THE LAPLACE TRANSFORM IN CIRCUIT ANALYSIS

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A Resistor in the s Domain



- In time domain v=Ri (Ohm's Law).
- Because R is constant, the Laplace transform of the equation is V(s)=RI(s). This is the voltage-current relation for a resistor in s-domain.





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An Inductor in the s Domain

Consider that the inductor carries an initial current of ^I₀ The v-i relation in the time-domain is $v = L \frac{di}{dt}$ The Laplace transform is V(s) $I(s) = \frac{V(s)}{sL} + \frac{I_0}{s}$ $V(s)=L[sI(s)-i(0^{-})]=sLI(s)-LI_{0}$ sL I V $\frac{I_0}{s}$ sL LI_0 +



A Capacitor in the s Domain





The Natural Response of an RC Circuit



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The Step Response of a Parallel Circuit



Opening the switch results in a step change in the current applied to the circuit

$$sCV + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}$$
$$V = \frac{\frac{I_{dc}}{s^2}}{s^2 + (\frac{1}{RC})s + (\frac{1}{LC})}$$
$$I_L = \frac{\frac{I_{dc}}{LC}}{s[s^2 + (\frac{1}{RC})s + (\frac{1}{LC})]}$$

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$$I_{L} = \frac{384 \times 10^{5}}{s(s^{2} + 64000s + 16 \times 10^{8})}$$

$$I_{L} = \frac{384 \times 10^{5}}{s(s + 32000 - j24000)(s + 32000 + j24000)}$$

$$I_{L} = \frac{K_{1}}{s} + \frac{K_{2}}{s + 32000 - j24000} + \frac{K_{2}^{*}}{s + 32000 + j24000}$$

$$K_{1} = \frac{384 \times 10^{5}}{16 \times 10^{8}} = 24 \times 10^{-3}$$

$$K_{2} = \frac{384 \times 10^{5}}{(-32000 + j24000)(j48000)} = 20 \times 10^{-3} / 126.87^{0}$$

$$i_{L}(t) = [24 + 40e^{-32000t} \cos(24000t + 126.87^{0})]u(t)mA$$

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Transient Response of a Parallel RLC Circuit

Replacing the dc current source with a sinusoidal current source

$$i_{g} = I_{m} \cos \omega t \text{ A} \Rightarrow I_{g}(s) = \frac{sI_{m}}{s^{2} + \omega^{2}}$$

$$V(s) = \frac{(\frac{1}{C})s}{s^{2} + (\frac{1}{RC})s + (\frac{1}{LC})} I_{g}(s)$$

$$V(s) = \frac{(I_{m}/c)s^{2}}{(s^{2} + \omega^{2})[s^{2} + (\frac{1}{RC})s + (\frac{1}{LC})]}$$

$$I_{L}(s) = \frac{V(s)}{sL} = \frac{(I_{m}/c)s}{(s^{2} + \omega^{2})[s^{2} + (\frac{1}{RC})s + (\frac{1}{LC})]}$$

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$$I_{m} = 24mA, \quad \omega = 40000rad / s$$

$$I_{L}(s) = \frac{384 \times 10^{5} s}{(s^{2} + 16 \times 10^{8})(s^{2} + 64000s + 16 \times 10^{8})}$$

$$I_{L}(s) = \frac{K_{1}}{s - j40000} + \frac{K_{1}^{*}}{s + j40000} + \frac{K_{2}}{s + 32000 - j24000} + \frac{K_{2}^{*}}{s + 32000 + j24000}$$

$$K_{1} = \frac{384 \times 10^{5} (j40000)}{(j80000)(32000 + j16000)(32000 + j64000)} = 7.5 \times 10^{-3} / -90^{0}$$

$$K_{2} = \frac{384 \times 10^{5} (-32000 + j24000)}{(-32000 - j16000)(-32000 + j64000)} = 12.5 \times 10^{-3} / 90^{0}$$

$$i_{L}(t) = (15 \sin 40000t - 25e^{-32000} \sin 24000t)u(t)mA$$

$$i_{Lss} = 15 \sin 40000t mA$$

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$$\frac{330}{s} = (42 + 8.4s)I_1 - 42I_2$$

$$0 = -42I_1 + (90 + 10s)I_2$$

$$I_1 = \frac{40(s+9)}{s(s+2)(s+12)} = \frac{15}{s} - \frac{14}{s+2} - \frac{1}{s+12}$$

$$I_2 = \frac{168}{s(s+2)(s+12)} = \frac{7}{s} - \frac{8.4}{s+2} + \frac{1.4}{s+12}$$





$$\begin{split} i_1 &= (15 - 14e^{-2t} - e^{-12t})u(t) \ A, \\ i_2 &= (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t) \ A. \\ i_1(\infty) &= \frac{336(90)}{42(48)} = 15 \ A, \\ i_2(\infty) &= \frac{15(42)}{90} = 7 \ A \end{split}$$

