ELECTROMAGNETICS – I FINAL EXAM

Dr Salih FADIL **August 19 2011**

#1) Let region 1 ($z \geq 0$) have a dielectric constant of 2 while region 2 ($z \leq 0$) have a dielectric constant of 7.5. Given that $\vec{D}_1 = 20\vec{a}_x - 60\vec{a}_y + 30\vec{a}_z \ nC/m^2$, find

a) $\vec{E}_1$ and $\vec{P}_2$

b) θ_1 and θ_2

c) the energy densities in both regions.

Take θ_1 and θ_2 as the angles $\vec{E}_1$ and $\vec{E}_2$ makes with the normal to the interface.

#2) Find $\vec{E}$ at (0, 0, 4) due to a charge of 2 *nC/m* distributed uniformly

a) on the line $0 \leq x \leq 3$, $y = z = 0$

b) on the arc $\rho = 3$, $\pi/4 \leq \phi \leq \pi/2$, $z = 0$

#3) A charge distribution with spherical symmetry has density

$$\rho_v = \begin{cases} \rho_0, & 0 \leq r \leq R \\ 0, & r > R \end{cases}$$

determine V everywhere and the energy stored in region $r < R$.

GOOD LUCK ☺

ELECTROMAGNETICS - I FINAL EXAM
SOLUTION MANUAL

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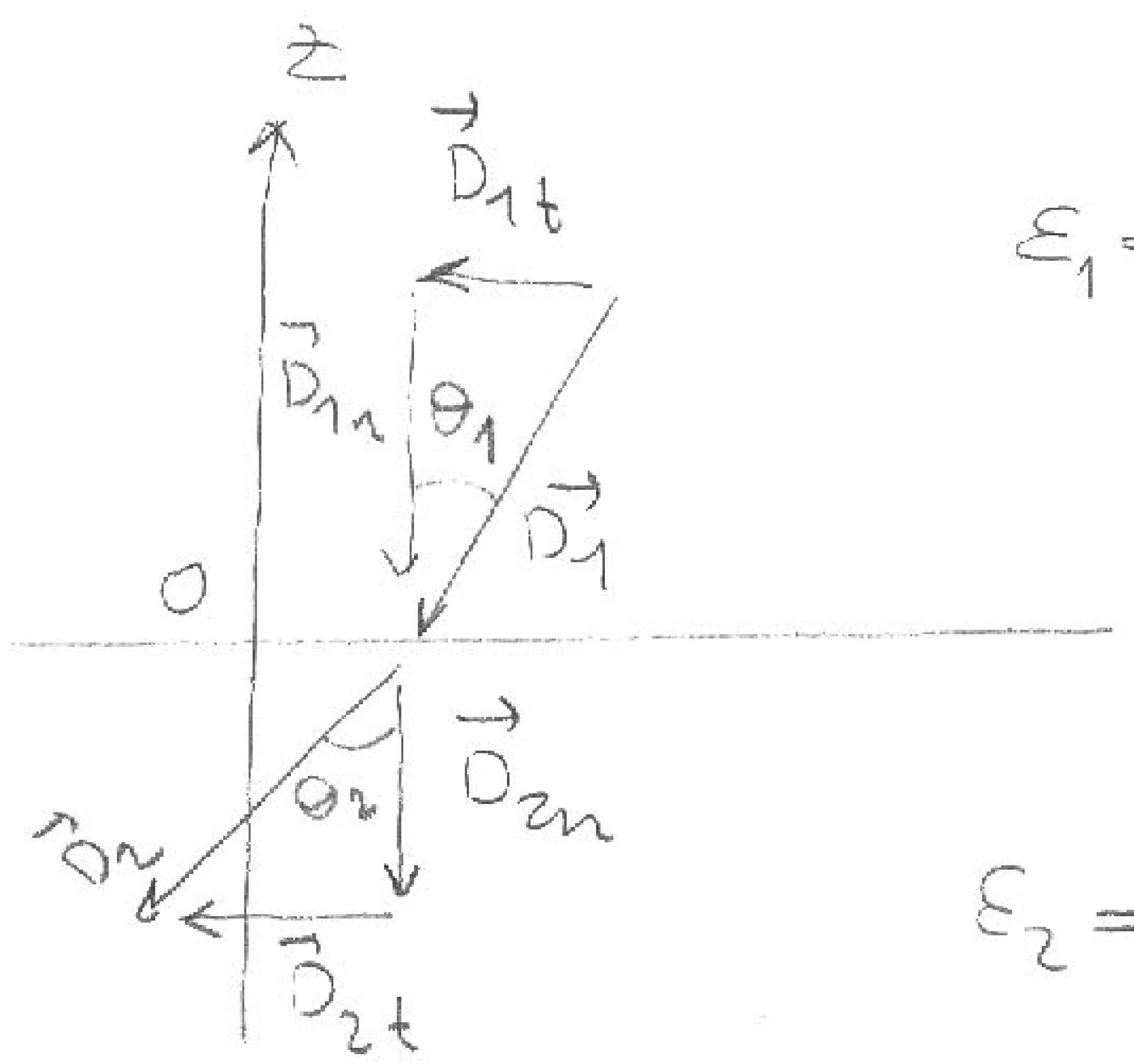
#1) 1

