Sinusoidal Steady-state Analysis

Osman Parlaktuna
Osmangazi University
Eskisehir, TURKEY

THE SINUSOIDAL SOURCE

A sinusoidal voltage (current) source produces a voltage (current) that varies sinusoidally with time.

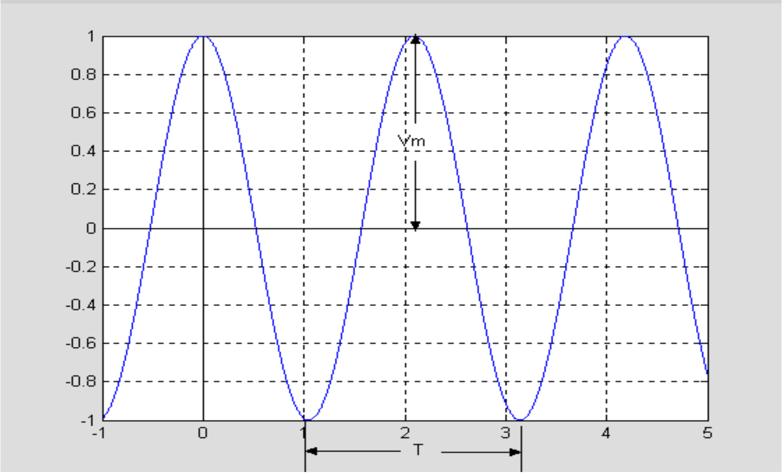
$$v(t) = V_m \cos(\omega t + \theta)$$

V_m is the amplitude,

ω is the frequency in rad/sec.

T: Period of the signal

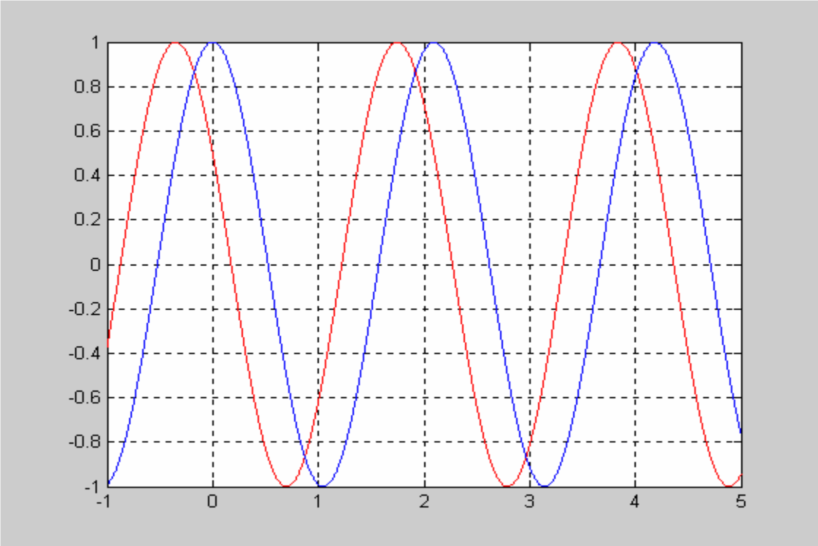
$f = \frac{1}{T}$ is the frequency in Hz. $\omega = 2\pi f$



Plot of cos(3t)

PHASE ANGLE

θ in the equation is called the phase angle of the sinusoidal voltage and it determines the value of the function at t=0. Changing the phase angle shifts the function along the time axis but has no effect on the amplitude or the angular frequency.



--cos3t, --cos(3t+30⁰)

cos(3t+30^0) waveform ***leads*** cos3t by 30^0

Or

cos3t ***lags*** cos(3t+30^0) by 30^0

RMS (root mean square) VALUE

The **rms value** of a periodic function is defined as the square root of the mean value of the squared function.

$$v(t) = V_m \cos(\omega t + \theta) \Rightarrow V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \theta)dt} = \frac{V_m}{\sqrt{2}}$$

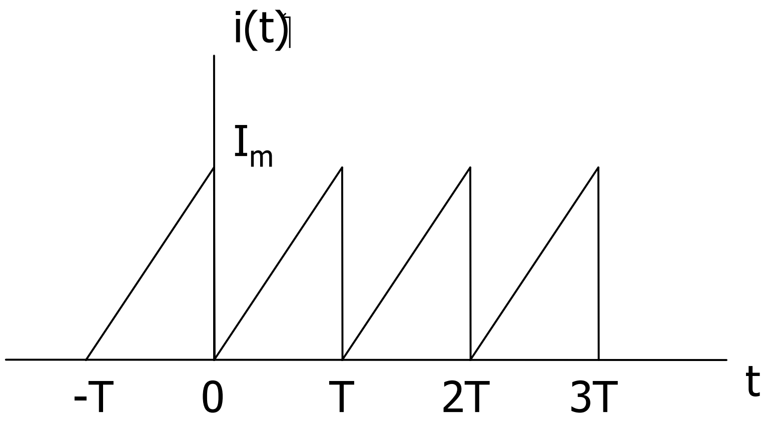
Example: $v(t) = 300\cos(120\pi t + 30^0)$

$$\omega = 120\pi \text{ rad/s}, \quad T = \frac{2\pi}{\omega} = \frac{1}{60} = 16.667 ms$$

