BODE DIAGRAMS

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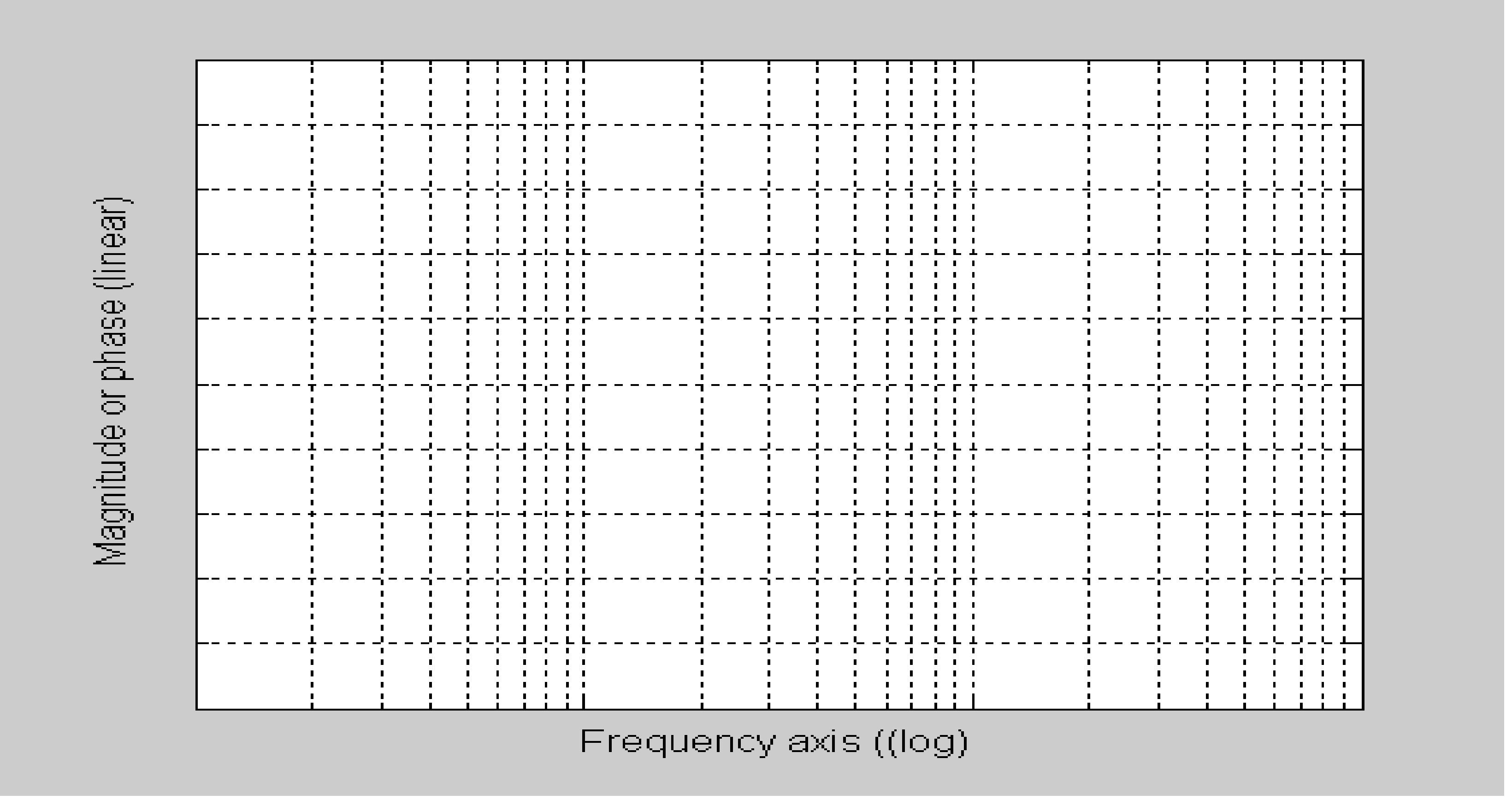
MAGNITUDE AND PHASE PLOTS

- A Bode diagram is a graphical technique that gives a feel for the frequency response of a circuit.
- It consists of two separate plots:
  1. |H(jw)| versus w
  2. Phase angle of H(jw) versus w.
- The plots are made on semilog graph paper to represent the wide range of frequency values. The frequency is plotted on the horizontal log axis, and the amplitude and phase angle are plotted on the linear vertical axis.

SEMILOG PAPER



Real, First-Order Poles and Zeros

Consider the following transfer function where all the poles and zeros are real and first-order

$$H(s)=\frac{K(s+z_1)}{s(s+p_1)}\Rightarrow H(j\omega)=\frac{K(j\omega+z_1)}{j\omega(j\omega+p_1)}$$

The first step is to put H(jw) in a standard form as:

$$H(j\omega)=\frac{Kz_1(1+{}^{j\omega}\!/_{z_1})}{p_1(j\omega)(1+{}^{j\omega}\!/_{p_1})}=\frac{K_0(1+{}^{j\omega}\!/_{z_1})}{(j\omega)(1+{}^{j\omega}\!/_{p_1})}$$

$$H(j\omega)=\frac{K_0\left|1+{}^{j\omega}\!/_{z_1}\right|\angle\psi_1}{|\omega|\angle 90^0\left|1+{}^{j\omega}\!/_{p_1}\right|\angle\beta_1}=\frac{K_0\left|1+{}^{j\omega}\!/_{z_1}\right|}{|\omega|\left|1+{}^{j\omega}\!/_{p_1}\right|}\angle(\psi_1-90^0-\beta_1)$$

$$|H(j\omega)| = \frac{K_0 \left|1 + {}^{j\omega}\!/\!{}_{z_1}\right|}{|\omega| \left|1 + {}^{j\omega}\!/\!{}_{p_1}\right|}$$

$$\theta(\omega) = \psi_1 - 90^0 - \beta_1$$

$$\psi_1 = \tan^{-1}(\omega / z_1)$$

$$\beta_1 = \tan^{-1}(\omega / p_1)$$

The Bode diagram consists of plotting |H(jw)| and θ(w) as functions of w.

