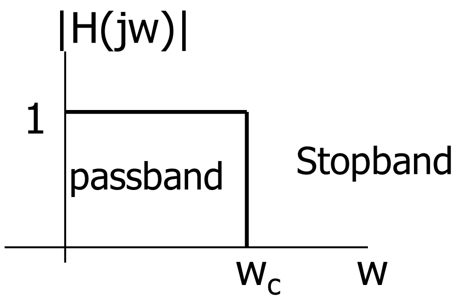
FREQUENCY SELECTIVE CIRCUITS

Osman Parlaktuna
Osmangazi University
Eskisehir, TURKEY
www.ogu.edu.tr/~oparlak

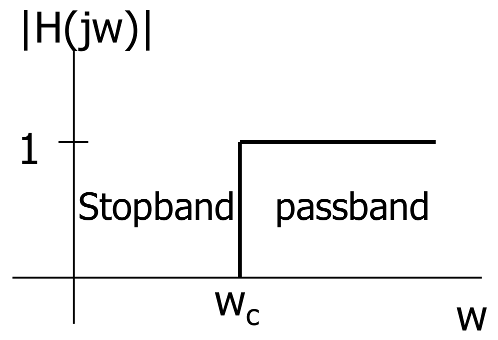
FILTERS

- Varying the frequency of a sinusoidal source alters the impedance of inductors and capacitors. The careful choice of circuit elements and their connections enables us to construct circuits that pass a desired range of frequencies. Such circuits are called **frequency selective circuits** or **filters**.
- The frequency range that frequencies allowed to pass from the input to the output of the circuit is called the **passband**.
- Frequencies not in a circuit's passband are in its **stopband**.

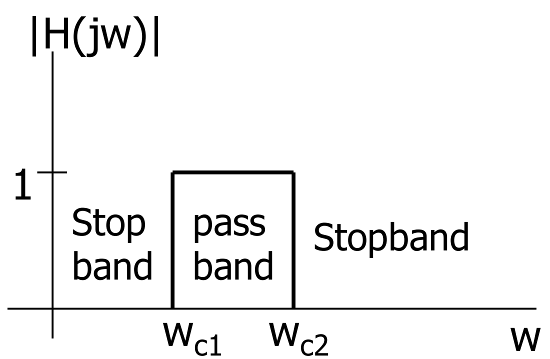
IDEAL FILTERS



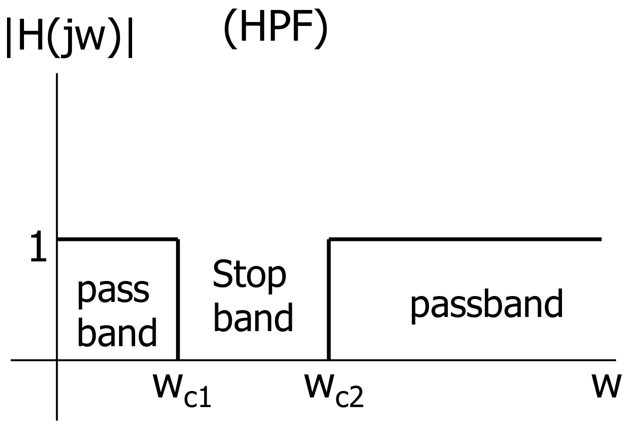
Ideal low-pass filter (LPF)



Ideal high-pass filter (HPF)

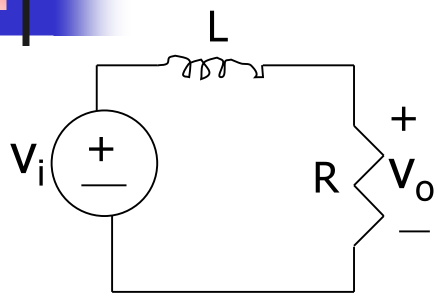


Ideal band-pass filter (BPF)

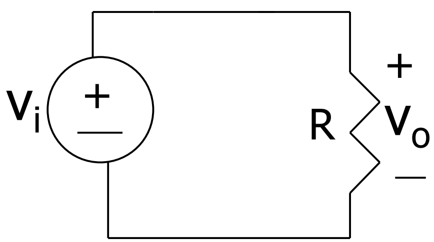


Ideal band-reject filter (BRF)

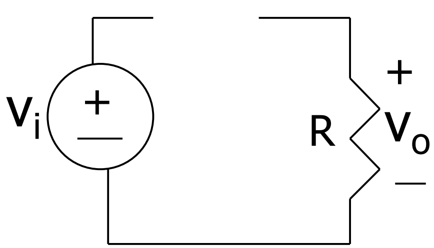
LOW-PASS FILTERS



The input is a sinusoidal voltage source with varying frequency. The impedance of an inductor is jwL.

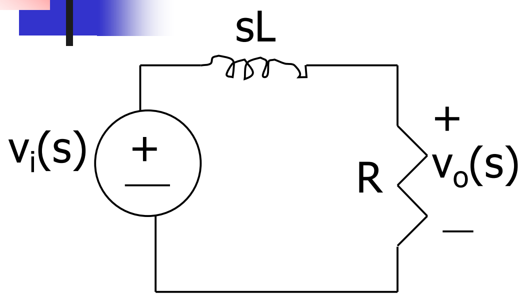


At low frequencies, the inductor’s impedance is very small compared to that of resistor impedance, and the inductor effectively functions as a short circuit. $v_o=v_i$.