$$f = \frac{1}{T} = 60 Hz$$

$$V_{rms} = \frac{300}{\sqrt{2}} = 212.13 V_{rms}$$

EXAMPLE



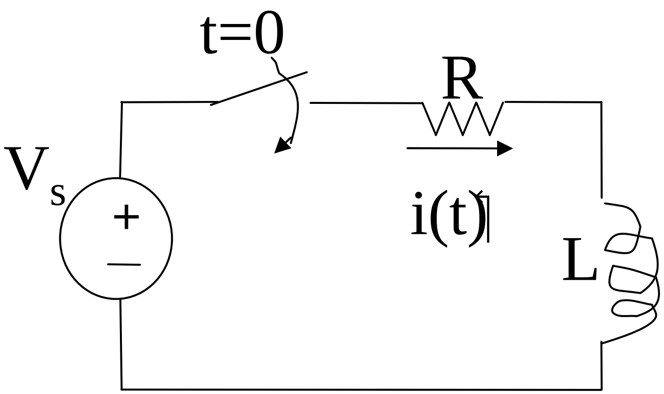
Find the rms value of i(t)

$$i(t) = \frac{I_m}{T} t \quad 0 \le t \le T$$

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2 dt} = \sqrt{\frac{1}{T}\int_0^T \frac{I_m^2}{T^2} t^2 dt} = \frac{I_m}{\sqrt{3}}$$

THE SINUSOIDAL RESPONSE

$$v_s(t) = V_m \cos(\omega t + \theta)$$



Assume that the initial current in the circuit is zero, find i(t) for t>=0.

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \theta)$$

The forced response must have the general form

$$i(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

$$L(-I_1\omega \sin \omega t + I_2\omega \cos \omega t) + R(I_1 \cos \omega t + I_2 \sin \omega t) = V_m \cos \omega t$$

$$(-LI_1\omega + RI_2)\sin \omega t + (LI_2\omega + RI_1 - V_m)\cos \omega t = 0$$

$$-LI_1\omega + RI_2 = 0 \qquad \omega LI_2 + RI_1 - V_m = 0$$

$$I_1 = \frac{RV_m}{R^2+\omega^2L^2} \qquad \mathrm{I}_2 = \frac{\omega LV_m}{R^2+\omega^2L^2}$$

$$i(t) = \frac{RV_m}{R^2+\omega^2L^2}\cos\omega t + \frac{\omega LV_m}{R^2+\omega^2L^2}\sin\omega t$$

A simpler equation may be obtained by expressing i(t) as a single sinusoid with a phase angle

$$i(t) = A\cos(\omega t - \theta) = A\cos\theta\cos\omega t + A\sin\theta\sin\omega t$$

$$A\cos\theta = \frac{RV_m}{R^2+\omega^2L^2} \qquad A\sin\theta = \frac{\omega LV_m}{R^2+\omega^2L^2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\omega L}{R} \Rightarrow \theta = \tan^{-1}\frac{\omega L}{R}$$

$$A^2\cos^2\theta + A^2\sin^2\theta = \frac{V_m^2}{R^2+\omega^2L^2}$$

$$A = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}$$

$$i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t - \tan^{-1}\frac{\omega L}{R})$$

Current lags the voltage by $\tan^{-1}\left(\frac{\omega L}{R}\right)$

If w=0 or L=0, the current must be in phase with the voltage.

If R=0, current lags the voltage by 90^0

THE PHASOR

The phasor is a complex number that carries the amplitude and phase angle information of a sinusoidal function.

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\cos\theta = \mathrm{Re}(e^{j\theta}) \quad \text{and} \quad \sin\theta = \mathrm{Im}(e^{j\theta})$$

$$v(t) = V_m \cos(\omega t + \theta) = V_m \mathrm{Re}\left\{e^{j(\omega t+\theta)}\right\}$$
$$= V_m \mathrm{Re}\left\{e^{j\omega t} e^{j\theta}\right\} = \mathrm{Re}\left\{V_m e^{j\theta} e^{j\omega t}\right\}$$

$V_m e^{j\theta}$ Carries the amplitude and phase angle of the given sinusoidal function and is defined as the phasor representation

$\mathbf{V} = V_m e^{j\theta}$ transforms the sinusoidal function from the time domain to the complex number domain, which is also called the frequency domain.

$$\mathbf{V} = V_m \cos\theta + jV_m \sin\theta$$ Rectangular form

$$\mathbf{V} = V_m \underline{/\theta}$$ Angle notation

Inverse Phasor transform: Given $\mathbf{V} = V_m e^{j\theta}$ what is the corresponding time domain function?

$$v(t) = V_m \cos(\omega t + \theta)$$

$$v(t) = v_1(t) + v_2(t) + \cdots + v_n(t)$$

Where all the voltages are sinusoidal voltages of the same frequency, then

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n$$

Example

$y_1 = 20\cos(\omega t - 30^0)$ and $y_2 = 40\cos(\omega t + 60^0)$

express $y = y_1 + y_2$ as a single sinusoid.

Working in the time domain:

$$y = 20\cos(\omega t - 30^0) + 40\cos(\omega t + 60^0)$$

$$= (20\cos 30^0 + 40\cos 60^0)\cos\omega t + (20\sin 30^0 - 40\sin 60^0)\sin\omega t$$

$$= 37.32\cos\omega t - 24.64\sin\omega t = \sqrt{37.32^2 + 24.64^2}\cos(\omega t + \tan^{-1}\frac{24.64}{37.32})$$

$$= 44.72\cos(\omega t + 33.43^0)$$

Using phasors

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = 20\angle -30^0 + 40\angle 60^0$$