STRAIGHT-LINE AMPLITUDE PLOTS

The amplitude plot involves the multiplication and division of factors associated with the poles and zeros of H(s). This multiplication and division is reduced to addition and subtraction by expressing |H(jw)| in terms of a logarithmic value: the decibel (dB). The amplitude of H(jw) in decibels is $A_{dB} = 20\log_{10}\left|H(j\omega)\right|$

For the transfer function H(s) in the example

$$A_{dB} = 20\log_{10}\frac{K_0\left|1+{}^{j\omega}\!/\!_{z_1}\right|}{\left|\omega\right|\left|1+{}^{j\omega}\!/\!_{p_1}\right|}$$

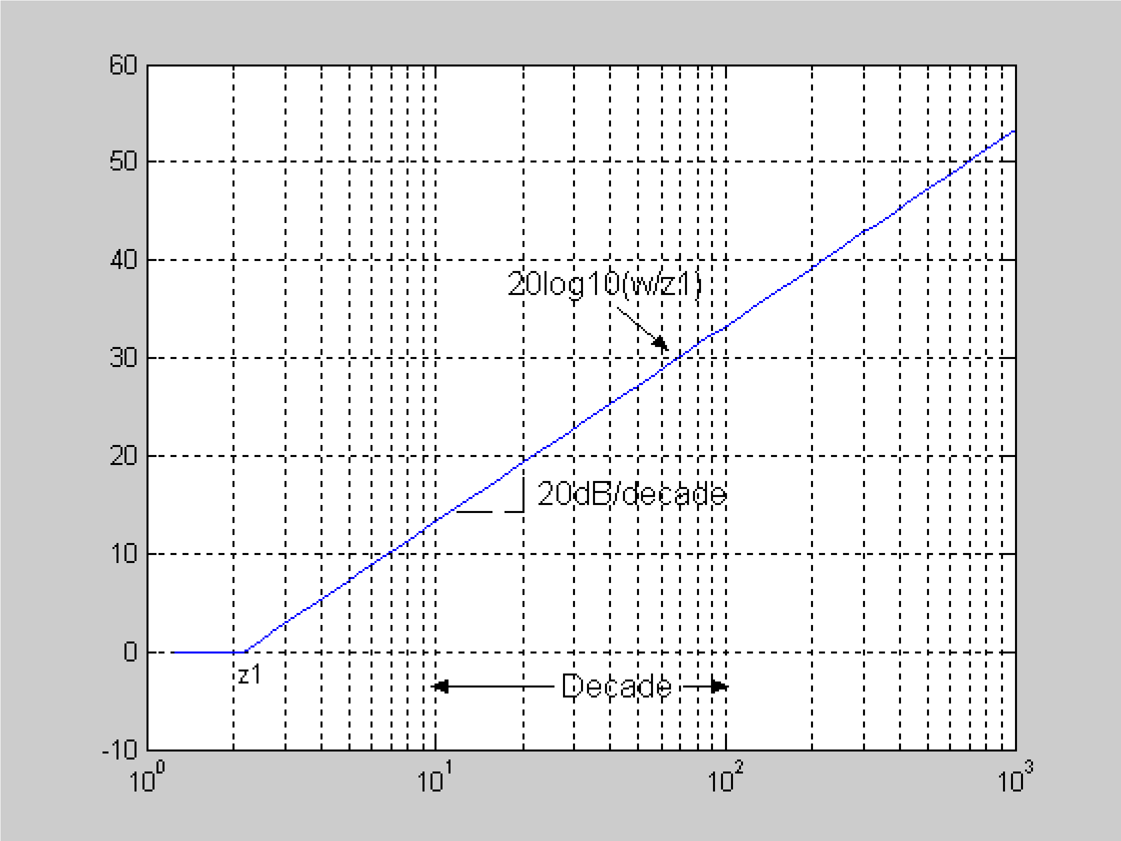
$$= 20\log_{10}K_0 + 20\log_{10}\left|1+{}^{j\omega}\!/\!_{z_1}\right| - 20\log_{10}\left|\omega\right| - 20\log_{10}\left|1+{}^{j\omega}\!/\!_{p_1}\right|$$

The key point is to plot each term separately and then combine the separate plots graphically.

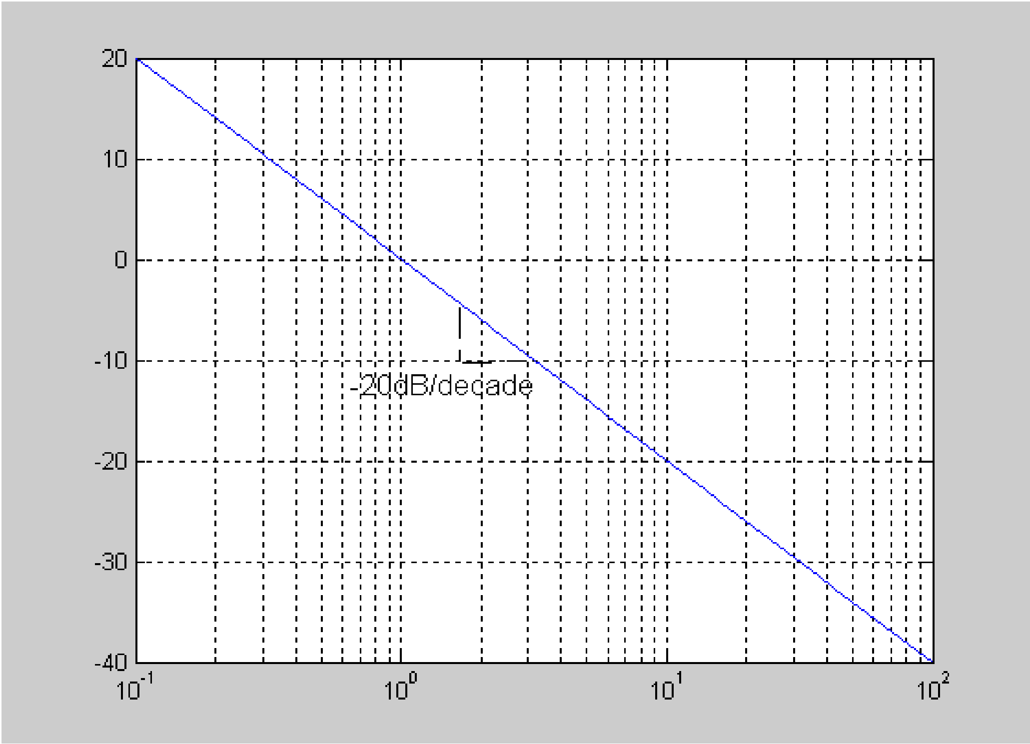
1) The plot of $20\log_{10}K_0$ is a horizontal line because K_0 is not a function of frequency. The value of this term is positive if $K_0>1$, zero for $K_0=1$, and negative for $K_0<1$.