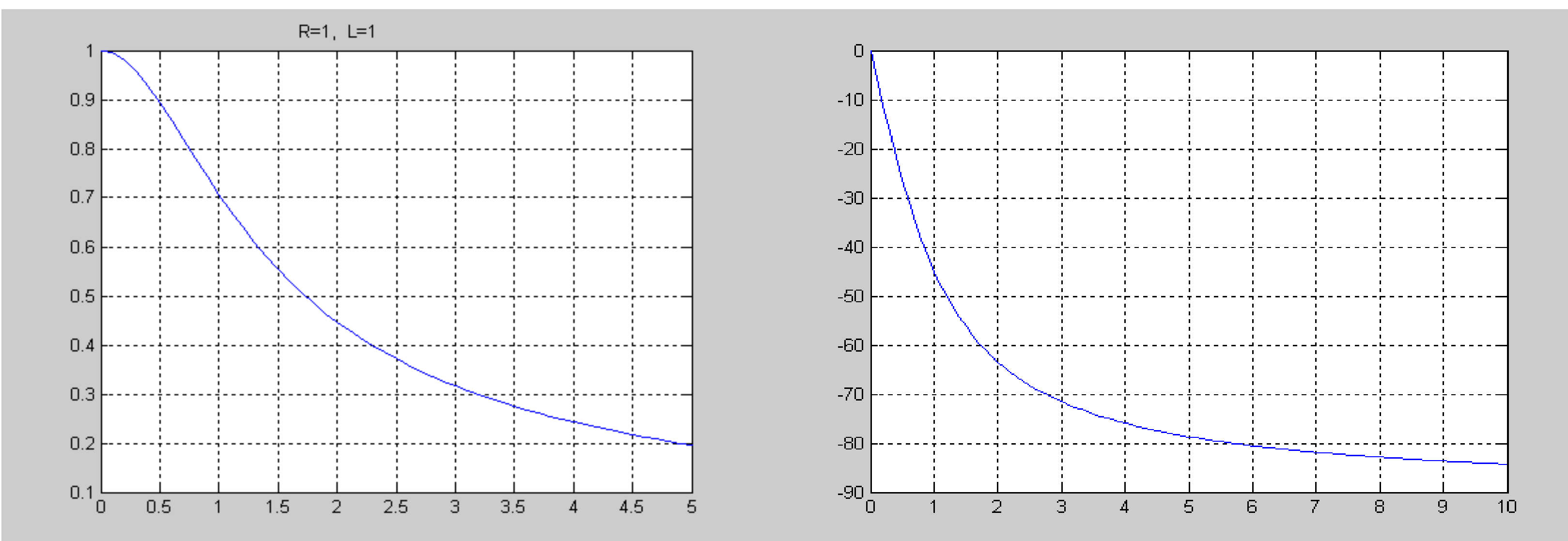
As the frequency increases, the impedance of the inductor increases relative to that of the resistor. The magnitude of the output voltage decreases. As $w \longrightarrow \infty$, inductor approaches to open circuit. Output voltage approaches zero.

QUANTITATIVE ANALYSIS



$$H(s)=\frac{\frac{R}{L}}{s+\frac{R}{L}} \Rightarrow H(j\omega)=\frac{\frac{R}{L}}{j\omega+\frac{R}{L}}$$

$$\left|H(j\omega)\right|=\frac{\frac{R}{L}}{\sqrt{\omega^2+\left(\frac{R}{L}\right)^2}}, \qquad \theta(\omega)=-\tan^{-1}\left(\frac{\omega L}{R}\right)$$



CUTOFF FREQUENCY

Cutoff frequency is the frequency at which the magnitude of the transfer function is $H_{max}/\sqrt{2}$.

For the series RL circuit, $H_{max}=1$. Then,

$$\left|H(j\omega_c)\right| = \frac{1}{\sqrt{2}} = \frac{\frac{R}{L}}{\sqrt{\omega_c^2 + \left(\frac{R}{L}\right)^2}} \Rightarrow \omega_c = \frac{R}{L}$$

At the cutoff frequency $\theta(w_c)=-45^0$.

For the example $R=1\Omega$, $L=1H$, $w_c=1$ rad/s

EXAMPLE

Design a series RL low-pass filter to filter out any noise above 10 Hz.

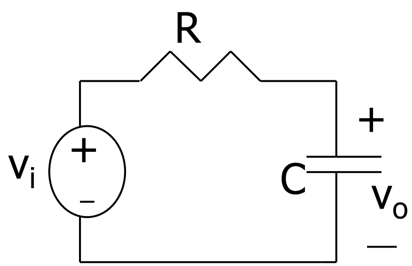
R and L cannot be specified independently to generate a value for w_c. Therefore, let us choose L=100 mH. Then,

$$R = \omega_c L = (2\pi)(10)(100\times 10^{-3}) = 6.28\Omega$$

$$|V_o(\omega)| = \frac{\frac{R}{L}}{\sqrt{\omega^2 + \left(\frac{R}{L}\right)^2}}|V_i| = \frac{20\pi}{\sqrt{\omega^2 + 400\pi^2}}|V_i|$$

F(Hz)	$\lvert V_i\rvert$	$\lvert V_o\rvert$
1	1.0	0.995
10	1.0	0.707
60	1.0	0.164

A SERIES RC CIRCUIT



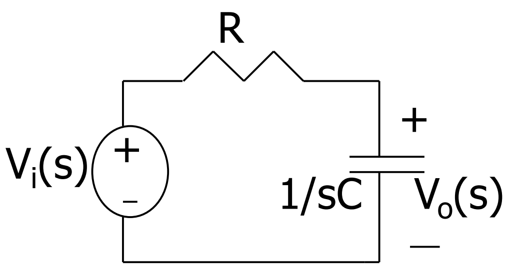
1) Zero frequency w=0: The impedance of the capacitor is infinite, and the capacitor acts as an open circuit. $v_o=v_i$.

2) Increasing frequency decreases the impedance of the capacitor relative to the impedance of the resistor, and the source voltage divides between the resistor and capacitor. The output voltage is thus smaller than the source voltage.

3) Infinite frequency $w=\infty$: The impedance of the capacitor is zero, and the capacitor acts as a short circuit. $v_o=0$.