a) $\boxed{111}$

region-1

$\varepsilon_1 = 2\varepsilon_0$



region-2

$\varepsilon_2 = 7.5\,\varepsilon_0$

$$\vec{D}_1 = 20\,\vec{a}_x - 60\,\vec{a}_y + 30\,\vec{a}_z \quad nC/m^2$$

$$\vec{D}_{1n} = 30\,\vec{a}_z \ nC/m^2 \quad \vec{D}_{1t} = 20\,\vec{a}_x - 60\,\vec{a}_y \quad nC/m^2$$

since $\rho_s = 0$ on the boundary

$$\vec{D}_{2n} = \vec{D}_{1n} = 30\,\vec{a}_z \quad nC/m^2$$

$$\vec{E}_{1t} = \frac{\vec{D}_{1t}}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(20\,\vec{a}_x - 60\,\vec{a}_y\right) = \frac{\cancel{10^{-9}}}{\cancel{2}\,\frac{\cancel{10^{-9}}}{36\pi}}\left(\cancel{20}^{10}\,\vec{a}_x - \cancel{60}^{30}\,\vec{a}_y\right) \ V/m = \vec{E}_{2t}$$

$$\vec{E}_{2t} = 360\pi\,\vec{a}_x - 1080\,\vec{a}_y \quad V/m$$

$$\vec{E}_{2n} = \frac{\vec{D}_{2n}}{\varepsilon_2} = \frac{\cancel{10^{-9}}\ \cancel{30}^{4}\,\vec{a}_z}{\frac{\cancel{10^{-9}}}{36\pi}\ \cancel{7.5}} = 144\pi\,\vec{a}_z \quad V/m$$

$$\vec{E}_2 = 360\pi\,\vec{a}_x - 1080\pi\,\vec{a}_y + 144\pi\,\vec{a}_z \quad V/m$$

$$\vec{P}_2 = \chi_e\,\varepsilon_0\,\vec{E}_2 \qquad \chi_e = \varepsilon_{r2} - 1 = 6.5$$

$$\vec{P}_2 = 6.5\,\frac{10^{-9}}{36\cancel{\pi}}\left(360\cancel{\pi}\,\vec{a}_x - 1080\cancel{\pi}\,\vec{a}_y + 144\cancel{\pi}\,\vec{a}_z\right) = 6.5\,10^{-9}\left(10\,\vec{a}_x - 30\,\vec{a}_y + 4\,\vec{a}_z\right)$$

$$\vec{P}_2 = \left(65\ \vec{a}_x - 195\ \vec{a}_y + 26\ \vec{a}_z\right)\ nC/m^2 \quad 8$$

