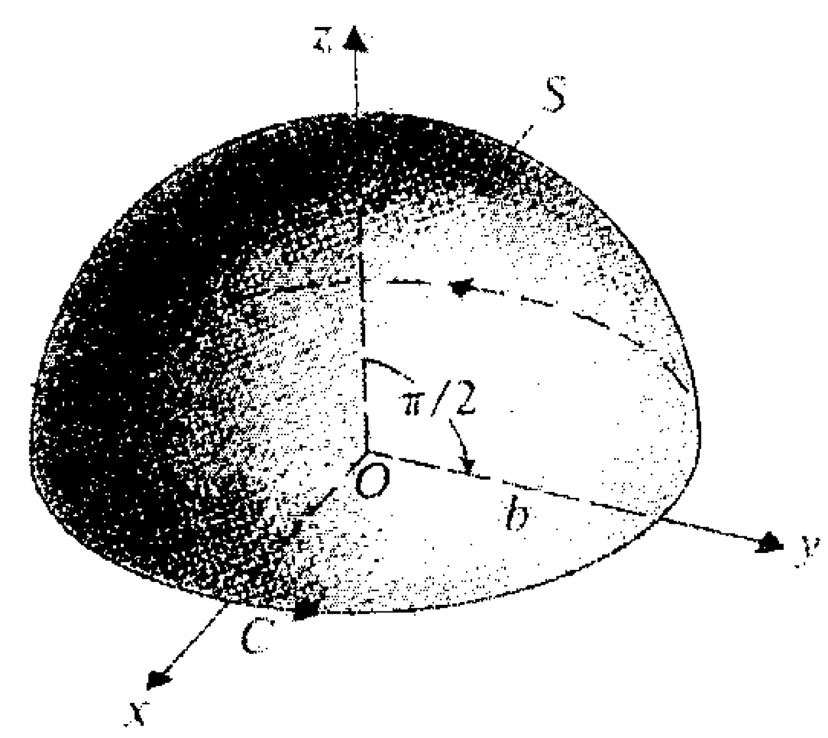
ELECTROMAGNETICS – I FIRST MIDTERM EXAM

Dr. Salih FADIL **March 06, 2009**

#1)

a) For the vector function $\vec{A} = \vec{a}_\rho \rho^2 + \vec{a}_z 2z$, verify the divergence theorem for the cylindrical region enclosed by $\rho = 5$, $z = 0$ and $z = 5$.

b) Given the vector function $\vec{A} = \sin(\phi/2)\vec{a}_\phi$, verify Stokes's theorem over the hemispherical surface and its circular contour that are shown in the figure.



#2) Let $\vec{A} = \rho\cos(\phi)\vec{a}_\rho + z\sin(\phi)\vec{a}_\phi - \rho z^2\vec{a}_z$ and $\vec{B} = r\sin(\theta)\vec{a}_r + r^2\cos(\theta)\sin(\phi)\vec{a}_\theta$, find a unit vector, in cylindrical coordinates, that is perpendicular to both $\vec{A}$ and $\vec{B}$ at $P(3,-2,6)$.

#3)

a) If $\vec{A} = \vec{a}_x + 2\vec{a}_y - \vec{a}_z$ and $\vec{B} = B_x\vec{a}_x + B_y\vec{a}_y + 3\vec{a}_z$, find B_x and B_y such that $\vec{A}$ and $\vec{B}$ are parallel.

b) The scalar component of a vector $\vec{D}$ along $\vec{E} = 3\vec{a}_x - 4\vec{a}_z$ is 10. If $\vec{D}$ is parallel to $\vec{C} = \vec{a}_x + 2\vec{a}_y + \vec{a}_z$, determine $\vec{D}$.