Based on the above analysis, the series RC circuit functions as a low-pass filter.

QUANTITATIVE ANALYSIS



$$H(s)=\frac{\frac{1}{RC}}{s+\frac{1}{RC}} \Rightarrow \left|H(j\omega)\right|=\frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

$$H_{\max}=\left|H(j0)\right|=1$$

$$\left|H(j\omega_c)\right|=\frac{1}{\sqrt{2}}(1)=\frac{\frac{1}{RC}}{\sqrt{\omega_c^2+\left(\frac{1}{RC}\right)^2}}$$

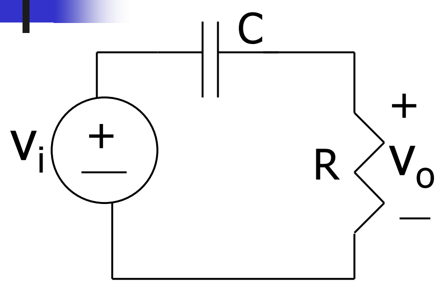
$$\omega_c=\frac{1}{RC}$$

Relating the Frequency Domain to the Time Domain

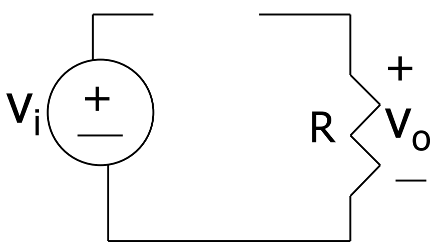
- Remember the natural response of the first-order RL and RC circuits. It is an exponential with a time constant of $\tau = \frac{L}{R}$ or $\tau = RC$
- Compare the time constants to the cutoff frequencies for these circuits and notice that

$$\tau = \frac{1}{\omega_c}$$

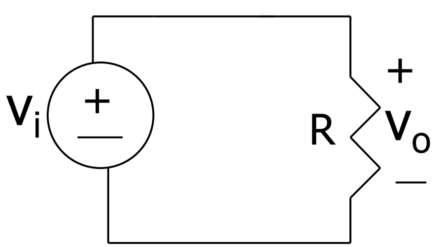
HIGH-PASS FILTERS



At w=0, the capacitor behaves like an open circuit. No current flows through resistor. Therefore v_o=0.

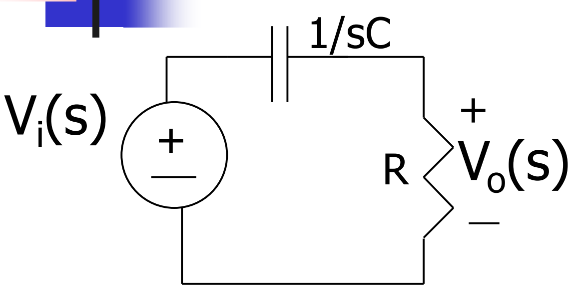


As the frequency increases, the impedance of the capacitor decreases relative to the impedance of the resistor and the voltage across the resistor increases.



When the frequency of the source is infinite, the capacitor behaves as a short circuit and v_o=v_i.

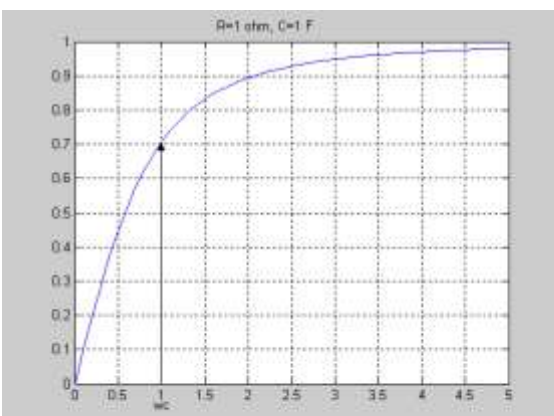
QUANTITATIVE ANALYSIS

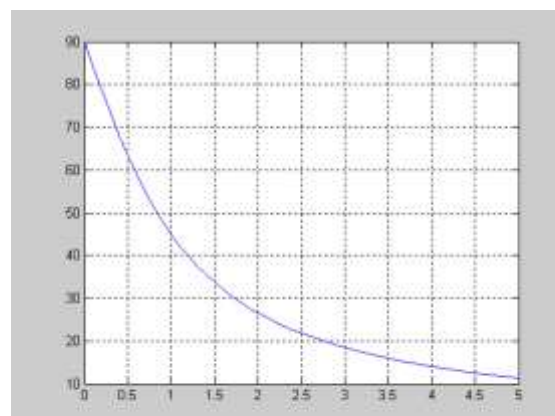


$$H(s)=\frac{s}{s+\frac{1}{RC}}\Rightarrow H(j\omega)=\frac{j\omega}{j\omega+\frac{1}{RC}}$$

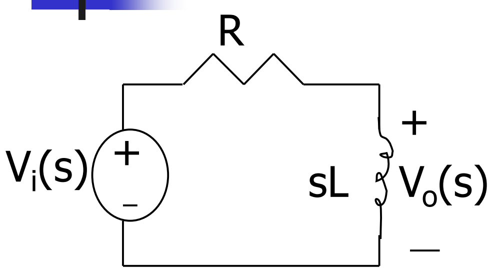
$$\left|H(j\omega)\right|=\frac{\omega}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

$$\theta(j\omega)=90^0-\tan^{-1}\omega RC$$





SERIES RL CIRCUIT



$$H(s)=\frac{s}{s+\frac{R}{L}} \Rightarrow H(j\omega)=\frac{j\omega}{j\omega+\frac{R}{L}}$$

