1}$$

$$\frac{d^2 V_0}{dt^2} = -\frac{1}{R_2 C_2} \frac{dV_{o1}}{dt}$$

$$\frac{d^2 V_0}{dt^2} = \frac{1}{R_1 C_1} \frac{1}{R_2 C_2} V_g$$

EXAMPLE



No energy is stored in the circuit when the input voltage V_g jumps instantaneously from 0 to 25 mV.

Derive the expression for $V_o(t)$ for $0 \le t \le t_{sat}$.

How long is it before the circuit saturates?

$$\frac{1}{R_1C_1} = \frac{1000}{(250)(0.1)} = 40$$

$$\frac{1}{R_2C_2} = \frac{1000}{(500)(1)} = 2$$

$$\frac{d^2V_o}{dt^2} = 40(2)(25 \times 10^{-3}) = 2$$

$$\frac{dV_o}{dt} = \int_0^t 2dx = 2t$$

$$V_o = \int_0^t 2xdx = t^2 \quad 0 \leq \mathrm{t} \leq \mathrm{t}_{\mathrm{sat}}$$

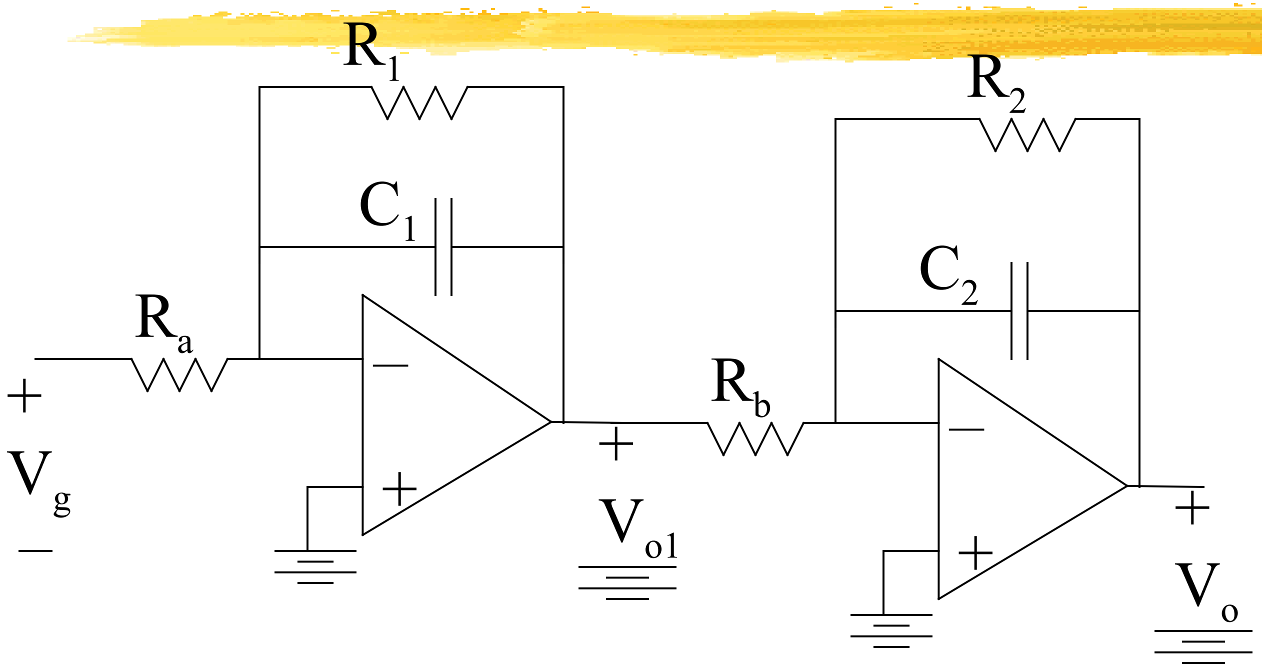
The second integrating amplifier saturates when V_o reaches 9V or t=3s. But it is possible that the first opamp saturates before t=3s. To explore this possibility use the following equation

$$\frac{dV_{o1}}{dt} = -\frac{1}{R_1 C_1} V_g = -40(25 \times 10^{-3}) = -1$$

$$V_{o1} = -t$$

Thus, at t=3s, V_{o1}=-3V. The first opamp does not reach saturation at t=3s. The circuit reaches saturation when the second amplifier saturates.