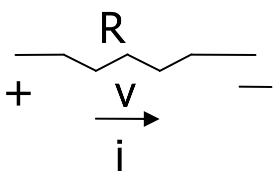
$$= (17.32 - j10) + (20 + j34.64)$$

$$= 37.32 + j24.64$$

$$= 44.72\angle 33.43^0$$

$$y = 44.72\cos(\omega t + 33.43^0)$$

The V-I Relationship for a Resistor

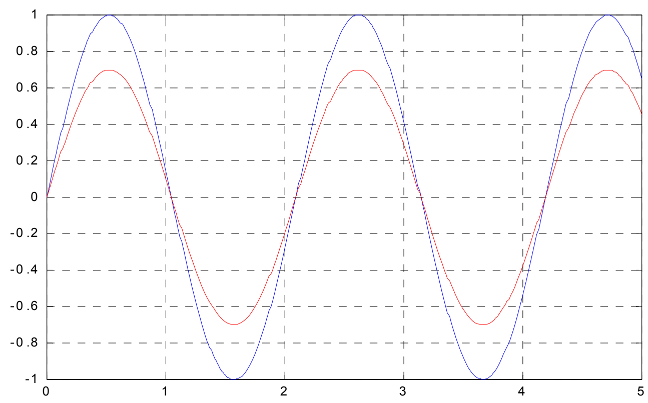


$$i = I_m \cos(\omega t + \theta_i) \Rightarrow v = RI_m \cos(\omega t + \theta_i)$$

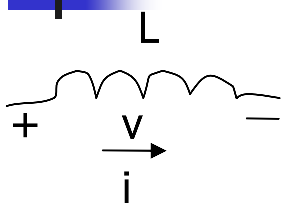
$$\mathbf{V} = RI_m e^{j\theta_i} = RI_m \angle \theta_i$$

$$\mathbf{I} = I_m \angle \theta_i \Rightarrow \mathbf{V} = R\mathbf{I}$$

The phase angle between voltage and current for a resistor is zero for a resistor. The signals are said to be **in phase**



The V-I Relationship for an Inductor

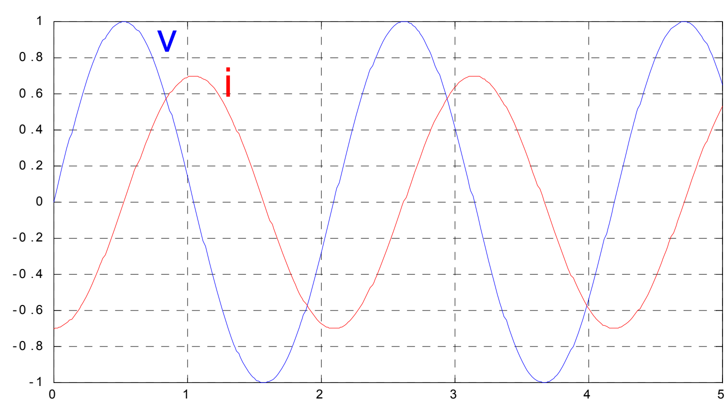


$$i = I_m \cos(\omega t + \theta_i) \Rightarrow v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i)$$

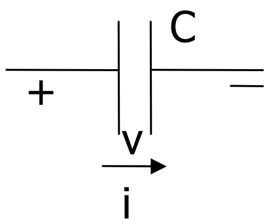
$$v = -\omega L I_m \cos(\omega t + \theta_i - 90^0)$$

$$\mathbf{V} = -\omega L I_m e^{j(\theta_i - 90^0)} = -\omega L I_m e^{j\theta_i} e^{-j90^0} = j\omega L I_m e^{j\theta_i} = j\omega L\mathbf{I}$$

The phase difference between voltage and current for an inductor is exactly 90^0 and voltage leads current or current lags voltage



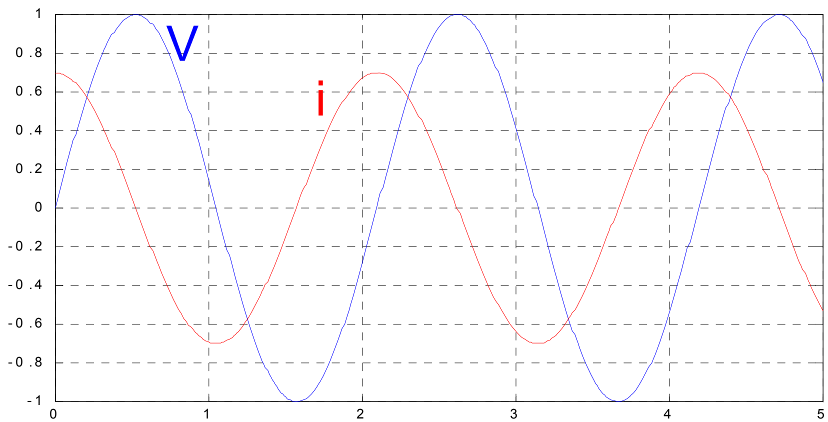
The V-I Relationship for a Capacitor



$$v = V_m \cos(\omega t + \theta_v) \Rightarrow i = C\frac{dv}{dt}$$

$$\mathbf{I} = j\omega C\mathbf{V} \Rightarrow \mathbf{V} = \frac{1}{j\omega C}\mathbf{I}$$

Voltage lags the current exactly by 90^0 for a capacitor.



