Introduction to Laplace Transforms

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Introduction

- Laplace transform:
  - important method of representing circuits and signals
  - used to analyze circuits with complicated signals
- To analyze circuits using Laplace transform:
  - change signals from time-domain to frequency domain
  - analyze circuits in frequency domain
  - convert signals in frequency-domain back to time domain
- More general than phasor method

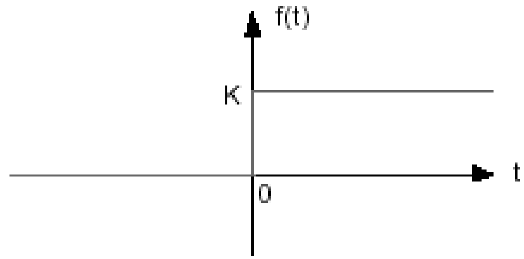
Definition of the Laplace Transform

One-sided Laplace transform of a time function f(t) is defined as

$$F(s) = L\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

- Some functions may not have Laplace transforms but we do not use them in circuit analysis.
- F(s) is a one sided transform (starts from t=0).
- Choose 0^- as the lower limit (to capture discontinuity in f(t) due to an event such as closing a switch).

The Step Function



if $K = 1$, unit step function is defined as

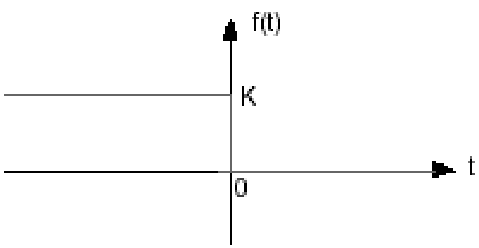
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

if $K \neq 1$, the step function is defined as

$$Ku(t) = \begin{cases} 0 & t < 0 \\ K & t > 0 \end{cases}$$

Note: the step function is not defined for t=0.

$$Ku(-t) = \begin{cases} K & t < 0 \\ 0 & t > 0 \end{cases}$$



Laplace Transform of the Step Function

$$L\{u(t)\} = \int_{0^-}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^{\infty} = \frac{1}{s}$$

The Impulse Function

Impulse = signal of infinite amplitude and zero duration

the unit impulse function :

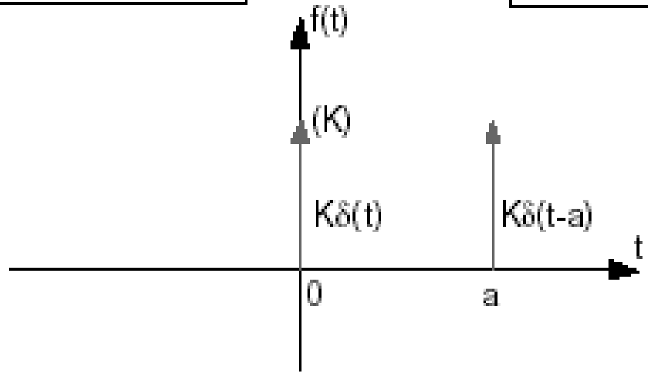
$$\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) = 1$$

the impulse function :

$$K\delta(t) = 0 \quad t \neq 0$$

$$\int_{-\infty}^{\infty} K\delta(t) = K$$



K – strength of the impulse function

The sifting property: $\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = f(a)$

The impulse function is a derivative of the step function:

$$\delta(t) = \frac{du(t)}{dt} \qquad \int_{-\infty}^{t} \delta(x)dx = u(t)$$

The Laplace transform of the impulse function:

$$L\{\delta(t-a)\} = \int_{0^-}^{\infty} \delta(t-a)e^{-st}dt = e^{-as}$$

$$\text{For } a = 0, \qquad L\{\delta(t)\} = 1$$

Operational Transforms

1) Multiplication by a constant

$$\text{If } \; L\{f(t)\} = F(s) \quad \text{then } \; L\{Kf(t)\} = KF(s)$$

2) Addition (subtraction)

$$\text{If } \; L\{f_1(t)\} = F_1(s), \quad L\{f_2(t)\} = F_2(s), \; L\{f_3(t)\} = F_3(s)$$

$$\text{then } \; L\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s)\text{-}F_3(s)$$