GOOD LUCK….☺

ELECTROMAGNETICS - I FIRST MIDTERM EXAM
SOLUTION MANUAL

Dr. Salih FADIL

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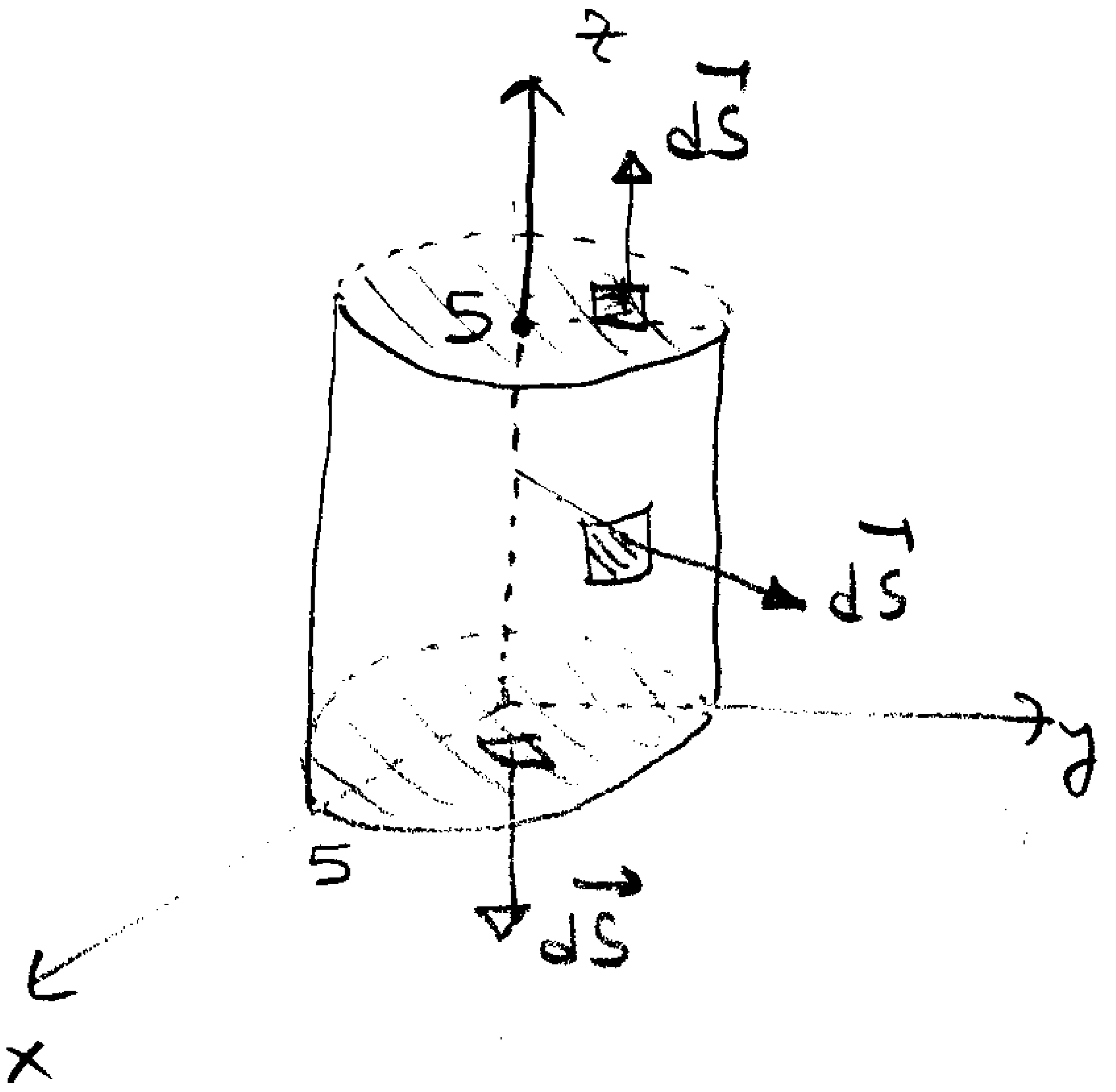
#1)

a) $\vec{A} = \rho^2 \vec{a}_\rho + 2z\,\vec{a}_z$ cylindrical region defined by $\rho = 5$, $z=0$ and $z=5$



$$\oint_S \vec{A}\cdot d\vec{S} = \int_v (\nabla\cdot\vec{A})\, dv \quad : \text{Divergence theorem}$$

$$\left(\int_{top} + \int_{bottom} + \int_{curved}\right)\vec{A}\cdot d\vec{S} = \oint_S \vec{A}\cdot d\vec{S}$$

top surface

$d\vec{S} = \rho\, d\phi\, d\rho\, \vec{a}_z$, $z=5$ $\quad 0\leq\rho\leq 5 \quad 0\leq\phi\leq 2\pi$.