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Thevenin's Theorem



Use the Thevenin's theorem to find v_c(t).

$$V_{Th} = \frac{(480 / s)(0.002s)}{20 + 0.002s} = \frac{480}{s + 10^4}$$

$$Z_{Th} = 60 + \frac{0.002s(20)}{20 + 0.002s} = \frac{80(s + 7500)}{s + 10^4}$$





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$$I_{c} = \frac{480/(s+10^{4})}{[80(s+7500)/(s+10^{4})] + [(2 \times 10^{5})/s]}$$

$$I_{c} = \frac{6s}{s^{2} + 10000s + 25 \times 10^{6}} = \frac{6s}{(s+5000)^{2}}$$

$$I_{c} = \frac{-30000}{(s+5000)^{2}} + \frac{6}{s+5000}$$

$$i_{c}(t) = (-30000te^{-5000t} + 6e^{-5000t})u(t)A$$

$$V_{c} = \frac{1}{sC}I_{c} = \frac{2 \times 10^{5}}{s} \frac{6s}{(s+5000)^{2}} = \frac{12 \times 10^{5}}{(s+5000)^{2}}$$

$$v_{c}(t) = 12 \times 10^{5}te^{-5000t}u(t)V$$

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MUTUAL INDUCTANCE EXAMPLE





The switch has been in position a for a long time. At t=0, the switch moves instantaneously to position b. Find $i_2(t)$

$$i_1(0^-) = \frac{60}{12} = 5A, \quad i_2(0^-) = 0$$

Using the T-equivalent of the inductors, and s-domain equivalent gives the following circuit





$$(3+2s)I_1 + 2sI_2 = 10$$

 $2sI_1 + (12+8s)I_2 = 10$

$$I_2 = \frac{2.5}{(s+1)(s+3)} = \frac{1.25}{s+1} - \frac{1.25}{s+3}$$
$$i_2(t) = 1.25(e^{-t} - e^{-3t})u(t)A$$

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THE TRANSFER FUNCTION

The **transfer function** is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input when all the **initial conditions are zero**.

 $H(s) = \frac{Y(s)}{X(s)}$ Where Y(s) is the Laplace transform of the output, and X(s) is the Laplace transform of the input.











Find the transfer function V_0/V_g and determine the poles and zeros of H(s).

$$\frac{V_0 - V_g}{1000} + \frac{V_0}{250 + 0.05s} + \frac{V_0 s}{10^6} = 0$$

$$V_0 = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6} V_g$$

$$H(s) = \frac{V_0}{V_g} = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}$$

$$p_1 = -3000 + j4000, \quad p_2 = -3000 - j4000$$

$$z_1 = -5000$$

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Assume that $v_g(t)=50tu(t)$. Find $v_0(t)$. Identify the transient and steady-state components of $v_0(t)$.

$$\begin{split} V_0(s) &= H(s)V_g(s) = \frac{1000(s+5000)}{(s^2+6000s+25\times10^6)} \frac{50}{s^2} \\ &= \frac{K_1}{s+3000-j4000} + \frac{K_1^*}{s+3000+j4000} + \frac{K_2}{s^2} + \frac{K_3}{s} \\ K_1 &= 5\sqrt{5}\times10^{-4} \ / \frac{79.7^0}{2}, \ K_2 &= 10, \ K_3 &= -4\times10^{-4} \\ v_0 &= [10\sqrt{5}\times10^{-4} e^{-3000t} \cos(4000t+79.7^0) \\ &+ 10t - 4\times10^{-4}]u(t)V \end{split}$$

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The transient component is generated by the poles of the transfer function and it is

 $10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.7^{\circ})$

The steady-state components are generated by the poles of the driving function (input).