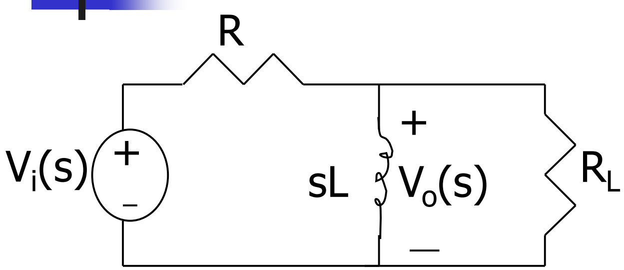
$$\left|H(j\omega)\right|=\frac{\omega}{\sqrt{\omega^2+\left(\frac{R}{L}\right)^2}}$$

$$H_{\max}=\left|H(j\infty)\right|=1$$

$$\frac{1}{\sqrt{2}}=\left|H(j\omega_c)\right|=\frac{\omega_c}{\sqrt{\omega_c^2+\left(\frac{R}{L}\right)^2}}$$

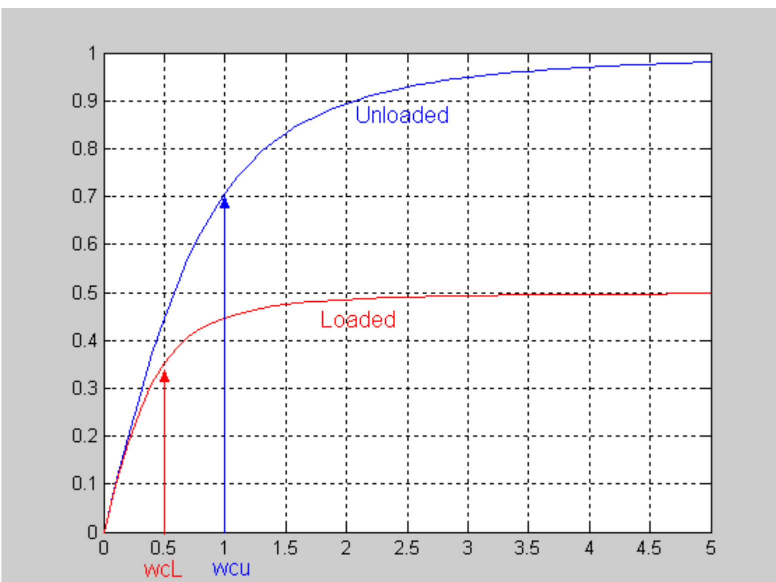
$$\omega_c=\frac{R}{L}$$

LOADED SERIES RL CIRCUIT



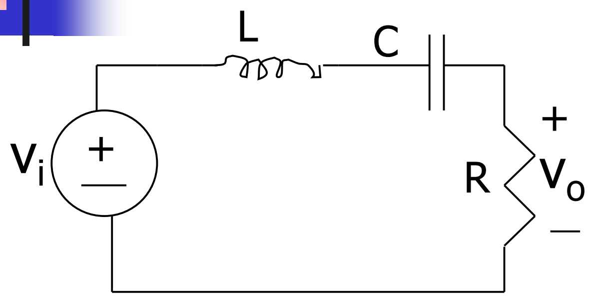
$$H(s) = \frac{\frac{R_L sL}{R_L + sL}}{R + \frac{R_L sL}{R_L + sL}} = \frac{\left(\frac{R_L}{R+R_L}\right)s}{s + \left(\frac{R_L}{R+R_L}\right)\left(\frac{R}{L}\right)}$$

$$H(s) = \frac{Ks}{s+\omega_c}, \quad K = \left(\frac{R_L}{R+R_L}\right), \quad \omega_c = KR/L$$

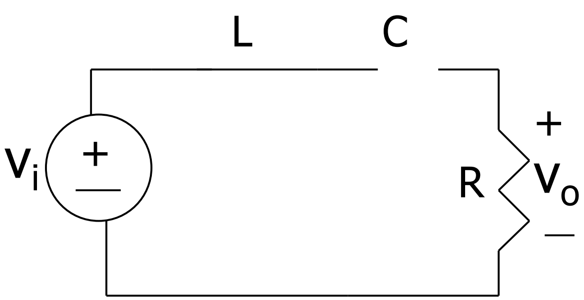


The effect of the load resistor is to reduce the passband magnitude by the factor K and to lower the cutoff frequency by the same factor.

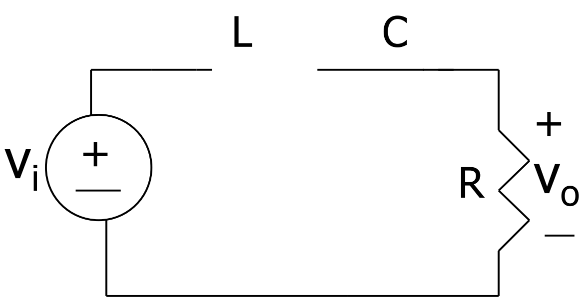
BANDPASS FILTERS



At w=0, the capacitor behaves like an open circuit, and the inductor behaves like a short circuit. The current through the circuit is zero, therefore v_o=0.



At w=∞, the capacitor behaves like a short circuit, and the inductor behaves like an open circuit. The current through the circuit is zero, therefore v_o=0.

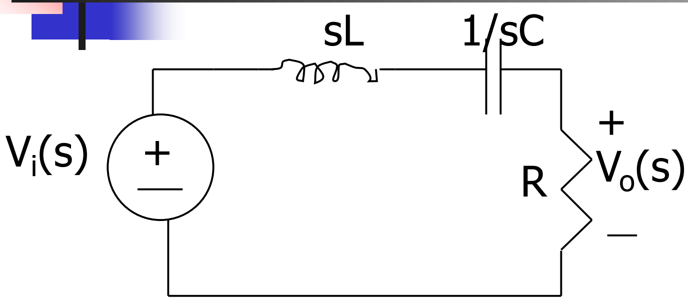


Between these two frequencies, both the capacitor and the inductor have finite impedances. Current will flow through the circuit and some voltage reaches to the load.