$$\vec{E}_1 = \frac{\vec{D}_1}{2\varepsilon_0} = \frac{\cancel{10^{-9}}}{\cancel{2}\frac{\cancel{10^{-9}}}{36\pi}}\left(\cancel{20}^{10}\ \vec{a}_x - \cancel{60}^{30}\ \vec{a}_y + \cancel{30}^{15}\ \vec{a}_z\right)$$

$$\vec{E}_1 = 360\pi\ \vec{a}_x - 1080\pi\ \vec{a}_y + 540\pi\ \vec{a}_z \quad V/m$$

$$\vec{E}_1 = \underline{1131\ \vec{a}_x - 3393\ \vec{a}_y + 1696{,}4\ \vec{a}_z} \quad V/m \quad 3$$

b) 11

$$\tan\theta_1 = \frac{D_{1t}}{D_{1n}} = \frac{(20^2+60^2)^{1/2}}{30} = 2.108 \qquad \theta_1 = \tan^{-1}(2.108)$$

$$\underline{\theta_1 = 64.23^\circ} \quad 5$$

$$\tan\theta_2 = \frac{\vec{E}_{2t}}{\vec{E}_{2n}} = \frac{\left[(360\pi)^2 + (1080\pi)^2\right]^{1/2}}{144\pi} = 7{,}905$$

$$\underline{\theta_2 = \tan^{-1}(7{,}905) = 82{,}79^\circ} \quad 6$$

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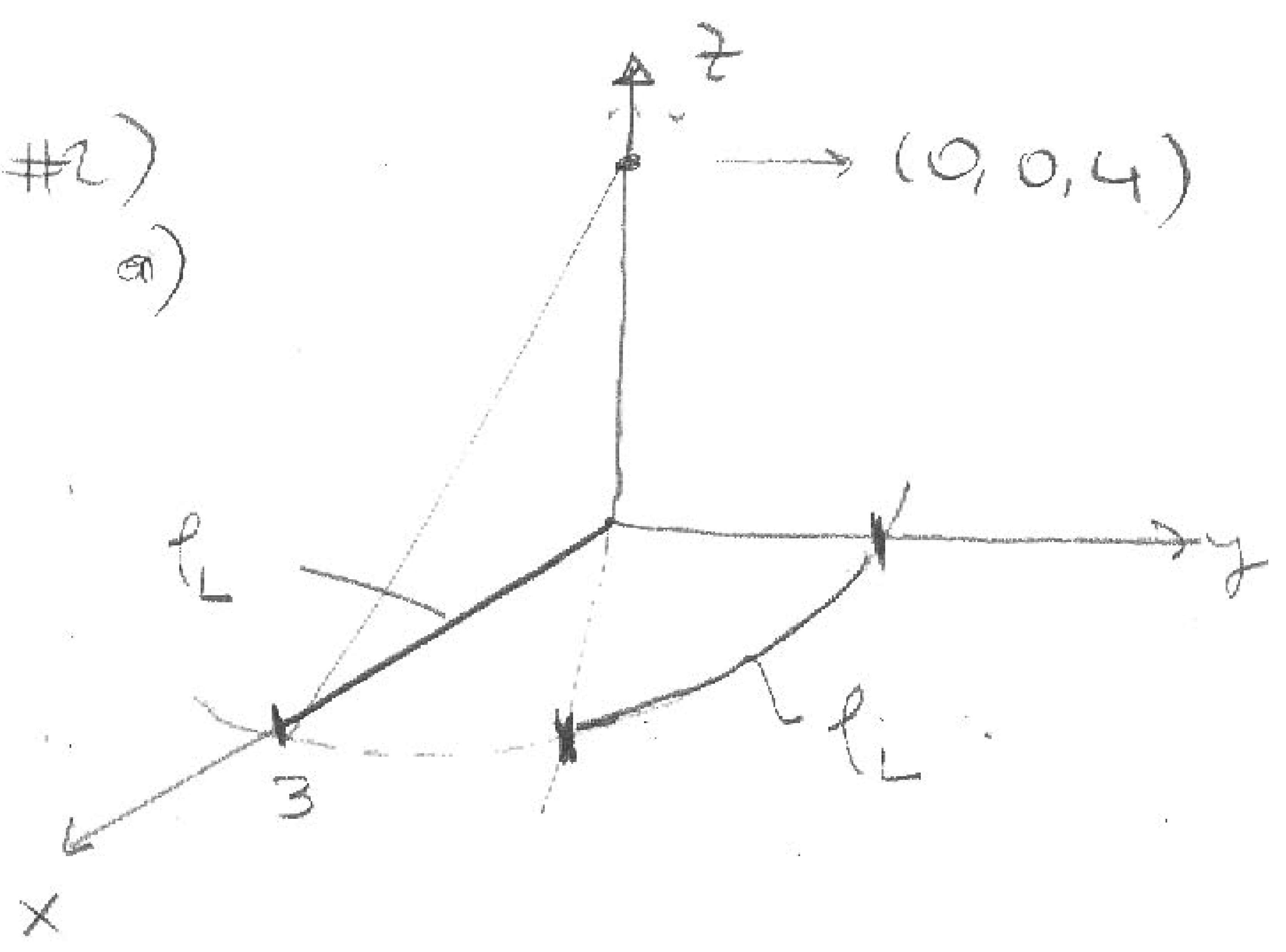
c) $$W_1 = \frac{1}{2}\vec{D}_1\vec{E}_1 = \frac{1}{2}\left(20 \times 360\pi + 60 \times 1080\pi + 30 \times 540\pi\right) 10^{-9}\ J/m^3$$