 $(10t - 4 \times 10^{-4})u(t)$



Time Invariant Systems

If the input delayed by a seconds, then

$$L\{x(t-a)u(t-a)\} = e^{-as}X(s)$$
$$Y(s) = H(s)X(s)e^{-as}$$
$$y(t) = L^{-1}\{Y(s)\} = y(t-a)u(t-a)$$

Therefore, delaying the input by a seconds simply delays the response function by a seconds. A circuit that exhibits this characteristic is said to be **time invariant**.



Impulse Response

If a unit impulse source drives the circuit, the response of the circuit equals the inverse transform of the transfer function.

$$x(t) = \delta(t) \Longrightarrow X(s) = 1$$

$$Y(s) = H(s)$$

$$y(t) = L^{-1} \{ H(s) \} = h(t)$$

Note that this is also the natural response of the circuit because the application of an impulsive source is equivalent to instantaneously storing energy in the circuit.



Circuit N is linear with no initial stored energy. **If we know the form of x(t), then how is y(t) described?** To answer this question, we need to know something about N. Suppose we know the impulse response of the system.





Instead of applying the unit impulse at t=0, let us suppose that it is applied at t= λ . The only change in the output is a time delay.

$$\delta(t-\lambda) \longrightarrow \mathbb{N} \longrightarrow h(t-\lambda)$$

Next, suppose that the unit impulse has some strength other than unity. Let the strength be equal to the value of x(t) when $t = \lambda$. Since the circuit is linear, the response should be multiplied by the same constant $x(\lambda)$

$$x(\lambda)\delta(t-\lambda) \longrightarrow \mathbb{N} \longrightarrow x(\lambda)h(t-\lambda)$$



Now let us sum this latest input over all possible values of λ and use the result as a forcing function for N. From the linearity, the response is the sum of the responses resulting from the use of all possible values of λ

$$\int_{-\infty}^{+\infty} x(\lambda) \delta(t-\lambda) d\lambda \longrightarrow \mathbb{N} \longrightarrow \int_{-\infty}^{+\infty} x(\lambda) h(t-\lambda) d\lambda$$

From the sifting property of the unit impulse, we see that the input is simply x(t)

$$X(t) \longrightarrow \mathbb{N} \longrightarrow \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$$



Our question is now answered. When x(t) is known, and h(t), the unit impulse response of N is known, the response is expressed by

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda) h(t - \lambda) d\lambda$$

This important relation is known as the **convolution integral.** It is often abbreviated by means of y(t) = x(t) * h(t)

Where the asterisk is read "convolved with". If we let $z=t-\lambda$, then $d\lambda=-dz$, and the expression for y(t) becomes

$$y(t) = \int_{-\infty}^{\infty} -x(t-z)h(z)dz = \int_{-\infty}^{+\infty} x(t-z)h(z)dz$$
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(z)h(t-z)dz = \int_{-\infty}^{\infty} x(t-z)h(z)dz$$

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Convolution and Realizable Systems

For a physically realizable system, the response of the system cannot begin before the forcing function is applied. Since h(t)is the response of the system when the unit impulse is applied at t=0, h(t) cannot exist for t<0. It follows that, in the second integral, the integrand is zero when z<0; in the first integral, the integrand is zero when (t-z) is negative, or when z>t. Therefore, for **realizable systems** the convolution integral becomes

$$y(t) = x(t) * h(t) = \int_{-\infty}^{t} x(z)h(t-z)dz = \int_{0}^{\infty} x(t-z)h(z)dz$$

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Graphical Method of Convolution



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Since h(z) does not exist prior to t=0 and $v_i(t-z)$ does not exist for z>t, product of these functions has nonzero values only in the interval of 0<z<t for the case shown where t<1.

$$y(t) = \int_0^t 2e^{-z} dz = 2(1 - e^{-t}) \quad 0 \le t \le 1$$

When t>1, the nonzero values for the product are obtained in the interval (t-1) < z < t.

$$y(t) = \int_{t-1}^{t} 2e^{-z} dz = 2(e-1)e^{-t} \quad t > 1$$

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Apply a unit-step function, x(t)=u(t), as the input to a system whose impulse response is h(t)=u(t)-2u(t-1)+u(t-2), and determine the corresponding output y(t)=x(t)*h(t).