CENTER FREQUENCY

At some frequency, the impedance of the capacitor and the impedance of the inductor have equal magnitudes and opposite signs; the two impedances cancel out, causing the output voltage to equal the source voltage. This special frequency is called the **center frequency** w_o. On either side of w_o, the output voltage is less than the source voltage. Note that at the center frequency, the series combination of the inductor and capacitor appears as a short circuit and circuit behaves as a purely resistive one.

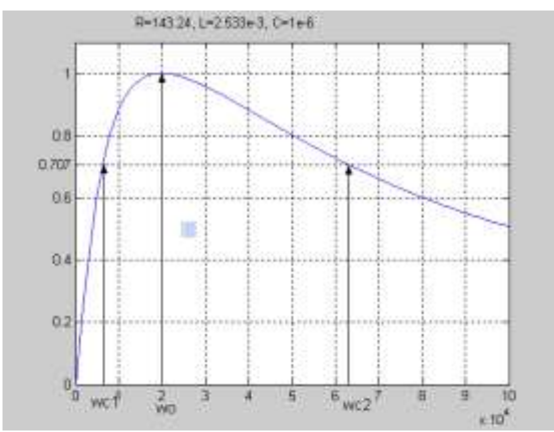
QUANTITATIVE ANALYSIS

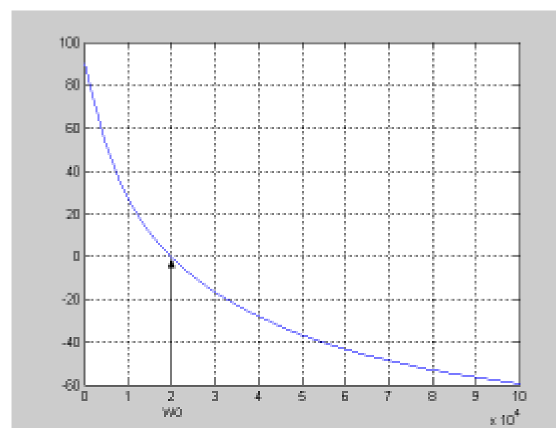


$$H(s)=\frac{\frac{R}{L}s}{s^2+\left(\frac{R}{L}\right)s+\frac{1}{LC}}$$

$$\left|H(j\omega)\right|=\frac{\omega\frac{R}{L}}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2+\left(\omega\frac{R}{L}\right)^2}}$$

$$\theta(j\omega)=90^0-\tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC}-\omega^2}\right)$$





CENTER FREQUENCY AND CUTOFF FREQUENCIES

At the center frequency, the transfer function will be real, that means:

$$j\omega_o L + \frac{1}{j\omega_o C} = 0 \Rightarrow \omega_o = \frac{1}{\sqrt{LC}}$$

At the cutoff frequencies, the magnitude of H(jw) is $H_{max}/\sqrt{2}$.

$$H_{\max} = \left|H(j\omega_o)\right| = \frac{\omega_o \frac{R}{L}}{\sqrt{\left(\frac{1}{LC} - \omega_o^2\right)^2 + \left(\frac{\omega_o R}{L}\right)^2}}$$

$$= \frac{\sqrt{\frac{1}{LC}}\,\frac{R}{L}}{\sqrt{\left(\frac{1}{LC} - \frac{1}{LC}\right)^2 + \left(\sqrt{\frac{1}{LC}}\,\frac{R}{L}\right)^2}} = 1$$

$$\frac{1}{\sqrt{2}} = \frac{\omega_c \frac{R}{L}}{\sqrt{\left(\frac{1}{LC} - \omega_c^2\right)^2 + \left(\omega_c \frac{R}{L}\right)^2}} = \frac{1}{\sqrt{\left(\omega_c \frac{R}{L} - \frac{1}{\omega_c RC}\right)^2 + 1}}$$

$$\pm 1 = \omega_c \frac{L}{R} - \frac{1}{\omega_c RC} \Rightarrow \omega_c^2 L \pm \omega_c R - 1/C = 0$$

$$\omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

$$\omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

It is easy to show that the center frequency is the geometric mean of the two cutoff frequencies

$$\omega_o = \sqrt{\omega_{c1} \cdot \omega_{c2}}$$

BANDWIDTH AND THE QUALITY FACTOR

The bandwidth of a bandpass filter is defined as the difference between the two cutoff frequencies.

$$\beta = \omega_{c2} - \omega_{c1}$$

$$= \left[\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \right] - \left[-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)} \right]$$

$$= \frac{R}{L}$$

The quality factor is defined as the ratio of the center frequency to bandwidth

$$Q = \frac{\omega_o}{\beta} = \frac{\sqrt{\frac{1}{LC}}}{\frac{R}{L}} = \sqrt{\frac{L}{CR^2}}$$

CUTOFF FREQUENCIES IN TERMS OF CENTER FREQUENCY, BANDWIDTH AND QUALITY FACTOR

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}$$

$$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}$$

$$\omega_{c1} = \omega_o\left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right]$$

$$\omega_{c2} = \omega_o\left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right]$$

EXAMPLE

Design a series RLC bandpass filter with cutoff frequencies 1kHz and 10 kHz.

Cutoff frequencies give us two equations but we have 3 parameters to choose. Thus, we need to select a value for either R, L, and C and use the equations to find other values. Here, we choose C=1μF.