2) Two straight lines approximate the plot of $20\log_{10}|1+jw/z_1|$. For small values of w, the magnitude |1+jw/z1| is approximately 1, and therefore 20log10|1+jw/z1| $\rightarrow$ 0 as w $\rightarrow$ 0. For large values of w, the magnitude |1+jw/z1| is approximately (w/z_1), and therefore 20log10|1+jw/z1| $\rightarrow 20\log_{10}(w/z_1)$ as $w \rightarrow \infty$.

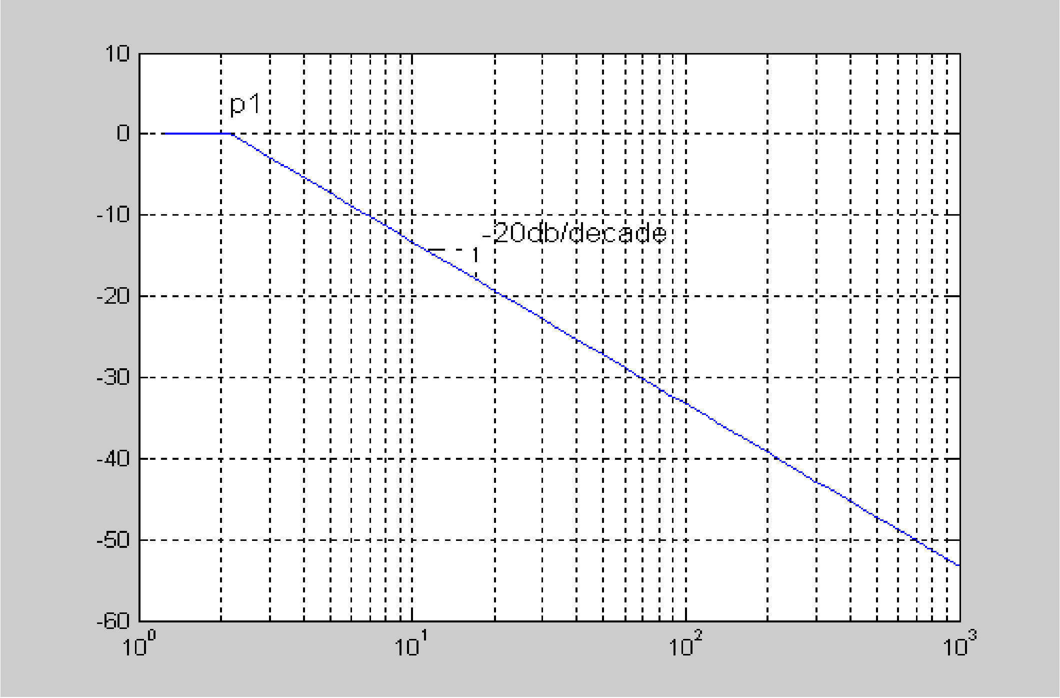
On a log scale, 20log10(w/z1) is a straight line with a slope of 20dB/decade (a decade is a 10-to-1 change in frequency). This straight line intersects the 0dB axis at $w=z_1$. This value of w is called the **corner frequency**.


60
50
40
30
20
10
0
-10
20log10(w/z1)
20dB/decade
z1
Decade
10^0
10^1
10^2
10^3

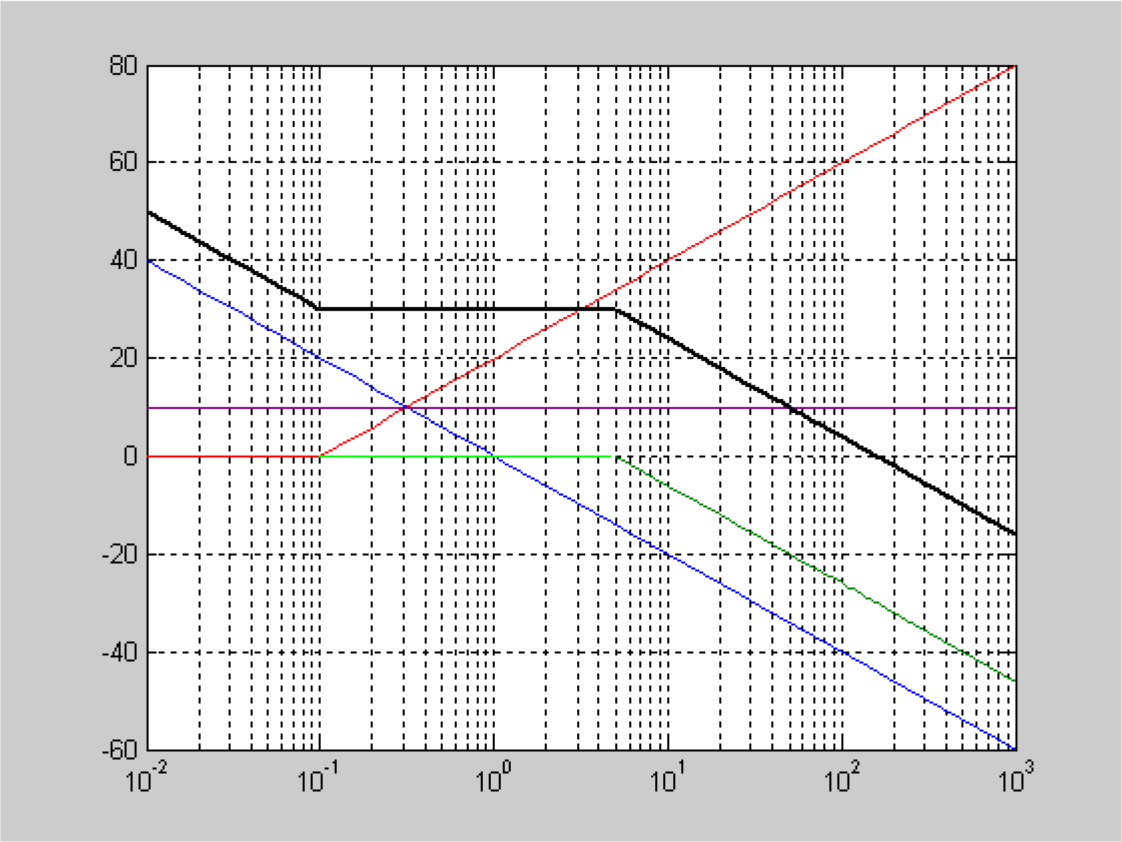
3) The plot of $-20\log_{10}w$ is a straight line having a slope of -20dB/decade that intersects the 0 dB axis at w=1.