$$\int_{top}\vec{A}\cdot d\vec{S} = \int_{\rho=0}^{5}\int_{\phi=0}^{2\pi} 2z\,\rho\, d\phi\, d\rho = 10\,\frac{\rho^2}{2}\Big|_0^5\,\phi\Big|_0^{2\pi} = 5\,(25-0)(2\pi-0)$$
$$= 250\pi$$

bottom surface

$d\vec{S} = \rho\, d\phi\, d\rho\,(-\vec{a}_z)$, $z=0$, $0\leq\rho\leq 5$, $0\leq\phi\leq 2\pi$

$$\int_{bottom}\vec{A}\cdot d\vec{S} = 0$$

curved surface

$\vec{dS} = \rho\, d\phi\, dz\, \vec{a}_\rho$, $\rho=5$, $0\leq z\leq 5$, $0\leq\phi\leq 2\pi$

$$\int_{curved}\vec{A}\,\vec{dS} = \int_{curved}\rho^3\, d\phi\, dz = 125\; 2\pi . 5 = 1250\pi$$

$$\oint_S \vec{A}\cdot d\vec{S} = 250\pi + 1250\pi = 1500\pi$$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot\rho^2) + \frac{\partial}{\partial z}(2z) = \frac{1}{\rho}3\rho^2 + 2 = 3\rho + 2$$

$$\int_v (3\rho+2)\,dv = \int_{\rho=0}^{5}\int_{z=0}^{5}\int_{\phi=0}^{2\pi} (3\rho+2)\,\rho\,d\phi\,dz\,d\rho$$

$$= \cancel{3}\frac{\rho^3}{\cancel{3}}\Big|_0^5 \times 2\pi\times 5 + \cancel{2}\frac{\rho^2}{\cancel{2}}\Big|_0^5 2\pi\times 5$$

$$= 1250\pi + 25\;10\pi = 1500\pi$$

$$\oint_S \vec{A}\cdot d\vec{S} = \int_v (\nabla\cdot\vec{A})\,dv \;\longrightarrow$$ so, divergence theorem is satisfied.

b) $$\oint_C \vec{A}\cdot d\vec{l} = \int_S (\nabla\times\vec{A})\cdot d\vec{S}$$; Stokes's theorem

$$\oint_C \vec{A}\cdot d\vec{l}\,,\quad d\vec{l} = r\sin\theta\,d\phi\,\vec{a}_\phi = b\,d\phi\,\vec{a}_\phi \qquad \theta = \pi/2,\; r = b$$

$$\int_{\phi=0}^{2\pi} \sin\left(\frac{\phi}{2}\right)\cdot b\,d\phi = b\,2\left\{-\cos\left(\frac{\phi}{2}\right)\right\}_{\phi=0}^{2\pi}$$

$$= 2b\,(1+1) = \underline{\underline{4b}}$$

$$\nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\,\vec{a}_\theta & r\sin\theta\,\vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & r\sin\theta\cdot\sin(\phi/2) \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\Big\{\vec{a}_r\,(r\cos\theta\,\sin(\phi/2) - 0) - r\,\vec{a}_\theta\,(\sin\theta\cdot\sin(\phi/2) - 0)$$
$$+ r\sin\theta\,\vec{a}_\phi\,(0-0)\Big\}$$

$$\nabla \times \vec{A} = \frac{\cancel{r} \cos\theta \sin(\phi/2)}{\cancel{r^2}_{\,r} \sin\theta} \vec{a}_r - \frac{r \sin\theta \sin(\phi/2)}{r^2 \sin\theta} \vec{a}_\theta$$

$$\vec{dS} = r\,d\theta\; r\sin\theta\, d\phi\, \vec{a}_r \qquad \text{(according to right hand rule)}$$

$$\vec{dS} = r^2 \sin\theta\, d\theta\, d\phi\, \vec{a}_r \qquad r = b,\ 0 \le \theta \le \pi/2,\ 0 \le \phi \le 2\pi$$