$$= 0{,}13854\ 10^{-3}\ J/m^3$$
$$= 0{,}13854\ mJ/m^3 \quad 6$$

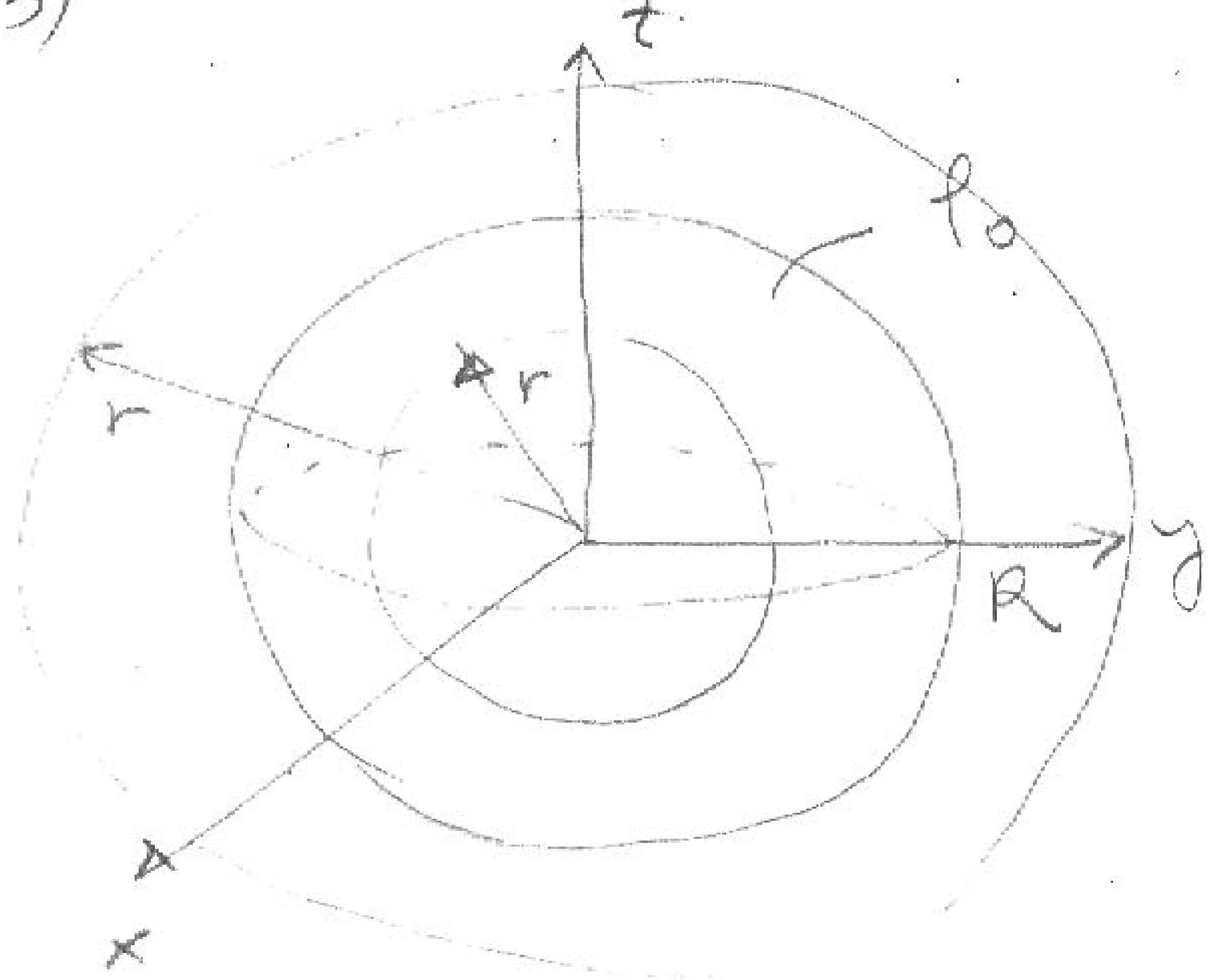
$$W_2 = \frac{1}{2}\vec{D}_2\vec{E}_2 = \frac{1}{2}\varepsilon_2|E_2|^2 = \frac{1}{2}\,7{,}5\,\frac{10^{-9}}{36\pi}\left((360\pi)^2 + (1080\pi)^2 + (144\pi)^2\right)$$