3) Differentiation

$$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-), \qquad L\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0^-) - f'(0^-)$$

$$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - sf^{(n-2)}(0^-) - f^{(n-1)}(0^-)$$

4) Integration $$L\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s}$$

5) Translation in the Time Domain

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s), \qquad a \succ 0$$

6) Translation in the Frequency Domain

$$L\{e^{-at} f(t)\} = F(s+a)$$

7) Scale Changing

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \qquad a \succ 0$$

Laplace Transform of Cos and Sine

$$L\{\cos \omega t u(t)\} = \int_{0^-}^{\infty} \cos \omega t u(t) e^{-st} dt = \int_0^{\infty} \cos \omega t e^{-st} dt$$

$$= \int_0^{\infty} \frac{e^{j\omega t} + e^{-j\omega t}}{2} e^{-st} dt = \frac{1}{2}[\int_0^{\infty} e^{-(s-j\omega)t} dt + \int_0^{\infty} e^{-(s+j\omega)t} dt]$$

$$= -\frac{1}{2}\frac{1}{s-j\omega} e^{-(s-j\omega)t}\Big|_0^{\infty} - \frac{1}{2}\frac{1}{s+j\omega} e^{-(s+j\omega)t}\Big|_0^{\infty}$$

$$= \frac{1}{2}\frac{1}{s-j\omega} + \frac{1}{2}\frac{1}{s+j\omega} = \frac{s}{s^2+\omega^2}$$

$$L\{\sin \omega t u(t)\} = \frac{\omega}{s^2+\omega^2}$$

Inverse Laplace Transform

- Goal: find the inverse Laplace transform of a function that has the following form:

$$F(s)=\frac{N(s)}{D(s)}=\frac{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}$$

  - If m>n
    - F(s) is a proper rational function

$$\text{example : } F(s)=\frac{(s-4)(s+4)}{s(s-6)(s^2+6s+25)}$$

  - If m<=n
    - F(s) is an improper rational function

$$\text{example : } F(s)=\frac{(s-4)(s+4)(s^2+6s+25)}{s(s-6)}$$

Partial Fraction Expansion

Step1: Expand F(s) as a sum of partial fractions.
Step 2: Compute the expansion constants (four cases)
Step 3: Write the inverse transform

$$\frac{s+6}{s(s+3)(s+1)^2}$$

Denominator has four roots: s=0 and s=-3 are distinct. s=-1 is a multiple root of multiplicity 2.

$$\frac{s+6}{s(s+3)(s+1)^2} \equiv \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

$$L^{-1}\left\{\frac{s+6}{s(s+3)(s+1)^2}\right\}$$
$$= \left(K_1 + K_2 e^{-3t} + K_3 t e^{-t} + K_4 e^{-t}\right)u(t)$$

Distinct Real Roots

$$F(s)=\frac{96(s+5)(s+12)}{s(s+8)(s+6)}=\frac{K_1}{s}+\frac{K_2}{s+8}+\frac{K_3}{s+6}$$

$$K_1=sF(s)\Big|_{s=0}=s\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\Bigg|_{s=0}=\frac{96(5)(12)}{8(6)}=120$$

$$K_2=(s+8)F(s)\Big|_{s=-8}=(s+8)\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\Bigg|_{s=-8}=\frac{96(-3)(4)}{-8(-2)}=-72$$

$$K_3=(s+6)F(s)\Big|_{s=-6}=(s+6)\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\Bigg|_{s=-6}=\frac{96(-1)(6)}{-6(2)}=48$$

$$F(s)=\frac{120}{s}+\frac{48}{s+6}-\frac{72}{s+8}$$

$$f(t)=(120+48e^{-6t}-72e^{-8t})u(t)$$

Distinct Complex Roots

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{100(s+3)}{(s+6)(s+3+j4)(s+3-j4)}$$

$$= \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}$$

$$K_1 = \left.\frac{100(s+3)}{s^2+6s+25}\right|_{s=-6} = \frac{100(-3)}{25} = -12$$