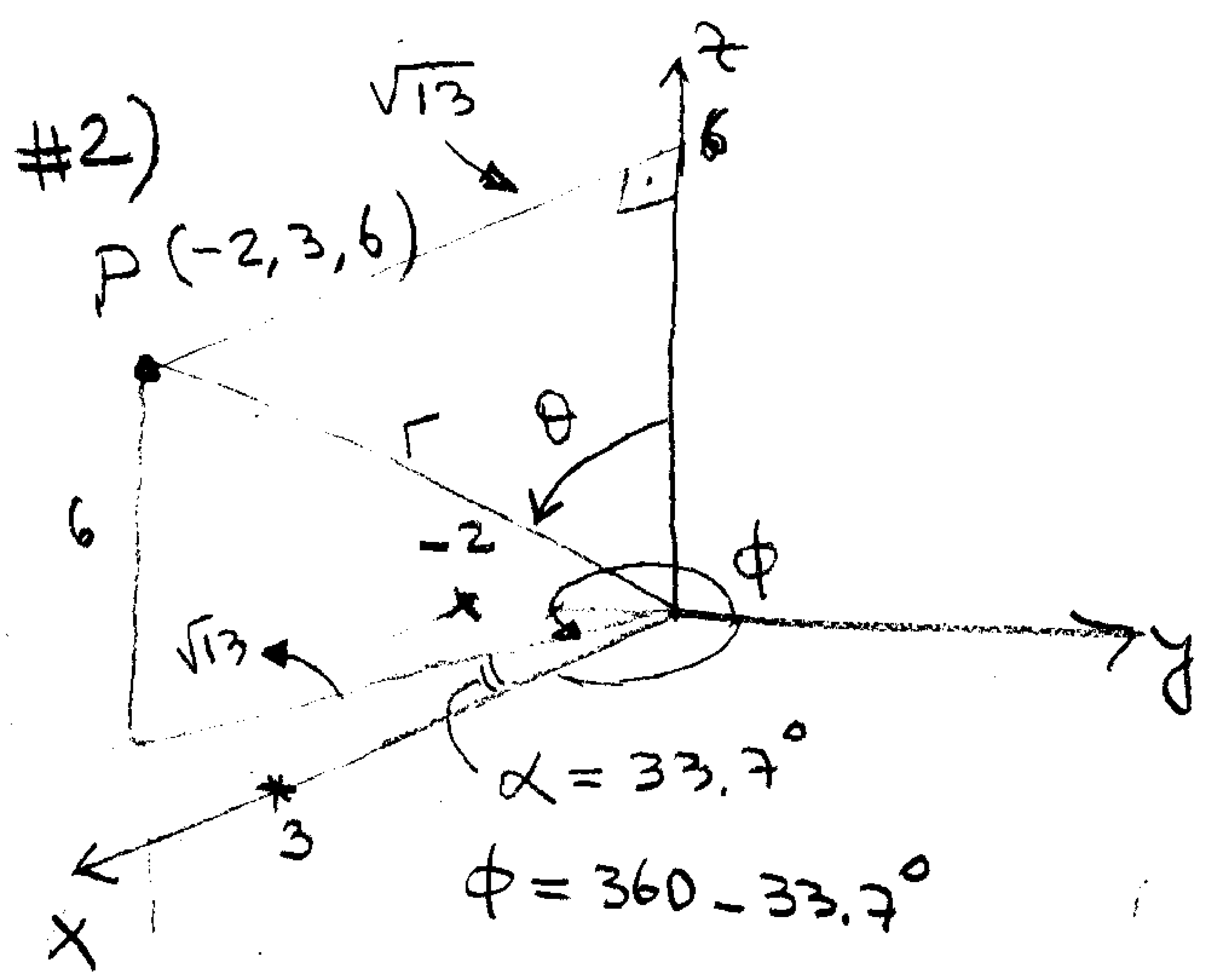
$$\int_S (\nabla \times \vec{A}) \cdot \vec{dS} = \int_S \frac{\cos\theta \cdot \sin(\phi/2)}{\cancel{r}\ \cancel{\sin\theta}} \cancel{r^2}^{\,r}\, \cancel{\sin\theta}\, d\theta\, d\phi$$

$$= b \cdot \sin\theta \Big|_0^{\pi/2} \cdot (-2) \cos\left(\frac{\phi}{2}\right)\Big|_0^{2\pi}$$

$$= b\,(1-0)\,(-2)\,(-1-1) = \underline{\underline{4b}}, \quad \text{So}$$

$$\oint_C \vec{A} \cdot \vec{dl} = \int_S (\nabla \times \vec{A}) \cdot \vec{dS} \longrightarrow \text{Stokes's theorem is satisfied.}$$

#2)