4) Two straight lines approximate the plot of $20\log_{10}|1+jw/p_1|$. Two straight lines intersect on the 0 dB axis at $w=p_1$. For large values of w, the straight line has a slope of -20dB/decade.

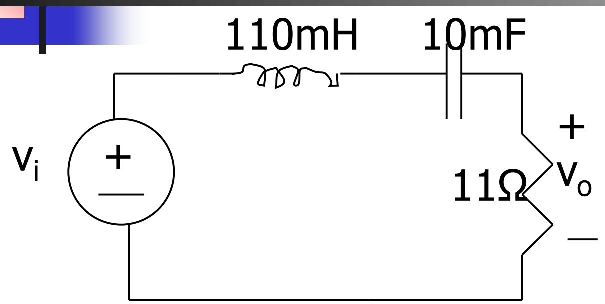


Magnitude Bode plot of $\dfrac{\sqrt{10}(1+{}^{j\omega}\!/_{0.1})}{j\omega(1+{}^{j\omega}\!/_{5})}$



- -- $20\log_{10}(1+jw/0.1)$
- -- $-20\log_{10}(1+jw/5)$
- -- $-20\log_{10}(w)$
- -- $20\log_{10}(\sqrt{10})$
- -- $20\log_{10}|H(jw)|$

EXAMPLE



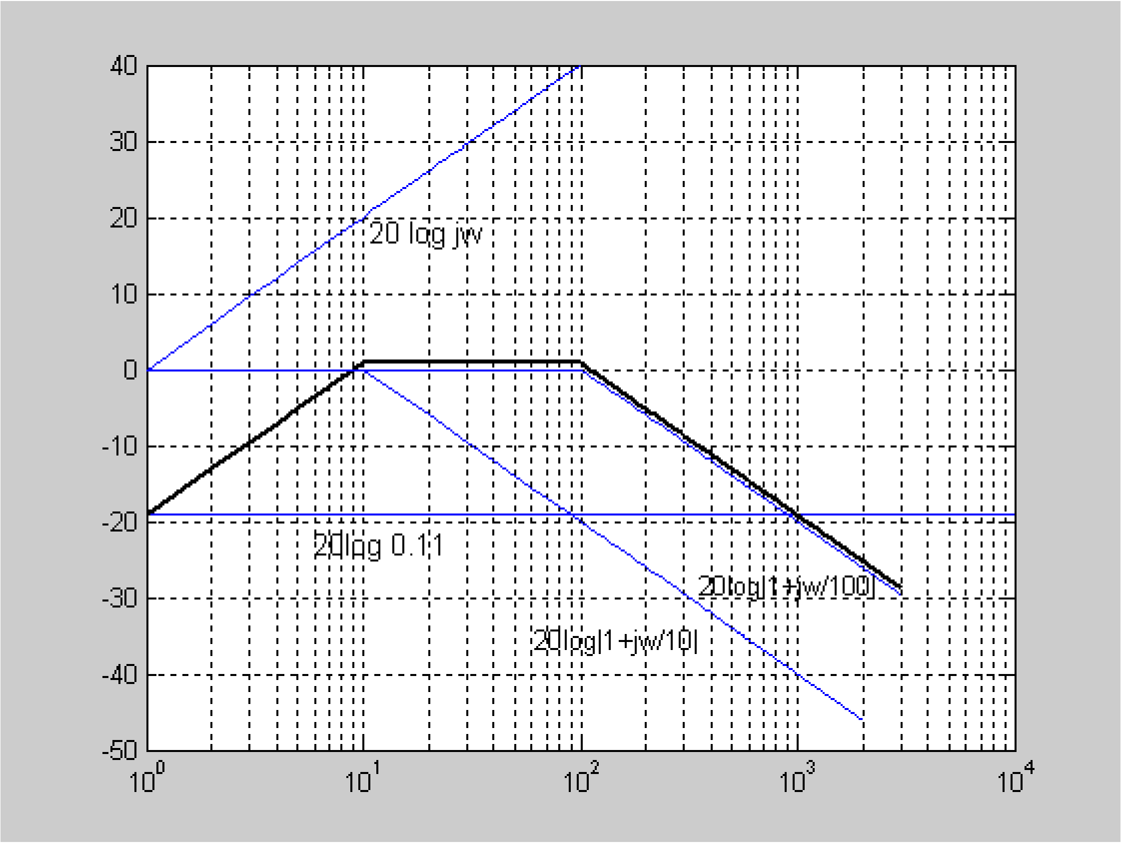
$$H(s)=\frac{(R/L)s}{s^2+(R/L)s+\frac{1}{LC}}=\frac{110s}{s^2+110s+1000}$$

$$=\frac{110s}{(s+10)(s+100)}$$

$$H(j\omega)=\frac{0.11j\omega}{[1+j(\omega/10)][1+j(\omega/100)]}$$

$$A_{dB}=20\log_{10}|H(j\omega)|$$

$$=20\log_{10}0.11+20\log_{10}|j\omega|-20\log_{10}\left|1+j\frac{\omega}{10}\right|-20\log_{10}\left|1+j\frac{\omega}{100}\right|$$