Impedance and Reactance

Voltage current relationship for a resistor, inductor or capacitor is

$\mathbf{V} = Z\mathbf{I}$ Where Z represents the **impedance** of the circuit element

Impedance in the frequency domain is the quantity analogous to resistance, inductance, and capacitance in the time domain. The imaginary part of the impedance is called **reactance**

Element	impedance	reactance
Resistor	R	---
Inductor	jwL	wL
Capacitor	1/jwC	1/wC

Impedance is measured in Ohms

Kirchhoff's Laws in the Frequency Domain

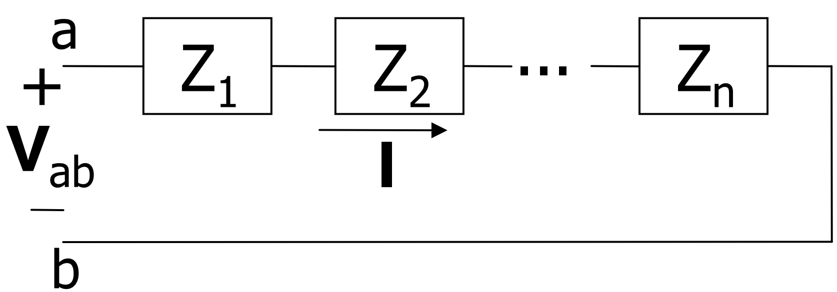
Voltage Law: Sum of the voltages around a closed path in a circuit is zero

$$v_1 + v_2 + \cdots + v_n = 0 \Rightarrow \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Current Law: Sum of the currents entering (leaving) a node is zero

$$i_1 + i_2 + \cdots + i_n = 0 \Rightarrow \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0$$

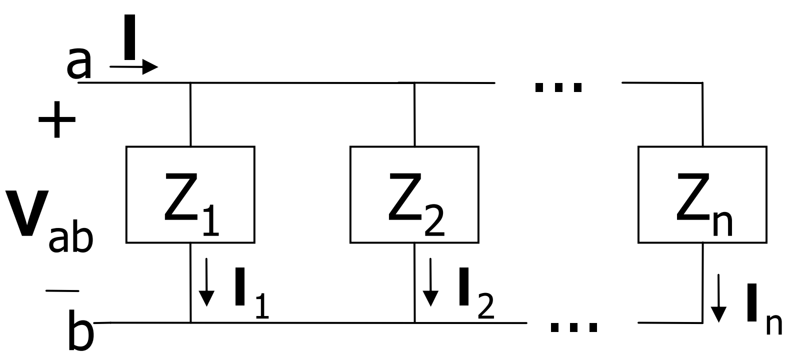
Series and Parallel Impedances



$$\mathbf{V}_{ab} = Z_1\mathbf{I} + Z_2\mathbf{I} + \ldots + Z_n\mathbf{I}$$

$$= (Z_1 + Z_2 + \ldots + Z_n)\mathbf{I}$$

$$Z_{ab} = Z_1 + Z_2 + \ldots + Z_n$$



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \ldots \mathbf{I}_n$$

$$\frac{\mathbf{V}_{ab}}{Z_{ab}} = \frac{\mathbf{V}_{ab}}{Z_1} + \frac{\mathbf{V}_{ab}}{Z_2} + \cdots + \frac{\mathbf{V}_{ab}}{Z_n}$$

$$\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$$

ADMITTANCE

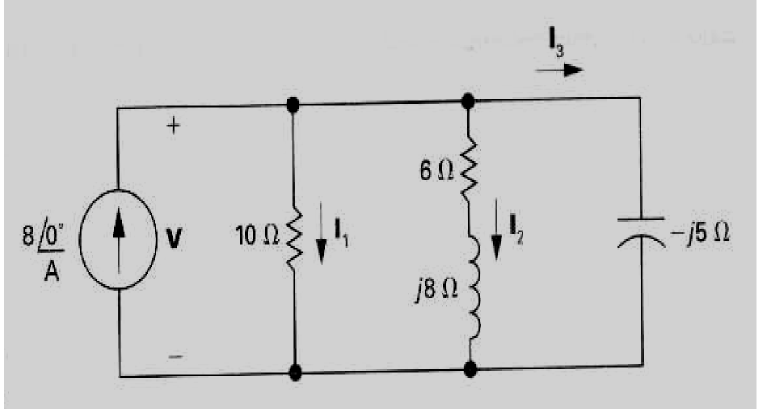
Admittance is defined as the reciprocal of impedance

$$Y = \frac{1}{Z} = G + jB$$

G is called conductance and B is called susceptance.

Unit of admittance is mhos

EXAMPLE



Determine the currents and voltage **V** in the circuit.

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = \frac{\mathbf{V}}{10} + \frac{\mathbf{V}}{6+j8} + \frac{\mathbf{V}}{-j5}$$

$$\mathbf{I} = 8\angle 0^0 \Rightarrow \mathbf{V} = 40\angle{-36.87^0} = 32 - j24V$$

$$\mathbf{I}_1 = \frac{40\angle{-36.87^0}}{10} = 4\angle{-36.87^0}A$$

$$\mathbf{I}_2 = \frac{40\angle{-36.87^0}}{6+j8} = 4\angle{-90^0} = -j4A$$

$$\mathbf{I}_3 = \frac{40\angle{-36.87^0}}{-j5} = 8\angle 53.13^0 = 4.8 + j6.4A$$