$$= \underline{0{,}4309}\ mJ/m^3 \quad 5$$

#2)
a)



#3)



$$\rho_v = \begin{cases} \rho_0 & 0 \leq r \leq R \\ 0 & r > R \end{cases}$$

<u>For $r \leq R$</u>

By applying Gauss law

$$\oint_S \vec{D} \cdot \vec{dS} = D_r \cdot 4\pi r^2 = \frac{4}{3}\pi r^3 \rho_0$$

$$D_r \cdot \cancel{4\pi} \cancel{r^2} = \frac{\cancel{4}}{3} \cancel{\pi} \cancel{r}^{3} \rho_0 \qquad \vec{D} = \frac{\rho_0}{3} r \vec{a}_r$$

$$\vec{E} = \frac{\rho_0}{3\varepsilon_0} r \vec{a}_r \ \text{V/m.} \quad 0 \leq r \leq R$$

<u>For $r \geq R$</u>

$$D_r \cdot \cancel{4\pi} r^2 = \frac{\cancel{4}}{3} \cancel{\pi} R^3 \rho_0 \qquad \vec{D} = \frac{R^3 \rho_0}{3 r^2} \vec{a}_r$$

$$\vec{E} = \frac{R^3 \rho_0}{3 \varepsilon_0 r^2} \vec{a}_r, \quad r \geq R$$

Potential for $r \geq R$

$$V = -\int_{-\infty}^{r} \frac{R^3 \rho_0}{3 \varepsilon_0 r^2} \vec{a}_r \cdot dr\, \vec{a}_r = \frac{R^3 \rho_0}{3 \varepsilon_0} \frac{1}{r} \Big|_{-\infty}^{r}$$

$$V(r) = \frac{\rho_0 R^3}{3 \varepsilon_0 r}, \quad r \geq R.$$

Potential for $r \leq R$

$$V(r) = -\int_{-\infty}^{R} \frac{R^3 \rho_0}{3\varepsilon_0 r^2}\, dr + \int_{R}^{r} \frac{\rho_0}{3\varepsilon_0} r \vec{a}_r \, dr \vec{a}_r$$