40
30
20
10
0
-10
-20
-30
-40
-50
10^0
10^1
10^2
10^3
10^4
20 log jw
20log 0.11
20log|1+jw/100|
20log|1+jw/10|

Calculate 20log$_{10}$ |H(jw)| at w=50 rad/s and w=1000 rad/s

$$H(j50) = \frac{0.11(j50)}{(1+j5)(1+j0.5)} = 0.9648\angle\text{-}15.25^0$$

$$20\log_{10}\left|H(j50)\right| = 20\log_{10}(0.9648) = -0.311\,\text{dB}$$

$$H(j1000) = \frac{0.11(j1000)}{(1+j100)(1+j10)} = 0.1094\angle -83.72^0$$

$$20\log_{10}\left|H(j1000)\right| = 20\log_{10}(0.1094) = -19.22\,\text{dB}$$

Using the Bode diagram, calculate the amplitude of v_o if $v_i(t)=5\cos(500t+15^0)$V.

From the Bode diagram, the value of A_{dB} at w=500 rad/s is approximately -12.5 dB. Therefore,

$$|A| = 10^{(-12.5/20)} = 0.24$$

$$V_{mo} = |A| V_{mi} = (0.24)(5) = 1.2V$$

MORE ACCURATE AMPLITUDE PLOTS

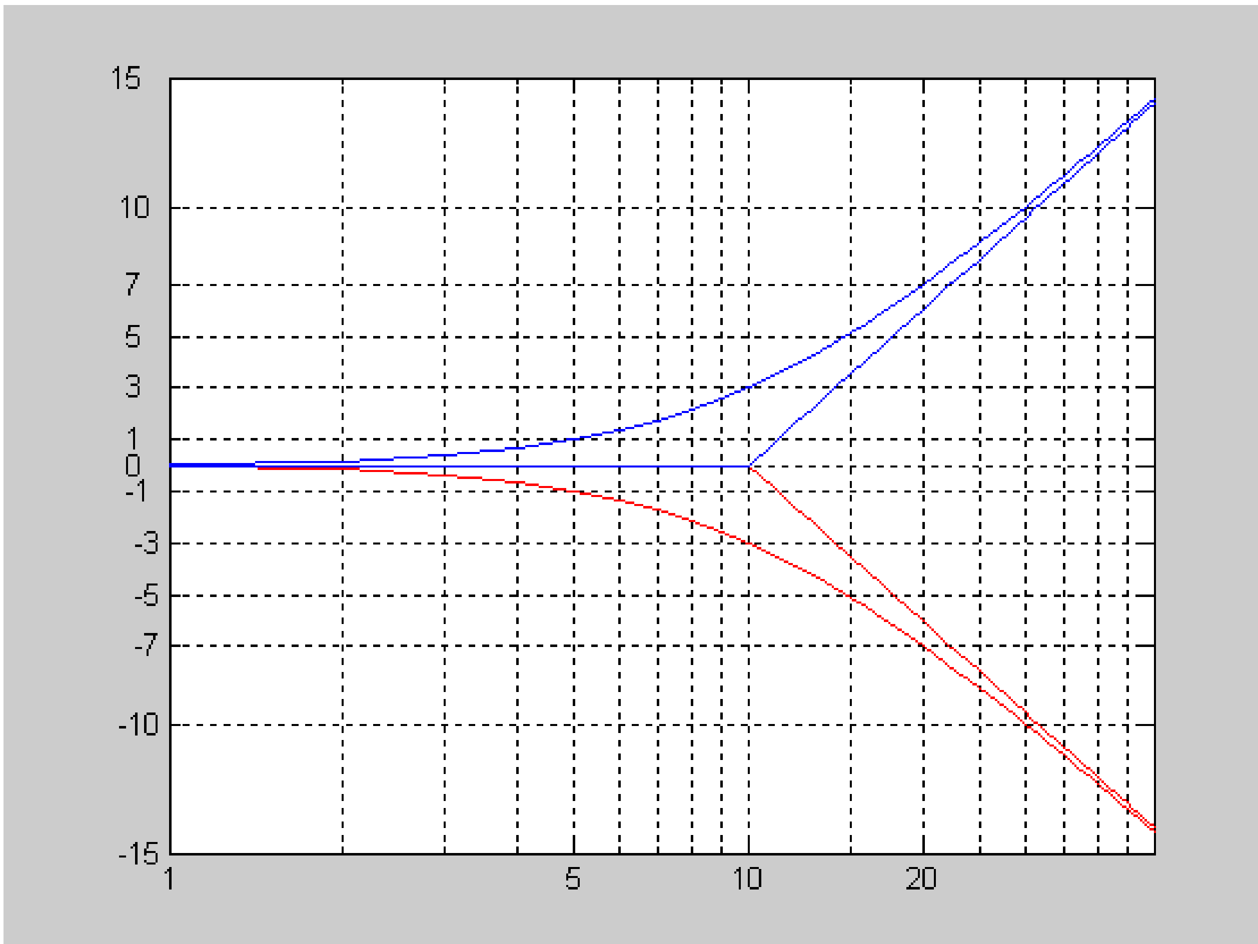
The straight-line plots for first-order poles and zeros can be made more accurate by correcting the amplitude values at the corner frequency, one half the corner frequency, and twice the corner frequency. The actual decibel values at these frequencies

$$A_{dB_c} = \pm 20\log_{10}\left|1 + j1\right| = \pm 20\log_{10}\sqrt{2} \approx \pm 3\,\mathrm{dB}$$

$$A_{dB_{c/2}} = \pm 20\log_{10}\left|1 + j1/2\right| = \pm 20\log_{10}\sqrt{5/4} \approx \pm 1\,\mathrm{dB}$$