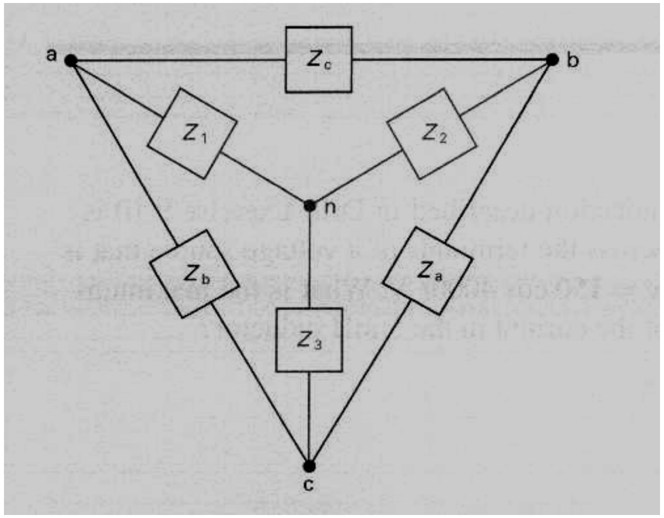
$$v = 40\cos(200000t - 36.87^0)V,$$

$$i_1 = 4\cos(200000t - 36.87^0)A,$$

$$i_2 = 4\cos(200000t - 90^0)A,$$

$$i_3 = 8\cos(200000t + 53.13^0)A$$

DELTA-to-Wye Transformations



$$\Delta \rightarrow Y$$

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c} \qquad Z_2 = \frac{Z_a Z_c}{Z_a + Z_b + Z_c}$$

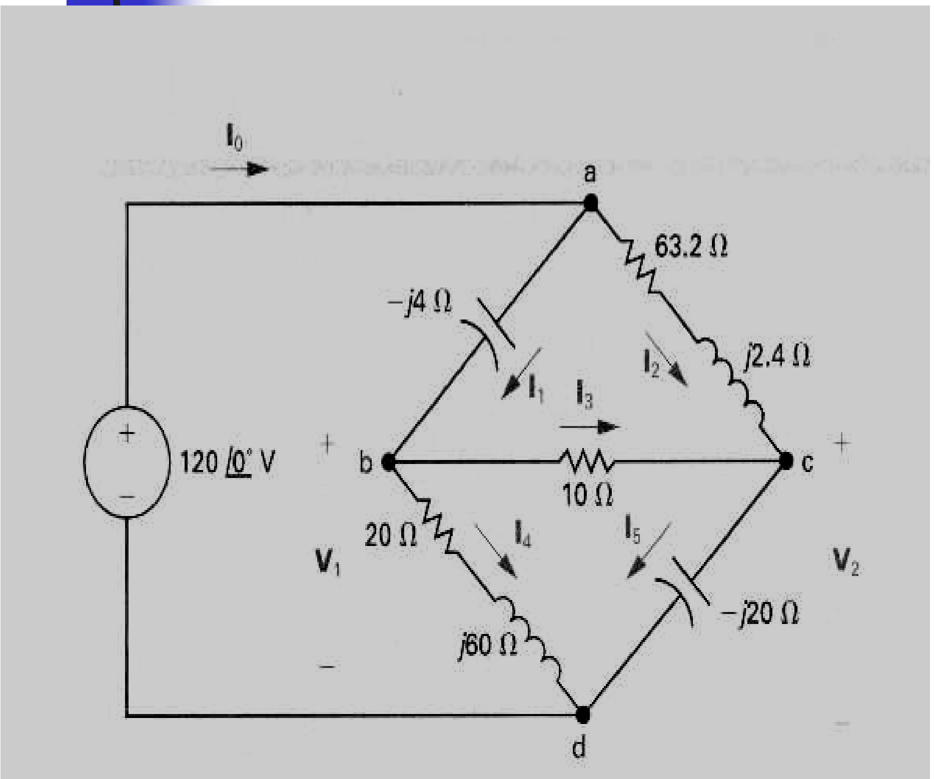
$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

$$Y \rightarrow \Delta$$

$$Z_a = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_1} \qquad Z_b = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3}{Z_3}$$

Example



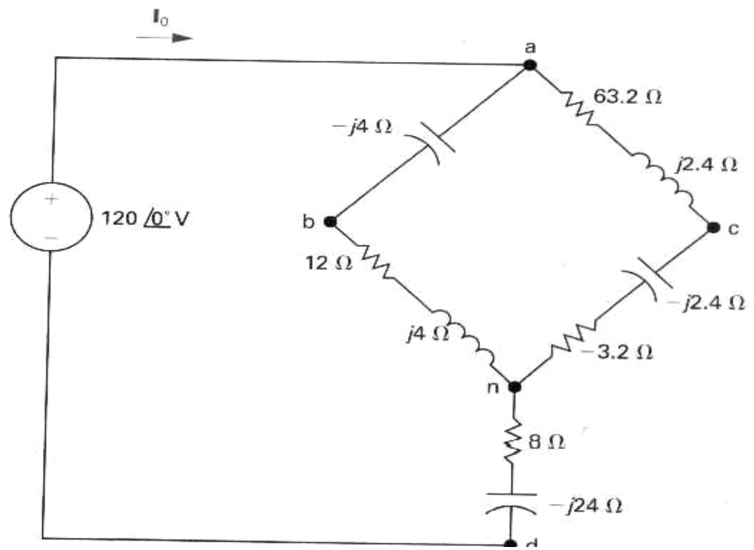
Use a delta-to-wye transformation to find currents and voltages in the circuit