$$= \frac{R^3 \rho_0}{3\varepsilon_0} \frac{1}{r}\Big|_{-\infty}^{R} + \frac{\rho_0}{3\varepsilon_0} \frac{r^2}{2}\Big|_{r}^{R}$$

$$= \frac{\cancel{R}^3 \rho_0}{3\varepsilon_0}\left(\frac{1}{\cancel{R}} - 0\right) + \frac{\rho_0}{6\varepsilon_0}(R^2 - r^2)$$

$$V(r) = \frac{\rho_0}{6\varepsilon_0}\left[R^2 - r^2 + 2R^2\right] = \frac{\rho_0}{6\varepsilon_0}(3R^2 - r^2) \qquad r \leq R$$

$$\boxed{V(r) = \begin{cases} \dfrac{\rho_0}{6\varepsilon_0}(3R^2 - r^2) & r \leq R \\ \dfrac{\rho_0 R^3}{3\varepsilon_0 r} & r \geq r \end{cases}}$$

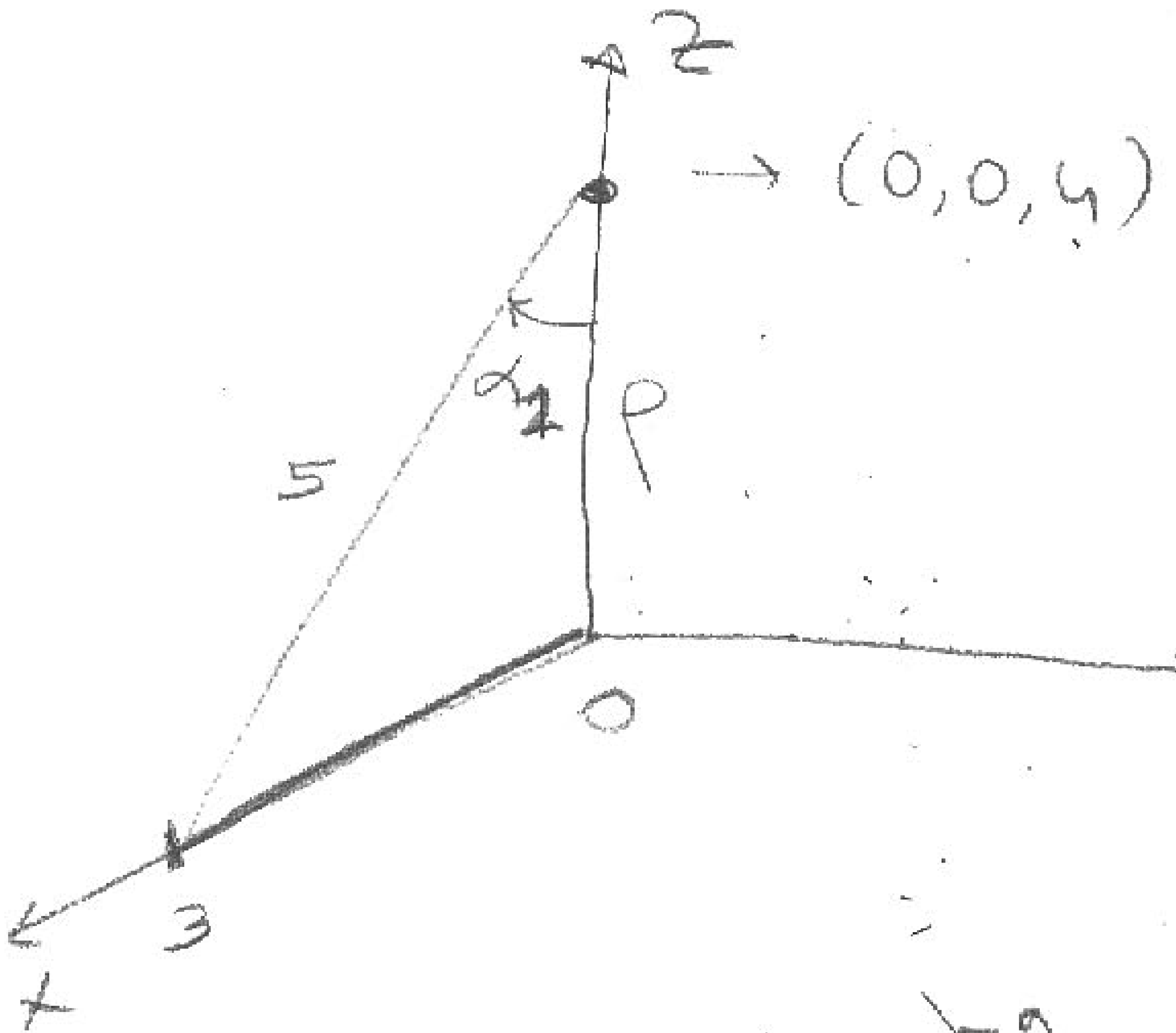
$$W_E = \frac{1}{2}\int_v \vec{D}\cdot\vec{E}\, dv = \frac{1}{2}\varepsilon_0 \int_v |\vec{E}|^2 dv = \frac{1}{2}\varepsilon_0 \left(\frac{\rho_0}{3\varepsilon_0}\right)^2 \int_v r^2\, r^2 \sin\theta \, d\theta \, d\phi \, d$$