When t<0, there is no overlap and y(t)=0 for t<0

For 0 < t < 1, the curves overlap from z=0 to z=t and product is 1. Thus,

$$y(t) = \int_0^t 1 dz = t$$
 $0 < t < 1$

When 1 < t < 2, h(t-z) has slid far enough to the right to bring under the step function that part of the negative square extending from 0 to z=t-1. Thus,

$$y(t) = \int_0^{t-1} -1dz + \int_{t-1}^t 1dz = -z \Big|_0^{t-1} + z \Big|_{t-1}^t = 2 - t, \quad 1 < t < 2$$

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Finally, when t>2, h(t-z) has slid far enough to the right so that it lies entirely to the right of z=0

$$y(t) = \int_{t-2}^{t-1} -1dz + \int_{t-1}^{t} 1dz = 0 \qquad t > 2$$



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Convolution and the Laplace Transform

Let $F_1(s)$ and $F_2(s)$ be the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively. Now, consider the laplace transform of $f_1(t)*f_2(t)$,

$$L\{f_1(t) * f_2(t)\} = L\{\int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda) d\lambda\}$$

Since we are dealing with the time functions that do not exist prior to $t=0^{-}$, the lower limit can be changed to 0^{-}

$$L\{f_{1}(t) * f_{2}(t)\} = \int_{0^{-}}^{\infty} \left[\int_{0^{-}}^{\infty} e^{-st} f_{1}(\lambda) f_{2}(t-\lambda) dt\right] d\lambda$$

Since e^{-st} does not depend on λ , we can move this inside the inner integral. Reversing the order of integration

$$L\{f_{1}(t) * f_{2}(t)\} = \int_{0^{-}}^{\infty} e^{-st} \left[\int_{0^{-}}^{\infty} f_{1}(\lambda) f_{2}(t-\lambda) d\lambda\right] dt$$

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 $f_{1}(\lambda) \text{ does not depend on } t, \text{ and it can be moved outside the inner integral}} L\{f_{1}(t) * f_{2}(t)\} = \int_{0^{-}}^{\infty} f_{1}(\lambda) [\int_{0^{-}}^{\infty} e^{-st} f_{2}(t-\lambda)dt] d\lambda$ $= \int_{0^{-}}^{\infty} f_{1}(\lambda) [\int_{0^{-}}^{\infty} e^{-s(x+\lambda)} f_{2}(x)dx] d\lambda$

$$= \int_{0^{-}}^{\infty} f_1(\lambda) e^{-s\lambda} \left[\int_{0^{-}}^{\infty} e^{-sx} f_2(x) dx \right] d\lambda$$
$$= \int_{0^{-}}^{\infty} f_1(\lambda) e^{-s\lambda} \left[F_2(s) \right] d\lambda$$

$$= F_2(s) \int_{0^-}^{\infty} f_1(\lambda) e^{-s\lambda} d\lambda$$
$$= F_1(s) \cdot F_2(s)$$

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STEADY-STATE SINUSOIDAL RESPONSE

If the input of a circuit is a sinusoidal function $x(t) = A\cos(\omega t + \theta)$

$$x(t) = A\cos\omega t\cos\theta - A\sin\omega t\sin\theta$$
$$X(s) = \frac{(A\cos\theta)s}{s^2 + \omega^2} - \frac{(A\sin\theta)\omega}{s^2 + \omega^2}$$
$$= \frac{A(s\cos\theta - \omega\sin\theta)}{s^2 + \omega^2}$$
$$Y(s) = H(s)\frac{A(s\cos\theta - \omega\sin\theta)}{s^2 + \omega^2}$$

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 $Y(s) = \frac{K_1}{s - j\omega} + \frac{K_1^*}{s + j\omega} + \sum \text{ terms generated by the poles of H(s)}$

If the poles of H(s) lie in the left half of the s plane, the corresponding time-domain terms approach zero as t increases and they do not contribute to the steady-state response. Thus only the first two terms determine the steady-state response.

$$K_{1} = \frac{H(s)A(s\cos\theta - \omega\sin\theta)}{s + j\omega} \bigg|_{s=j\omega}$$

$$= \frac{H(j\omega)A(j\omega\cos\theta - \omega\sin\theta)}{2j\omega}$$

$$= \frac{H(j\omega)A(\cos\theta + j\sin\theta)}{2} = \frac{1}{2}H(j\omega)Ae^{j\theta}$$

$$H(j\omega) = |H(j\omega)|e^{j\phi(\omega)}$$

$$K_{1} = \frac{A}{2}|H(j\omega)|e^{j[\theta + \phi(\omega)]}$$

$$y_{ss}(t) = A|H(j\omega)|\cos[\omega t + \theta + \phi(\omega)]$$

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$$EXAMPLE$$
If the input is 120 cos(5000t+30°)V, find the steady-state expression for v₀