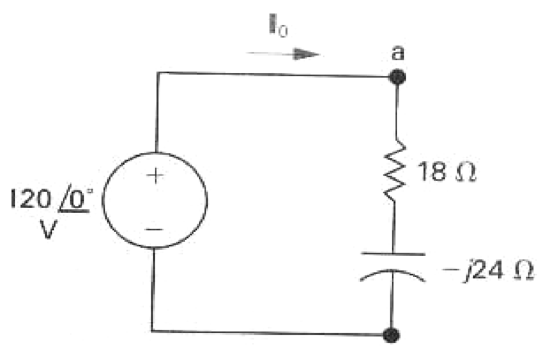
Replacing the lower delta by its Y equivalent:

$$Z_1 = \frac{(20 + j60)10}{30 + j40} = 12 + j4\Omega$$

$$Z_2 = \frac{10(-j20)}{30 + j40} = -3.2 - j2.4\Omega$$

$$Z_3 = \frac{(20 + j60)(-j20)}{30 + j40} = 8 - j2.4\Omega$$





$$Z_{abn} = 12 + j4 - j4 = 12\Omega,$$

$$Z_{acn} = 63.2 + j2.4 - 3.2 - j2.4 = 60\Omega$$

$$Z_{an} = \frac{60 \cdot 12}{72} = 10\Omega$$

$$\mathbf{I}_0 = \frac{120\angle 0^0}{18 - j24} = 4\angle 53.13^0 = 2.4 + j3.2A$$

$$\mathbf{V}_{nd} = (8 - j2.4)\mathbf{I}_0 = 96 - j32V$$

$$\mathbf{V}_{an} = 120 - 96 + j32 = 24 + j32V$$

$$\mathbf{I}_1 = \frac{24 + j32}{12} = 2 + j\frac{8}{3}A, \quad \mathbf{I}_2 = \frac{24 + j32}{60} = \frac{4}{10} + j\frac{8}{15}A$$

$$\mathbf{V}_1 = 120 - (-j4)\mathbf{I}_1 = \frac{328}{3} + j8V$$

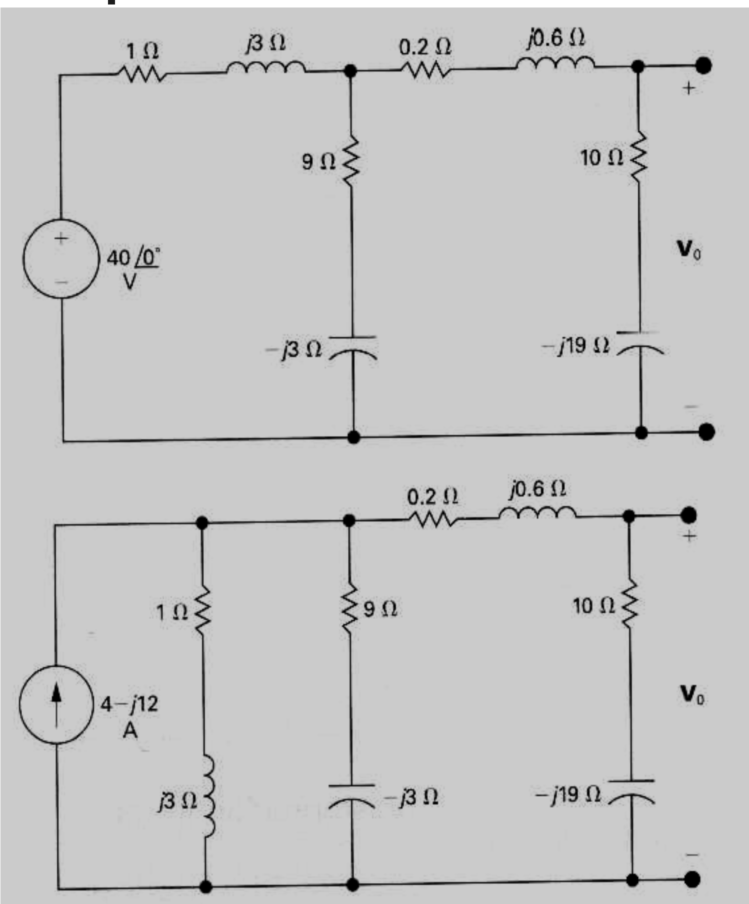
$$\mathbf{V}_2 = 120 - (63.2 + j3.2)\mathbf{I}_2 = 96 - j\frac{104}{3}V$$

$$\mathbf{I}_3 = \frac{\mathbf{V}_1 - \mathbf{V}_2}{10} = \frac{4}{3} + j\frac{12.8}{3}A$$

$$\mathbf{I}_4 = \frac{\mathbf{V}_1}{20 + j60} = \frac{2}{3} - j1.6A$$

$$\mathbf{I}_5 = \frac{\mathbf{V}_2}{-j20} = \frac{26}{15} + j4.8A$$

SOURCE TRANSFORMATIONS



Find $\mathbf{V}_0$ using successive source transformations

$$\mathbf{I}_0 = \frac{40}{1+j3} = 4 - j12A$$

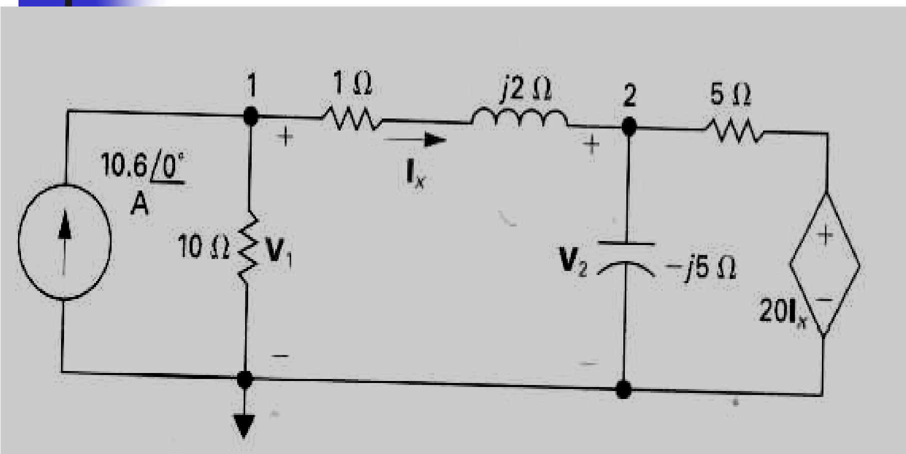
$$Z = \frac{(1+j3)(9-j3)}{10} = 1.8 + j2.4\Omega$$

$$\mathbf{V} = (4-j12)(1.8+j2.4) = 36 - j12V$$

$$\mathbf{I}_0 = \frac{36-j12}{12-j16} = 1.56 + j1.08A$$

$$\mathbf{V}_0 = (1.56+j0.8)(10-j19) = 36.12 - j18.84V$$

THE NODE-VOLTAGE METHOD



Use the node-voltage method to find $\mathbf{V}_1$ and $\mathbf{V}_2$ in the circuit.