$$K_2 = \left.\frac{100(s+3)}{(s+6)(s+3+j4)}\right|_{s=-3+j4} = \frac{100(j4)}{(3+j4)(j8)}$$

$$= 6 - j8 = 10e^{-j53.13^0}$$

$$K_3 = \left.\frac{100(s+3)}{(s+6)(s+3-j4)}\right|_{s=-3-j4} = \frac{100(-j4)}{(3-j4)(-j8)}$$

$$= 6 + j8 = 10e^{j53.13^0}$$

$$F(s)=\frac{-12}{s+6}+\frac{10\angle -53.13^0}{s+3-j4}+\frac{10\angle 53.13^0}{s+3+j4}$$

$$f(t)=(-12e^{-6t}+10e^{-j53.13^0}e^{-(3-j4)t}+10e^{j53.13^0}e^{-(3+j4)t})u(t)$$

$$10e^{-j53.13^0}e^{-(3-j4)t}+10e^{j53.13^0}e^{-(3+j4)t}$$

$$=10e^{-3t}(e^{j(4t-53.13^0)}+e^{-j(4t-53.13^0)})$$

$$=20e^{-3t}\cos(4t-53.13^0)$$

$$f(t)=[-12e^{-6t}+20e^{-3t}\cos(4t-53.13^0)]u(t)$$

Whenever F(s) contains distinct complex roots at the denominator as $(s+\alpha-j\beta)(s+\alpha+j\beta)$, a pair of terms of the form

$$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$$

appears in the partial fraction.

Where K is a complex number in polar form $K=|K|e^{j\theta}=|K|\angle\theta^0$ and K^* is the complex conjugate of K.

The inverse Laplace transform of the complex-conjugate pair is

$$L^{-1}=\left\{\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}\right\}$$

$$=2|K|e^{-\alpha t}\cos(\beta t+\theta)$$

Repeated Real Roots

$$F(s)=\frac{100(s+25)}{s(s+5)^3}=\frac{K_1}{s}+\frac{K_2}{(s+5)^3}+\frac{K_3}{(s+5)^2}+\frac{K_4}{s+5}$$

$$K_1=\left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0}=\frac{100(25)}{125}=20$$

$$K_2=\left.\frac{100(s+25)}{s}\right|_{s=-5}=\frac{100(20)}{-5}=-400$$

$$K_3=\frac{d}{ds}\left[(s+5)^3F(s)\right]_{s=-5}=100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5}=-100$$

$$K_4=\frac{1}{2}\frac{d^2}{ds^2}\left[(s+5)^3F(s)\right]_{s=-5}=\frac{1}{2}100\left[\frac{-2s(25)}{s^4}\right]_{s=-5}=-20$$

$$F(s)=\frac{20}{s}-\frac{400}{(s+5)^3}-\frac{100}{(s+5)^2}-\frac{20}{s+5}$$

$$f(t)=[20-200t^2e^{-5t}-100te^{-5t}-20e^{-5t}]u(t)$$

Repeated Complex Roots

$$F(s)=\frac{768}{(s^2+6s+25)^2}=\frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$=\frac{K_1}{(s+3-j4)^2}+\frac{K_1^*}{(s+3+j4)^2}+\frac{K_2}{(s+3-j4)}+\frac{K_2^*}{(s+3+j4)}$$

$$K_1=\left.\frac{768}{(s+3+j4)^2}\right|_{s=-3+j4}=\frac{768}{(j8)^2}=-12$$

$$K_2=\frac{d}{ds}\left[\frac{768}{(s+3+j4)^2}\right]_{s=-3+j4}=-\left.\frac{2(768)}{(s+3+j4)^3}\right|_{s=-3+j4}$$

$$=-\frac{2(768)}{(j8)^3}=-j3=3\angle{-90^0}$$

$$F(s)=\left[\frac{-12}{(s+3-j4)^2}+\frac{-12}{(s+3+j4)^2}\right]$$

$$+\left(\frac{3\angle{-90^0}}{s+3-j4}+\frac{3\angle{90^0}}{s+3+j4}\right)$$

$$f(t)=[-24te^{-3t}\cos 4t+6e^{-3t}\cos(4t-90^0)]u(t)$$

Improper Transfer Functions

An improper transfer function can always be expanded into a polynomial plus a proper transfer function.