$$H(s) = \frac{1000(s+5000)}{s^2+6000s+25\times10^6}$$

$$H(j5000) = \frac{1000(5000+j5000)}{-25\times10^6+j5000(6000)+25\times10^6}$$

$$= \frac{1+j1}{j6} = \frac{\sqrt{2}}{6} \ /-45^{\circ}$$

$$v_{0ss} = \frac{120\sqrt{2}}{6} \cos(5000t+30^{\circ}-45^{\circ})$$

$$= 20\sqrt{2}\cos(5000t-15^{\circ})V$$

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THE IMPULSE FUNCTION IN CIRCUIT ANALYSIS





The capacitor is charged to an initial voltage V_0 at the time the switch is closed. Find the expression for i(t) as $R \rightarrow 0$

$$I = \frac{\frac{V_0}{s}}{R + (\frac{1}{sC_1}) + (\frac{1}{sC_2})} = \frac{\frac{V_0}{R}}{s + (\frac{1}{RC_e})}$$
$$C_e = \frac{C_1 C_2}{C_1 + C_2} \qquad i(t) = \left(\frac{V_0}{R} e^{-t/RC_e}\right) u(t)$$

As R decreases, the initial current (V_0/R) increases and the time constant (RC_e) decreases. Apparently i is approaching an impulse function as R approaches zero.



The total area under the i versus t curve represents the total charge transferred to C_2 after the switch is closed.

Area =
$$q = \int_{0^{-}}^{\infty} \frac{V_0}{R} e^{-t/RC_e} dt = V_0 C_e$$

Thus, as R approaches zero, the current approaches an impulse strength V_0C_e . $i \rightarrow V_0C_e\delta(t)$



Series Inductor Circuit





Find v_0 . Note that opening the switch forces an instantaneous change in the current L_2 .

$$i_1(0^-) = 10A, \quad i_2(0^-) = 0$$

$$\frac{V_0}{2s+15} + \frac{V_0 - [(\frac{100}{s}) + 30]}{3s+10} = 0$$
$$V_0 = \frac{40(s+7.5)}{s(s+5)} + \frac{12(s+7.5)}{s+5}$$
$$V_0 = \frac{60}{s} + 12 + \frac{10}{s+5}$$
$$v_0(t) = 12\delta(t) + (60 + 10e^{-5t})u(t)V$$



Does this solution make sense? To answer this question, first let us determine the expression for the current.

$$I = \frac{\binom{100}{s} + 30}{5s + 25} = \frac{4}{s} + \frac{2}{s + 5}$$
$$i(t) = (4 + 2e^{-5t})u(t)A$$

Before the switch is opened, current through L_1 is 10A and in L_2 is 0 A, after the switch is opened both currents are 6A. Then the current in L_1 changes instantaneously from 10 A to 6 A, while the current in L_2 changes instantaneously from 0 to 6 A. How can we verify that these instantaneous jumps in the inductor current make sense in terms of the physical behavior of the circuit?

Switching operation places two inductors in series. Any impulsive voltage appearing across the 3H inductor must be balanced by an impulsive voltage across the 2H inductor. Faraday's law states that the induced voltage is proportional to the change in flux linkage $(v = \frac{d\lambda}{dt})$ before switching

$$\lambda = L_1 i_1 + L_2 i_2 = 3(10) + 2(0) = 30$$
 Wb - turns

After switching

$$\lambda = (L_1 + L_2)i(0^+) = 5i(0^+)$$

$$i(0^+) = \frac{30}{5} = 6A$$

Thus the solution agrees with the principle of the conservation of flux linkage.



Impulsive Sources



When the voltage source is applied, the initial energy in the inductor is zero; therefore the initial current is zero. There is no voltage drop across R, so the impulsive source appears directly across L

$$i = \frac{1}{L} \int_{0^{-}}^{t} V_0 \delta(x) dx \Longrightarrow i(0^+) = \frac{V_0}{L} A$$

Thus, in an infinitesimal moment, the impulsive voltage source has stored

$$w = \frac{1}{2} L \left(\frac{V_0}{L}\right)^2 = \frac{1}{2} \frac{V_0^2}{L} J$$

Current in the circuit decays to zero in accordance with the natural response of the circuit

$$i = \frac{V_0}{L} e^{-\binom{R}{L}t} u(t)$$

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EXAMPLE