$$-10.6+\frac{\mathbf{V}_1}{10}+\frac{\mathbf{V}_1-\mathbf{V}_2}{1+j2}=0$$

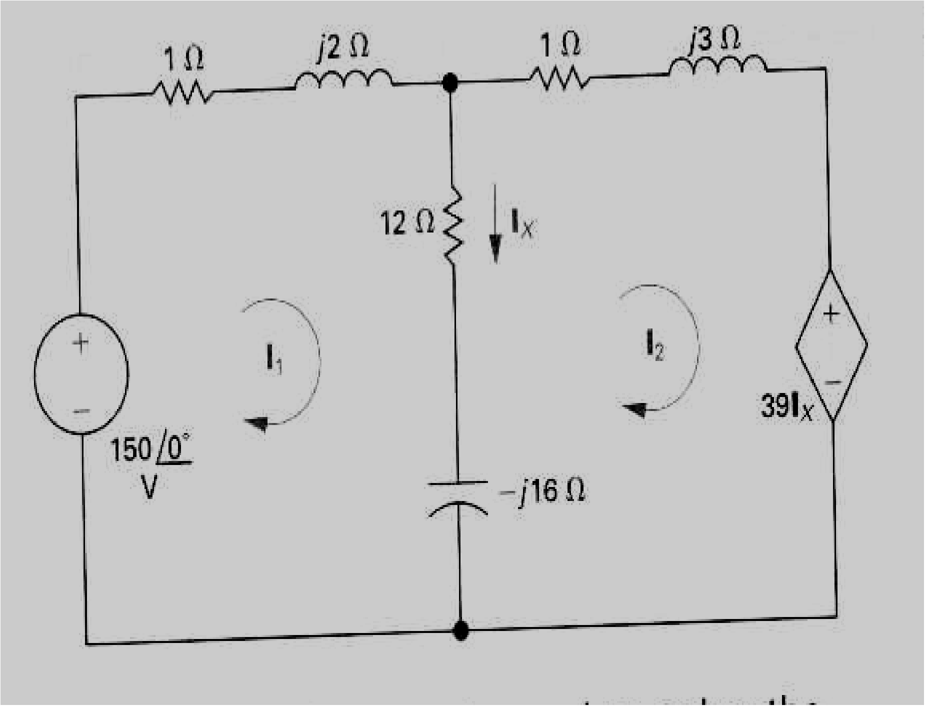
$$\frac{\mathbf{V}_2-\mathbf{V}_1}{1+j2}+\frac{\mathbf{V}_2}{-j5}+\frac{\mathbf{V}_2-20\mathbf{I}_x}{5}=0$$

$$\mathbf{I}_x=\frac{\mathbf{V}_1-\mathbf{V}_2}{1+j2}$$

$$\mathbf{V}_1=68.4-j16.8V$$

$$\mathbf{V}_2=68-j26V$$

THE MESH-CURRENT METHOD



Use the mesh-current method to find $\mathbf{I}_1$ and $\mathbf{I}_2$

$$150 = (1 + j2)\mathbf{I}_1 + (12 - j16)(\mathbf{I}_1 - \mathbf{I}_2)$$

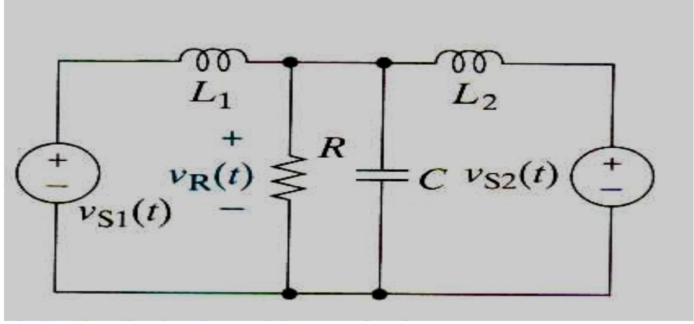
$$0 = (12 - j16)(\mathbf{I}_2 - \mathbf{I}_1) + (1 + j3)\mathbf{I}_2 + 39\mathbf{I}_x$$

$$\mathbf{I}_x = \mathbf{I}_1 - \mathbf{I}_2$$

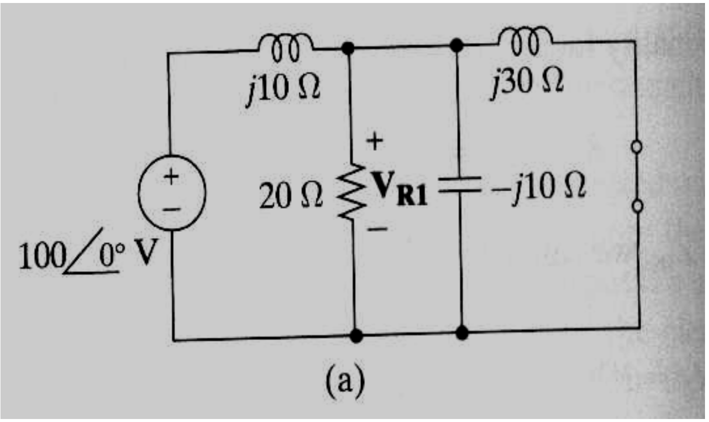
$$\mathbf{I}_1 = -26 - j52A$$

$$\mathbf{I}_2 = -24 - j58A$$

SUPERPOSITION



Use superposition to determine $v_R(t)$ for R=20 Ω, L_1=2 mH, L_2=6 mH, C=20 μF, $v_{s1}(t)$=100 cos 5000t V, and $v_{s2}(t)$=120 cos(5000t+30^0) V.