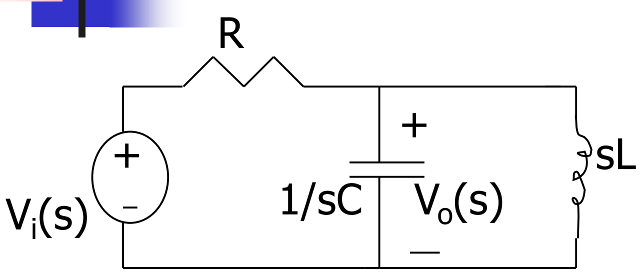
$$f_o = \sqrt{f_{c1} f_{c2}} = \sqrt{(1000)(10000)} = 3162.28\text{Hz}$$

$$L = \frac{1}{\omega_o^2 C} = \frac{1}{\left[2\pi(3162.28)\right]^2 (10^{-6})} = 2.533\,\text{mH}$$

$$Q = \frac{f_o}{f_{c2} - f_{c1}} = \frac{3162.28}{10000 - 1000} = 0.3514$$

$$R = \sqrt{\frac{L}{CQ^2}} = \sqrt{\frac{2.533(10^{-3})}{(10^{-6})(0.3514)^2}} = 143.24\Omega$$

EXAMPLE



$$Z_{eq}(s)=\frac{sL(\frac{1}{sC})}{sL+\frac{1}{sC}}=\frac{\frac{L}{C}}{sL+\frac{1}{sC}}$$

$$H(s)=\frac{\frac{s}{RC}}{s^2+\frac{s}{RC}+\frac{1}{LC}}$$

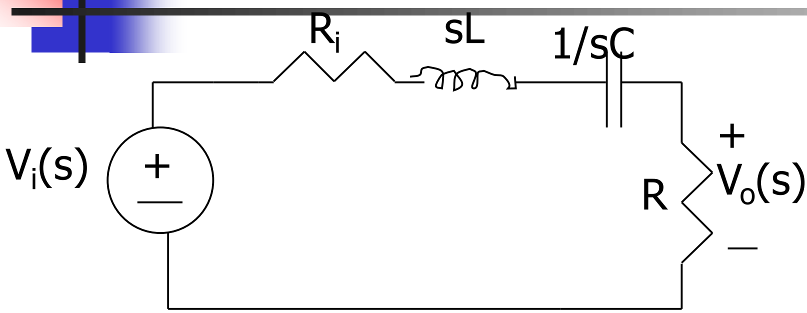
This transfer function and the transfer function for the series RLC bandpass filter are equal when R/L=1/RC. Therefore, all parameters of this circuit can be obtained by replacing R/L by 1/RC. Thus,

$$\omega_o=\sqrt{\frac{1}{LC}} \qquad \beta=\frac{1}{RC} \qquad Q=\sqrt{\frac{R^2C}{L}}$$

$$\omega_{c1}=-\frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\left(\frac{1}{LC}\right)}$$

$$\omega_{c2}=\frac{1}{2RC}+\sqrt{\left(\frac{1}{2RC}\right)^2+\left(\frac{1}{LC}\right)}$$

BANDPASS FILTER WITH PRACTICAL VOLTAGE SOURCE



$$H(s) = \frac{\frac{R}{L}s}{s^2 + \left(\frac{R+R_i}{L}\right)s + \frac{1}{LC}}$$

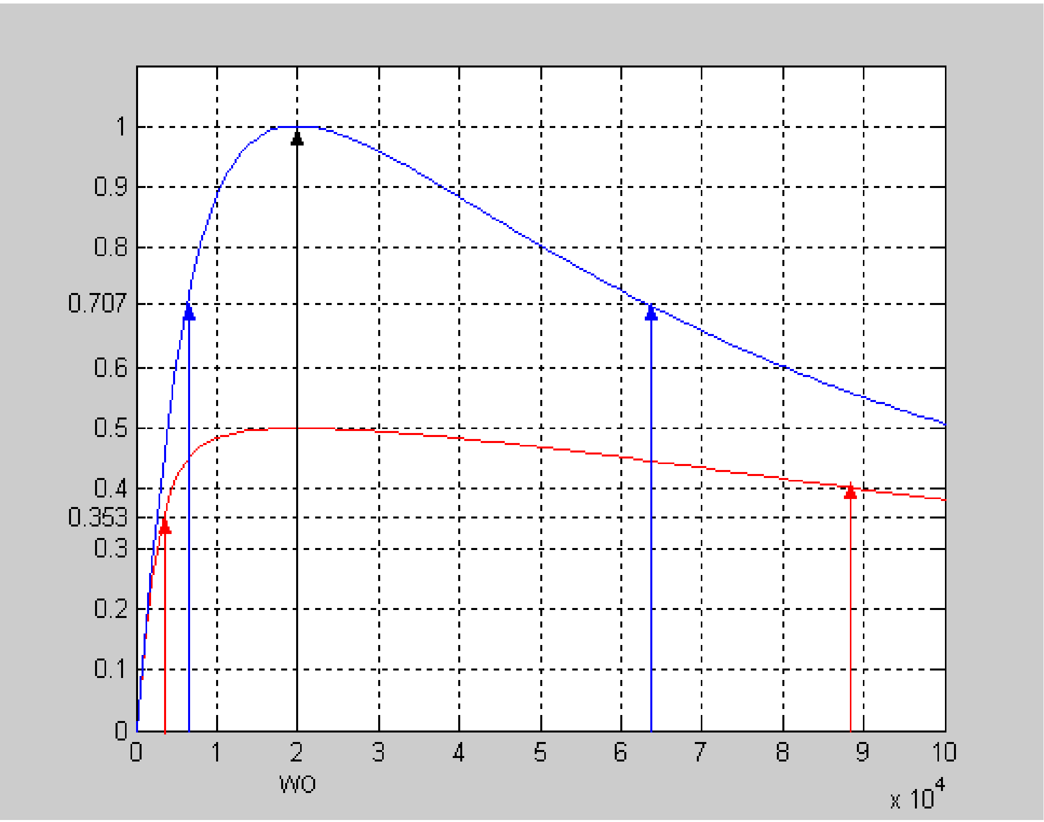
$$\left|H(j\omega)\right| = \frac{\frac{R}{L}\omega}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\omega\frac{R+R_i}{L}\right)^2}}$$

$$\omega_{c1} = -\frac{R+R_i}{2L} + \sqrt{\left(\frac{R+R_i}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_o = \frac{1}{\sqrt{LC}} \Rightarrow H_{max} = \left|H(j\omega_o)\right| = \frac{R}{R+R_i}$$

$$\omega_{c2} = \frac{R+R_i}{2L} + \sqrt{\left(\frac{R+R_i}{2L}\right)^2 + \frac{1}{LC}}$$

$$\beta = \frac{R+R_i}{L} \qquad Q = \frac{\sqrt{L/C}}{R+R_i}$$



The addition of a nonzero source resistance to a series RLC bandpass filter leaves the center frequency unchanged but widens the passband and reduces the passband magnitude.