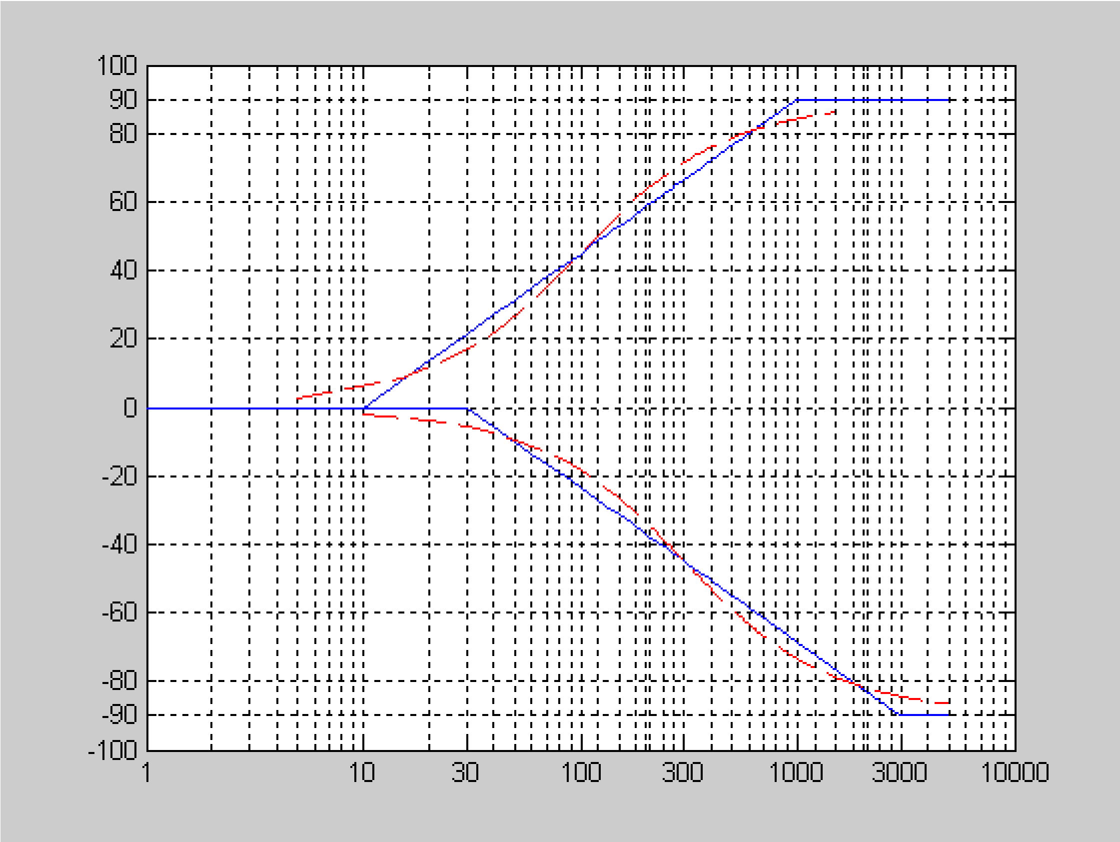
$$A_{dB_{2c}} = \pm 20\log_{10}\left|1 + j2\right| = \pm 20\log_{10}\sqrt{5} \approx \pm 7\,\mathrm{dB}$$

In these equations, + sign corresponds to a first-order zero, and – sign is for a first-order pole.


15
10
7
5
3
1
0
-1
-3
-5
-7
-10
-15
1
5
10
20

STRAIGHT-LINE PHASE ANGLE PLOTS

1. The phase angle for constant K_o is zero.
2. The phase angle for a first-order zero or pole at the origin is a constant $\pm 90^0$.
3. For a first-order zero or pole not at the origin,
   - For frequencies less than one tenth the corner frequency, the phase angle is assumed to be zero.
   - For frequencies greater than 10 times the corner frequency, the phase angle is assumed to be $\pm 90^0$.
   - Between these frequencies the plot is a straight line that goes from 0^0 to $\pm 90^0$ with a slope of $\pm 45^0$/decade.