$$W_E = \frac{1}{2}\cancel{\varepsilon_0} \frac{\rho_0^2}{9\cancel{\varepsilon_0^2}} \frac{r^5}{5}\Big|_0^R \cdot (-\cos\theta)\Big|_0^{\pi} 2\pi$$

$$W_E = \frac{1}{\cancel{2}} \frac{\rho_0^2}{9\varepsilon_0} \frac{R^5}{5} \cancel{2}\, 2\pi$$

$$\boxed{W_E = \frac{2\pi \rho_0^2 R^5}{45\varepsilon_0} \text{ Joule}}$$

3



$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0 \rho}\left[-(\sin\alpha_2 - \overset{0}{\sin\alpha_1})\,\underset{\vec{a}_z}{\vec{a}_\rho} + (\cos\alpha_2 - \cos\alpha_1)\,\vec{a}_x\right]$$

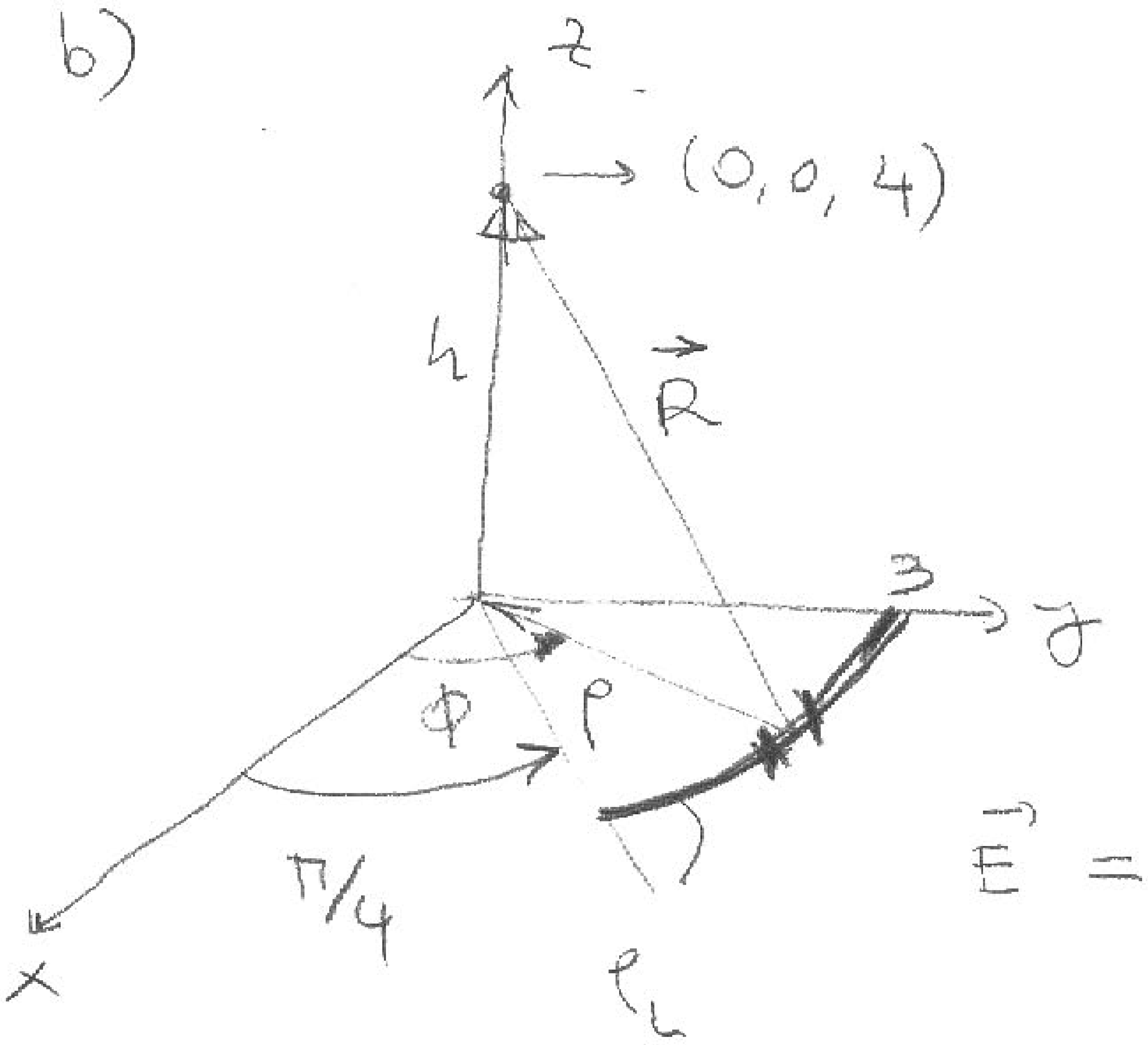
$$\rho = 4, \quad \vec{a}_\rho = \vec{a}_z$$

$$\alpha_1 = 0 \quad \alpha_2 < 0$$

$$\vec{E} = \frac{\cancel{2}\ \cancel{10^{-9}}}{\cancel{4\pi}\ \frac{\cancel{10^{-9}}}{\cancel{36\pi}_{9}} \cdot \cancel{4}_{2}}\left[\frac{3}{5}\vec{a}_z + \left(\frac{4}{5} - 1\right)\vec{a}_x\right]$$