For cylindrical coordinates

$$\rho = (9+4)^{1/2} = \sqrt{13}$$
$$\phi = 326.3^\circ$$
$$z = 6$$

For spherical coordinates

$$r = (13 + 36)^{1/2} = 7.$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{13}}{6}\right) = 31^\circ$$
$$\phi = 326.3^\circ$$

$$\vec{B} = 7 \sin(31^\circ)\, \vec{a}_r + 49 \cos(31^\circ) \sin(326.3^\circ)\, \vec{a}_\theta.$$

$$\vec{B} = 3.605\, \vec{a}_r - 23.304\, \vec{a}_\theta$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 3.605 \\ -23.304 \\ 0 \end{bmatrix}$$

$B_\rho = \sin(31^\circ) \times 3.605 - 23.304 \times \cos(31^\circ) = -18.118$

$B_\phi = 0$

$B_z = \cos(31^\circ) \times 3.605 + 23.304 \sin(31^\circ) = 15.092$

$A_\rho = \sqrt{13} \cos(326.3^\circ) = 3.0$

$A_\phi = 6 \cdot \sin(326.3^\circ) = -3.329$

$A_z = -\sqrt{13} \times 36 = -129.8$

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_\rho & \vec{a}_\phi & \vec{a}_z \\ 3.0 & -3.329 & -129.8 \\ -18.118 & 0 & 15.092 \end{vmatrix}$$

$$\vec{C} = (-15.092 \times 3.329 - 0)\,\vec{a}_\rho - (15.092 \times 3 - 129.8 \times 18.118)\,\vec{a}_\phi + (0 - 18.118 \times 3.329)$$

$\vec{C} = -50.241\,\vec{a}_\rho + 2306.44\,\vec{a}_\phi - 60.315\,\vec{a}_z \qquad (|\vec{C}| = 2307.77$

$$\vec{a}_c = \pm \frac{\vec{C}}{|\vec{C}|} = \pm(-0.02177\,\vec{a}_\rho + 0.99942\,\vec{a}_\phi - 0.02613\,\vec{a}_z)$$

#3) a) If $\vec{A} \parallel \vec{B} \longrightarrow \vec{A} \times \vec{B} = 0$

$$\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ 1 & 2 & -1 \\ B_x & B_y & 3 \end{vmatrix} = (6 + B_y)\,\vec{a}_x - (3 + B_x)\,\vec{a}_y + (B_y - 2B_x)\,\vec{a}_z = 0$$

$6 + B_y = 0 \Rightarrow \boxed{B_y = -6}$

$3 + B_x = 0 \Rightarrow \boxed{B_x = -3}$

$B_y - 2B_x = 0 \Rightarrow B_y = 2B_x$ O.K

$$\boxed{\vec{B} = -3\vec{a}_x - 6\vec{a}_y + 3\vec{a}_z}$$

b) $\vec{D} = D_x \vec{a}_x + D_y \vec{a}_y + D_z \vec{a}_z$, $\vec{E} = 3\vec{a}_x - 4\vec{a}_z$

$\vec{C} = \vec{a}_x + 2\vec{a}_y + \vec{a}_z$

$$\vec{E} \cdot \vec{D} = |\vec{E}|\,\underbrace{|\vec{D}| \cdot \cos\theta}_{D_E} = |\vec{E}| \cdot D_E \rightarrow D_E = \frac{\vec{E} \cdot \vec{D}}{|\vec{E}|}$$

$$D_E = 10 = \frac{3D_x - 4D_z}{(9+16)^{1/2}} = 10 \qquad 3D_x - 4D_z = 50$$

If $\vec{D} \parallel \vec{C} \rightarrow \vec{D} \times \vec{C} = 0$

$$\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ D_x & D_y & D_z \\ 1 & 2 & 1 \end{vmatrix} = \vec{a}_x (D_y - 2D_z) - (D_x - D_z)\vec{a}_y + (2D_x - D_y)\vec{a}_z \equiv 0$$

$D_y - 2D_z = 0 \rightarrow D_y = 2D_z$

$D_x - D_z = 0 \rightarrow D_x = D_z$.

$2D_x - D_y = 0 \rightarrow$ O.K

$3D_x - 4D_z = 50$

$3D_z - 4D_z = 50$

$-D_z = 50 \quad \boxed{D_z = -50}$

$\boxed{D_x = -50} \quad \boxed{D_y = -100}$

$$\boxed{\vec{D} = -50\,(\vec{a}_x + 2\vec{a}_y + \vec{a}_z)}$$