100
90
80
60
40
20
0
-20
-40
-60
-80
-90
-100
1
10
30
100
300
1000
3000
10000

EXAMPLE

$$H(j\omega) = \frac{0.11(j\omega)}{[1+j(\omega/10)][1+j(\omega/100)]}$$

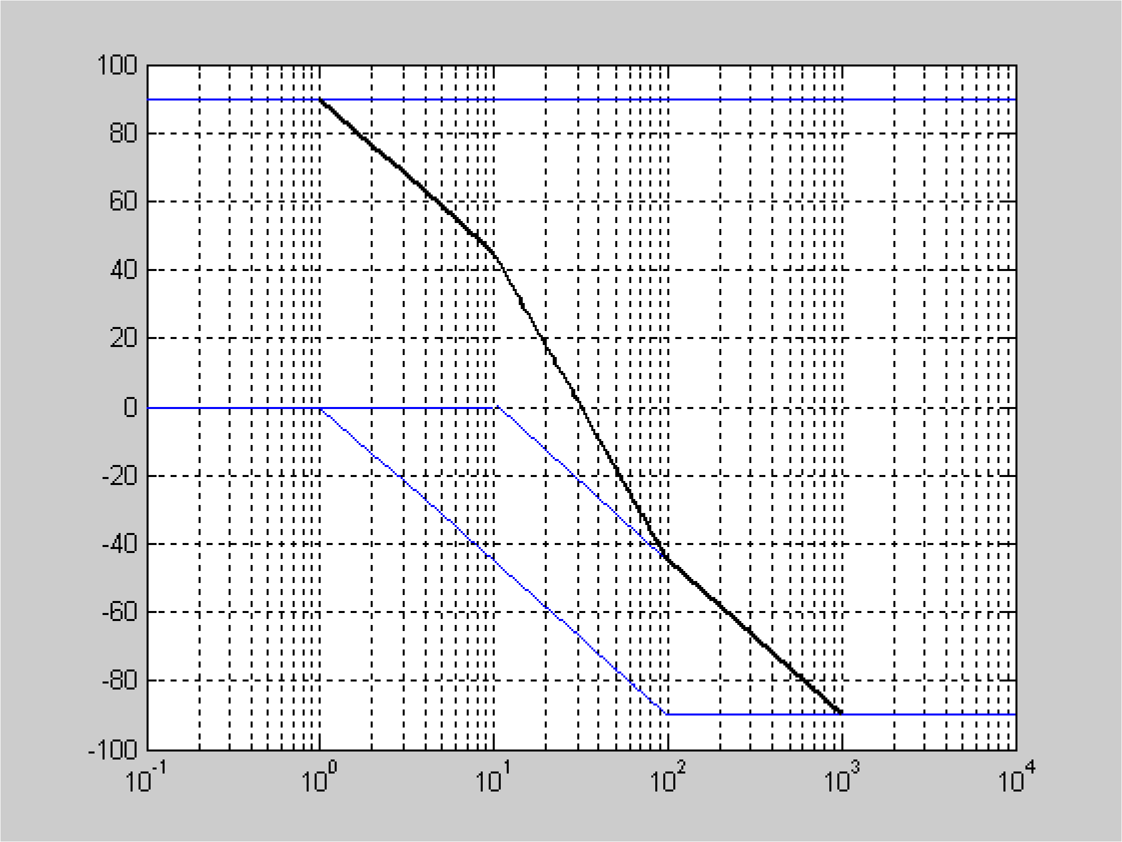
$$= \frac{0.11|j\omega|}{|1+j(\omega/10)||1+j(\omega/100)|}\angle(\psi_1 - \beta_1 - \beta_2)$$

$$\theta(\omega) = \psi_1 - \beta_1 - \beta_2$$

$$\psi_1 = 90^0$$

$$\beta_1 = \tan^{-1}(\omega/10)$$

$$\beta_2 = \tan^{-1}(\omega/100)$$



100
80
60
40
20
0
-20
-40
-60
-80
-100
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}

Compute the phase angle θ(w) at w=50, 500, and 1000 rad/s.

$$H(j50) = 0.96\angle -15.25^0 \Rightarrow \theta(j50) = -15.25^0$$

$$H(j500) = 0.22\angle -77.54^0 \Rightarrow \theta(j500) = -77.54^0$$

$$H(j1000) = 0.11\angle -83.72^0 \Rightarrow \theta(j1000) = -83.72^0$$

Compute the steady-state output voltage if the source voltage is given by $v_i(t)=10\cos(500t-25^0)$ V.

$$V_{mo} = |H(j500)|V_{mi} = (0.22)(10) = 2.2V$$

$$\theta_o = \theta(\omega) + \theta_i = -77.54^0 - 25^0 = -102.54^0$$

$$v_o(t) = 2.2\cos(500t - 102.54^0)V$$

COMPLEX POLES AND ZEROS

$$H(s)=\frac{K}{(s+\alpha-j\beta)(s+\alpha+j\beta)}=\frac{K}{s^2+2\alpha s+\alpha^2+\beta^2}$$

$$=\frac{K}{s^2+2\zeta\omega_n s+\omega_n^2}\Rightarrow \omega_n^2=\alpha^2+\beta^2,\quad \zeta\omega_n=\alpha$$

$$H(s)=\frac{K}{\omega_n^2}\frac{1}{1+\left(s/\omega_n\right)^2+2\zeta\left(s/\omega_n\right)}$$

$$H(j\omega)=\frac{K_0}{1-\left(\omega^2/\omega_n^2\right)+j\left(2\zeta\omega/\omega_n\right)}\Rightarrow K_0=\frac{K}{\omega_n^2}$$

$$H(j\omega)=\frac{K_0}{1-u^2+j2\zeta u}=\frac{K_0}{\left|(1-u^2)+j2\zeta u\right|\angle\beta_1}$$

$$A_{dB}=20\log_{10}K_0-20\log_{10}\left|(1-u^2)+j2\zeta u\right|$$

$$\theta(\omega)=-\beta_1=-\tan^{-1}\frac{2\zeta u}{1-u^2}$$

AMPLITUDE PLOTS

$$-20\log_{10}\left|(1-u^2)+j2\zeta u\right| = -20\log_{10}\sqrt{(1-u^2)^2+4u^2\zeta^2}$$