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

$$= s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

$$= s^2 + 4s + 10 + \frac{-20}{s+4} + \frac{50}{s+5}$$

$$f(t) = \frac{d^2\delta(t)}{dt^2} + 4\frac{d\delta(t)}{dt} + 10\delta(t) + (-20e^{-4t} + 50e^{-5t})u(t)$$

POLES AND ZEROS OF F(s)

The rational function F(s) may be expressed as the ratio of two factored polynomials as

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_m)}$$

The roots of the denominator polynomial $-p_1$, $-p_2$, ..., $-p_m$ are called the poles of F(s). At these values of s, F(s) becomes infinitely large. The roots of the numerator polynomial $-z_1$, $-z_2$, ..., $-z_n$ are called the zeros of F(s). At these values of s, F(s) becomes zero.

$$F(s)=\frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

The poles of F(s) are at 0, -10, -6+j8, and –6-j8.
The zeros of F(s) are at –5, -3+j4, -3-j4

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num=conv([1 5],[1 6 25]);
den=conv([1 10 0],[1 12 100]);
pzmap(num,den)
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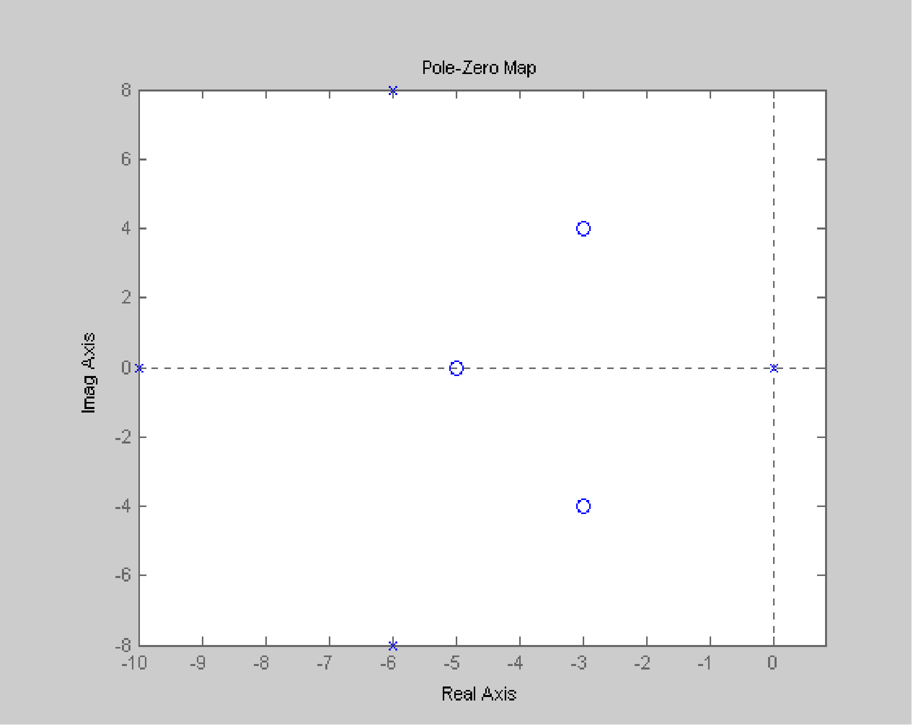

Initial-Value Theorem

The initial-value theorem enables us to determine the value of f(t) at t=0 from F(s). This theorem assumes that f(t) contains no impulse functions and poles of F(s), except for a first-order pole at the origin, lie in the left half of the s plane.

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$

$$f(t) = [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^0)]u(t)$$

Final-Value Theorem

The final-value theorem enables us to determine the behavior of f(t) at infinity using F(s).

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

The final-value theorem is useful only if f(∞) exists. This condition is true only if all the poles of F(s), except for a simple pole at the origin, lie in the left half of the s plane.

$$\lim_{s\to 0} F(s) = \lim_{s\to 0} s\frac{100(s+3)}{(s+6)(s^2+6s+25)} = 0$$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^0)]u(t) = 0$$