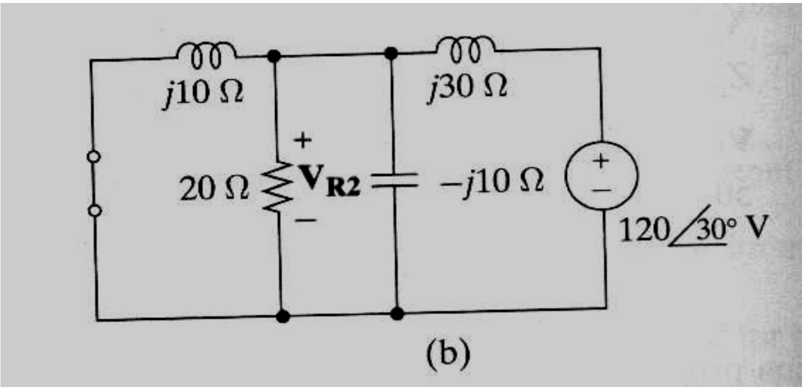


(a)

$$Z_{eq1} = \frac{1}{\frac{1}{20}+\frac{1}{-j10}+\frac{1}{j30}} = 7.2 - j9.6\Omega$$

$$\mathbf{V}_{R1} = \frac{Z_{eq1}}{j10 + Z_{eq1}} 100\angle 0^0$$

$$= 92.3 - j138 = 166\angle -56.3^0 V$$



$$Z_{eq2} = \frac{1}{\frac{1}{20} + \frac{1}{-j10} + \frac{1}{j10}} = 20 - j0\Omega$$

$$\mathbf{V}_{R2} = \frac{Z_{eq2}}{j30 + Z_{eq2}} 120\angle 30^0$$

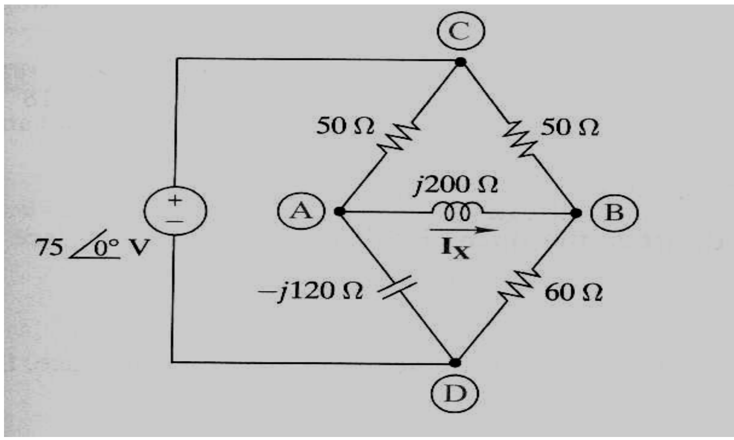
$$= 59.7 - j29.5 = 66.6 \angle -26.3^0 V$$

$$\mathbf{V}_R = \mathbf{V}_{R1} + \mathbf{V}_{R2} = 152 - j167 = 226 \angle -47.8^0 V$$

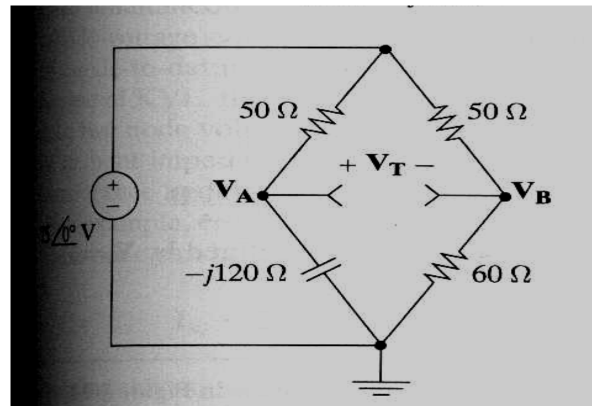
$$v_R(t) = 166\cos(5000t - 56.3^0) + 66.6\cos(5000t - 26.3^0)$$

$$= 226\cos(5000\text{t} - 47.8^0) V$$

THEVENIN EQUIVALENT



Use Thevenin's theorem to find the current $\mathbf{I}_x$ in the circuit.



$$\mathbf{V}_T = \mathbf{V}_A - \mathbf{V}_B = \frac{-j120}{50 - j120} 75\angle 0^0 - \frac{60}{60+50} 75\angle 0^0$$

$$= 23 - j26.6V$$

$$Z_{Th} = \frac{1}{\frac{1}{50} + \frac{1}{-j120}} + \frac{1}{\frac{1}{50} + \frac{1}{60}} = 69.9 - j17.8\Omega$$

$$\mathbf{I}_x = \frac{\mathbf{V}_T}{Z_{Th} + j200} = \frac{23 - j26.6}{69.9 + j182.2} = 0.18\angle -118^0 A$$