$$\vec{E} = \frac{9}{2}\left[\frac{3}{5}\vec{a}_z + \left(-\frac{1}{5}\right)\vec{a}_x\right] = 2{,}7\,\vec{a}_z - 0{,}9\,\vec{a}_x \ \text{V/m}$$

b)



$$d\vec{E} = \frac{\rho\, d\phi\, \rho_L\ (-\rho\,\vec{a}_\rho + h\,\vec{a}_z)}{4\pi\varepsilon_0\ (\rho^2 + h^2)^{3/2}}$$

$$\rho = 3 \ ; \quad h = 4$$

$$\vec{E} = \frac{3 \cdot 2\ \cancel{10^{-9}}\ (-3\vec{a}_\rho + 4\vec{a}_z)}{\cancel{4\pi}\ \frac{\cancel{10^{-9}}}{\cancel{36\pi}_{9}}\ \underbrace{(9+16)^{3/2}}_{5\sqrt{5}}} \cdot \phi\Big|_{\pi/4}^{\pi/2}$$

$$\vec{E} = \frac{6(-3\vec{a}_\rho + 4\vec{a}_z)}{\frac{5\sqrt{5}}{9}}\,\frac{\pi}{4} = \frac{54\pi}{20\sqrt{5}}\left(-3\vec{a}_\rho + 4\vec{a}_z\right)$$

$$= -11{,}38\,\vec{a}_\rho + 15{,}173\,\vec{a}_z \ \text{V/m}$$