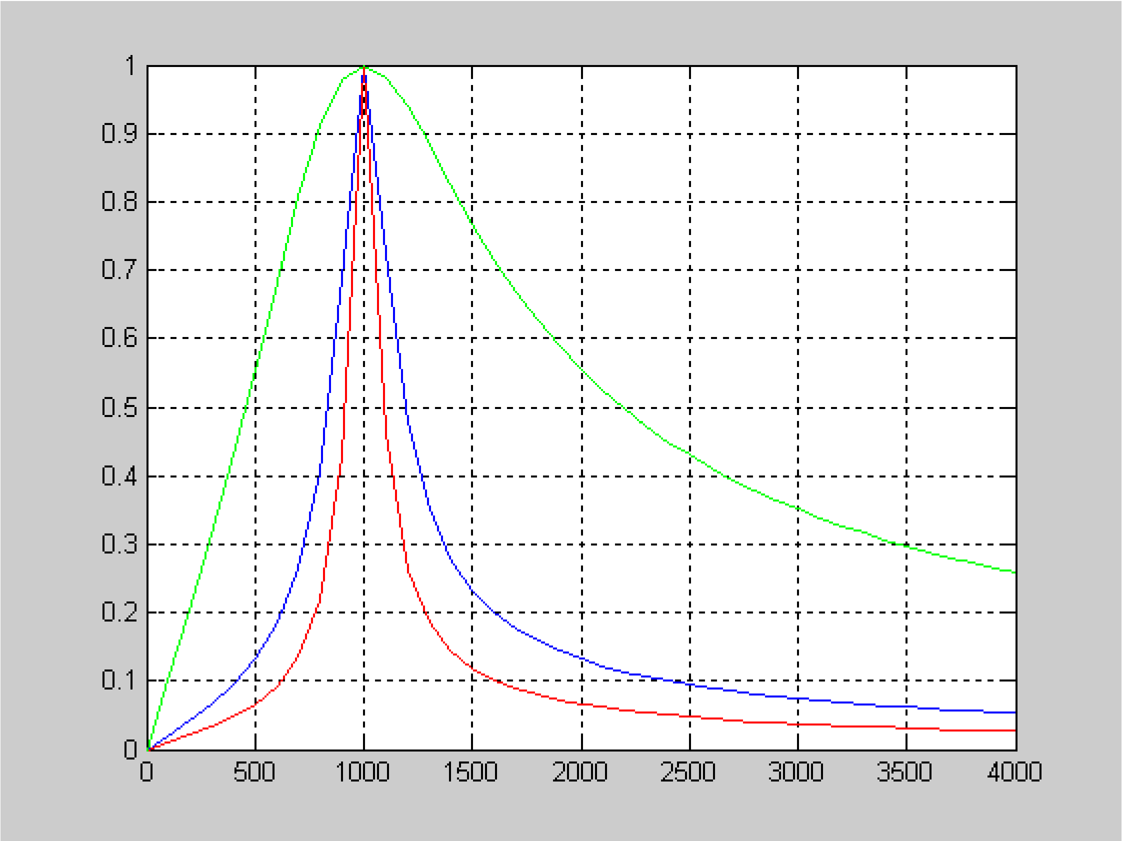
Relating the Frequency Domain to the Time Domain

The natural response of a series RLC circuit is related to the neper frequency $\alpha = \frac{R}{2L}$ and the resonant frequency $\omega_o = \sqrt{\frac{1}{LC}}$

ω_o is also used in frequency domain as the center frequency.

The bandwidth and the neper frequency are related by $\beta = 2\alpha$

The natural response of a series RLC circuit may be underdamped, overdamped, or critically damped. The transition from overdamped to criticallydamped occurs when $\omega_o^2 = \alpha^2$. The transition from an overdamped to an underdamped response occurs when Q=1/2. A circuit whose frequency response contains a sharp peak at the center frequency indicates a high Q and a narrow bandwidth, will have an underdamped natural response.