TWO INTEGRATING AMPLIFIERS WITH FEEDBACK RESISTORS



The reason that the op amp saturates in the integrating amplifier is the feedback capacitor's accumulation of charge. To overcome this problem, a resistor is placed in parallel with each feedback capacitor.

$$\frac{0-V_g}{R_a}+\frac{0-V_{o1}}{R_1}+C_1\frac{d}{dt}(0-V_{o1})=0$$

$$\frac{dV_{o1}}{dt}+\frac{1}{R_1C_1}V_{o1}=-\frac{V_g}{R_aC_1}$$

Let $\tau_1=R_1C_1 \Rightarrow \frac{dV_{o1}}{dt}+\frac{V_{o1}}{\tau_1}=-\frac{V_g}{R_aC_1}$

$$\frac{0-V_{o1}}{R_b}+\frac{0-V_o}{R_2}+C_2\frac{d}{dt}(0-V_o)=0$$

$$\frac{dV_o}{dt}+\frac{V_o}{\tau_2}=-\frac{V_{o1}}{R_bC_2}, \qquad \tau_2=R_2C_2$$

$$\frac{d^2V_0}{dt^2} + \frac{1}{\tau_2}\frac{dV_o}{dt} = -\frac{1}{R_bC_2}\frac{dV_{o1}}{dt}$$

$$\frac{dV_{o1}}{dt} = -\frac{V_{o1}}{\tau_1} - \frac{V_g}{R_aC_1}$$

$$V_{o1} = -R_bC_2\frac{dV_o}{dt} - \frac{R_bC_2}{\tau_2}V_o$$

$$\frac{d^2V_o}{dt^2} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\frac{dV_o}{dt} + \left(\frac{1}{\tau_1\tau_2}\right)V_o = \frac{V_g}{R_aC_1R_bC_2}$$

The characteristic equation is

$$s^2 + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)s + \frac{1}{\tau_1 \tau_2} = 0$$

The roots of the characteristic equation are real

$$s_1 = -\frac{1}{\tau_1} \qquad s_2 = -\frac{1}{\tau_2}$$

EXAMPLE

The parameters of the circuit are R_a=100 kΩ, R_b=25 kΩ, R_1=500 kΩ, R_2=100 kΩ, C_1=0.1μF, and C_2=1μF. The power supply of each op amp is ± 6V. The input voltage jumps from 0 to 250 mV at t=0. No energy is stored in the feedback capacitors at the instant the input is applied. Find $V_o(t)$ for t≥0.

$$\tau_1 = R_1C_1 = 0.05s \qquad \tau_{21} = R_2C_2 = 0.1s$$

$$\frac{V_g}{R_aC_1R_bC_2} = 1000 \text{ V / s}^2$$

$$\frac{d^2V_o}{dt^2} + 30\frac{dV_o}{dt} + 200V_o = 1000$$

The characteristic equation is $s^2+30s+200=0$. Then s_1=-20 rad/s and s_2=-10 rad/s. The final values of the output is the input voltage times the gain of each stage, because capacitors behave as open circuits as $t \rightarrow \infty$

$$V_o(\infty) = (250 \times 10^{-3})\frac{-500}{100}\frac{-100}{25} = 5V$$

$$V_o(t) = 5 + A_1 e^{-10t} + A_2 e^{-20t}$$

$$V_o(0^+) = 0 \quad \text{and} \quad \frac{dV_o(0^+)}{dt} = 0$$

$$A_1 = -10V \qquad A_2 = 5V$$

$$V_o(t) = (5 - 10e^{-10t} + 5e^{-20t})V \quad t \geq 0$$