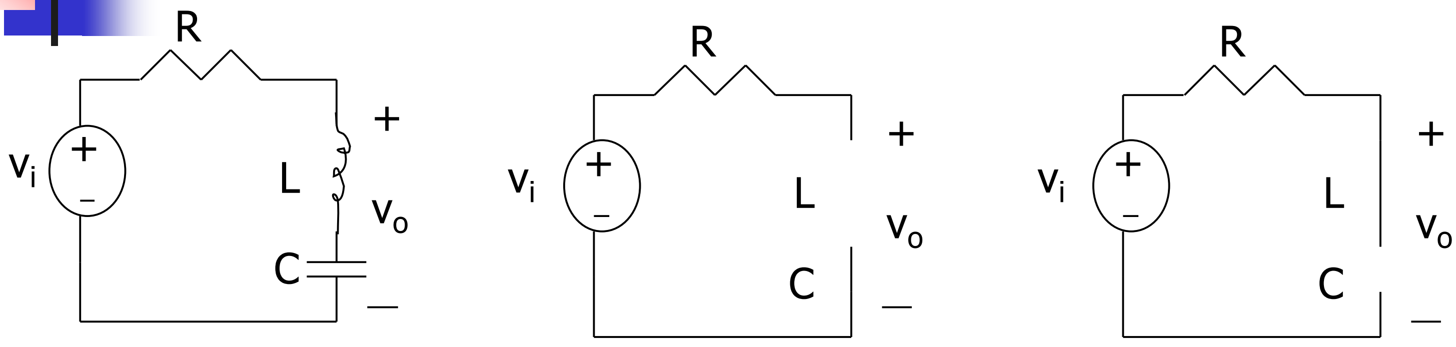


R=200Ω,
L=1H, C=1μF

R=200Ω,
L=2H, C=0.5μF

R=200Ω,
L=0.2H, C=5μF

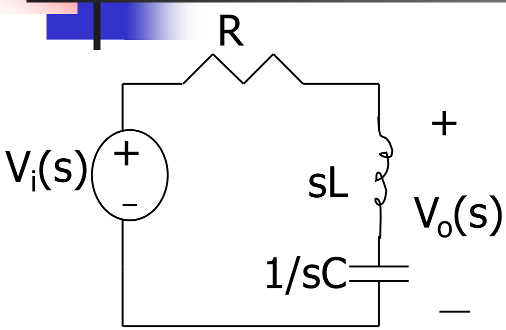
BANDREJECT FILTERS



At w=0, the capacitor behaves as an open circuit and inductor behaves as a short circuit. At w=∞, these roles switch. The output is equal to the input. This RLC circuit then has two passbands, one below a lower cutoff frequency, and the other is above an upper cutoff frequency.

Between these two passbands, both the inductor and the capacitor have finite impedances of opposite signs. Current flows through the circuit. Some voltage drops across the resistor. Thus, the output voltage is smaller than the source voltage.

QUANTITATIVE ANALYSIS



$$H(s) = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{s^2 + \frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$|H(j\omega)| = \frac{\left|\frac{1}{LC} - \omega^2\right|}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{\omega R}{L}\right)^2}}$$

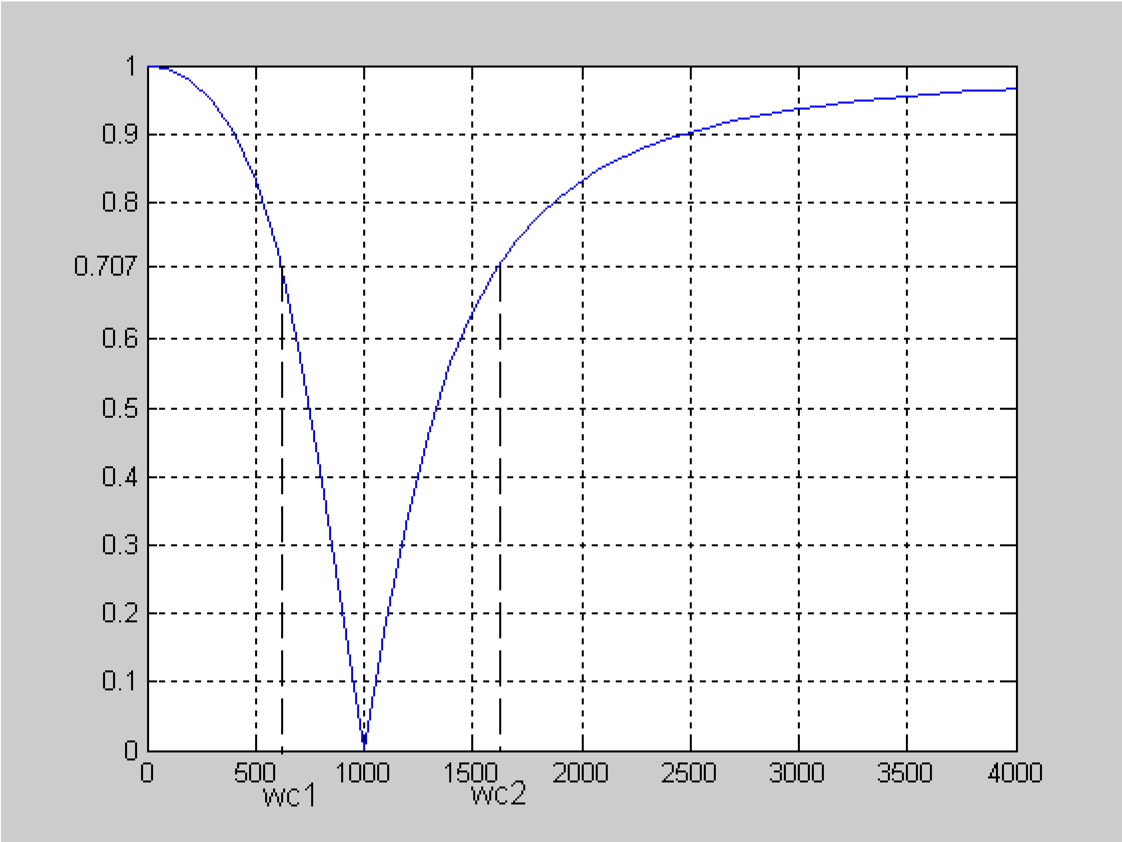
$$\theta(j\omega) = -\tan^{-1}\left(\frac{\frac{\omega R}{L}}{\frac{1}{LC} - \omega^2}\right)$$

$$\omega_o = \sqrt{\frac{1}{LC}} \qquad \beta = \frac{R}{L}$$

$$Q = \sqrt{\frac{L}{R^2 C}}$$

$$\omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

$$\omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$


1
0.9
0.8
0.707
0.6
0.5
0.4
0.3
0.2
0.1
0
0
500
1000
1500
2000
2500
3000
3500
4000
wc1
wc2