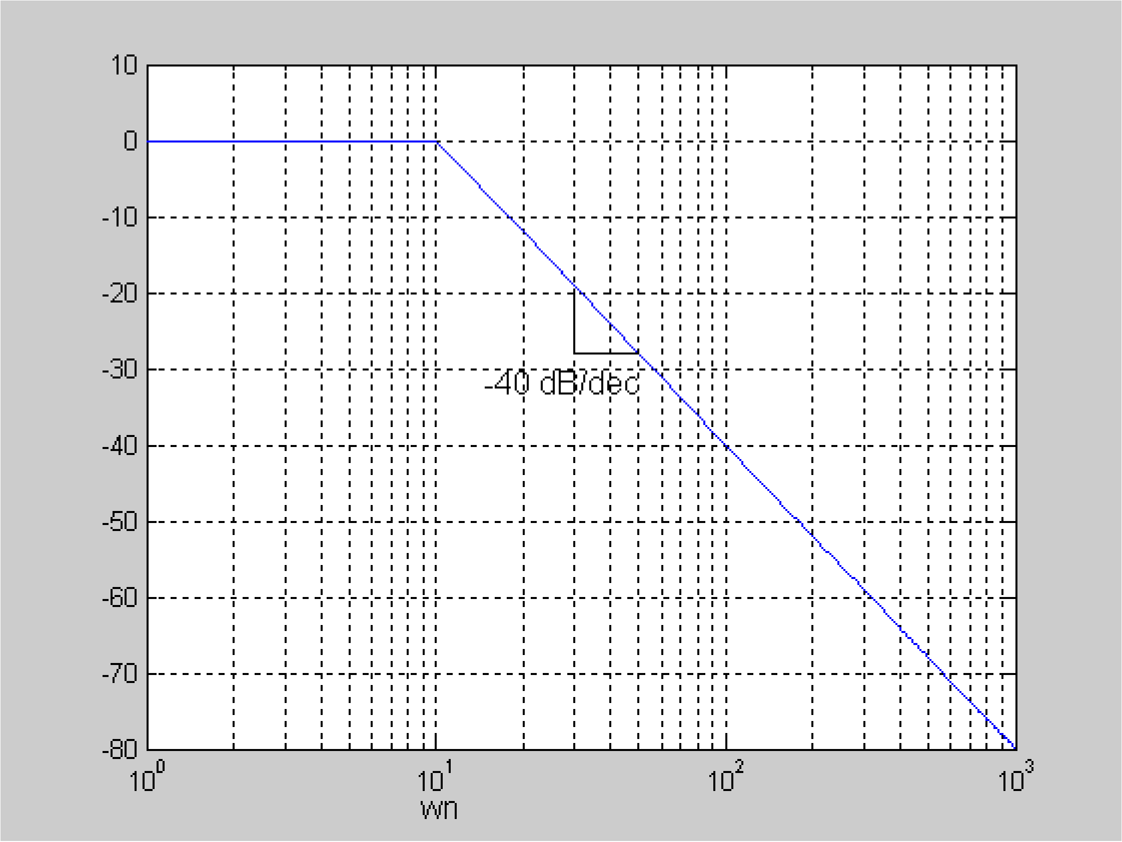
$$= -10\log_{10}\left[u^4+2u^2(2\zeta^2-1)+1\right]$$

$$u=\frac{\omega}{\omega_n}, \quad \text{then } u\to 0 \text{ as } \omega\to 0, \text{ and } u\to\infty \text{ as } \omega\to\infty$$

$$\text{as } u\to 0, \quad -10\log_{10}[u^4+2u^2(2\zeta^2-1)+1]\to 0$$

$$\text{as } u\to\infty, \quad -10\log_{10}[u^4+2u^2(2\zeta^2-1)+1]\to -40\log_{10} u$$

Thus, the approximate amplitude plot consists of two straight lines. For $w<w_n$, the straight line lies along the 0 dB axis, and for $w>w_n$, the straight line has a slope of -40 dB/decade. Thes two straight lines intersect at u=1 or $w=w_n$.



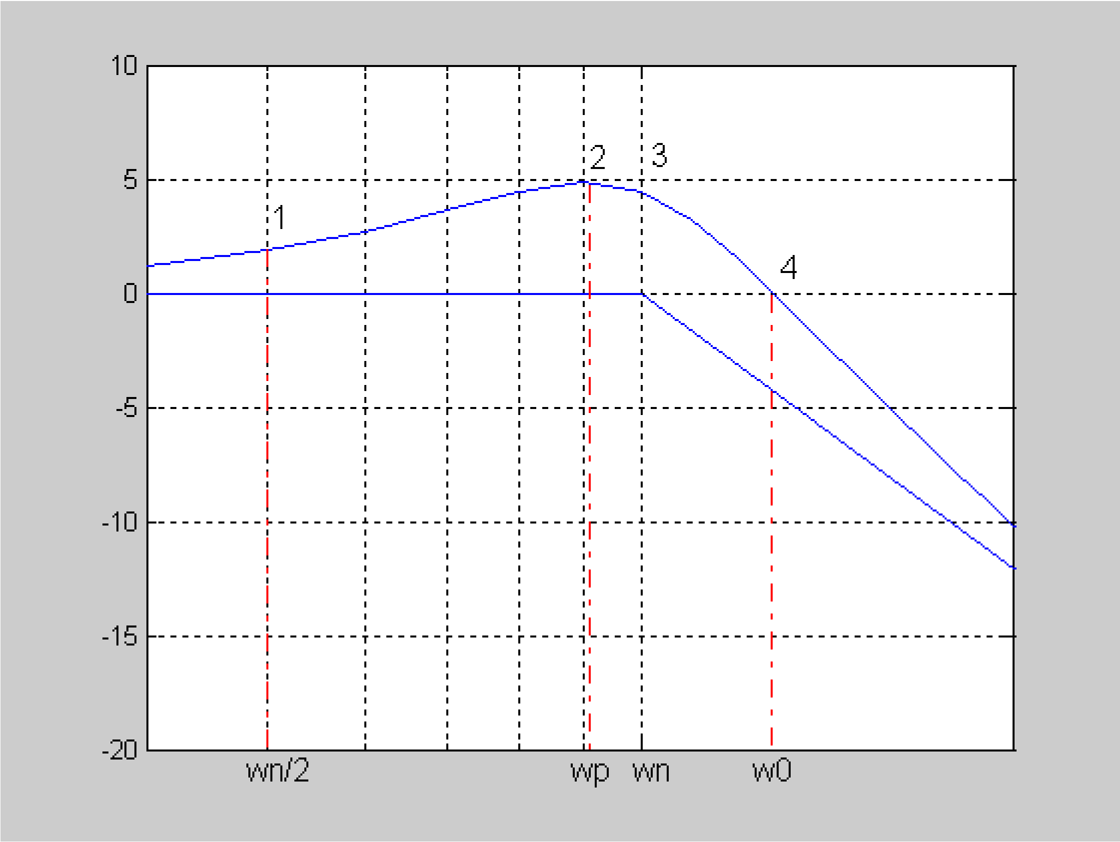
10
0
-10
-20
-30
-40
-50
-60
-70
-80
-40 dB/dec
10^0
10^1
10^2
10^3
wn

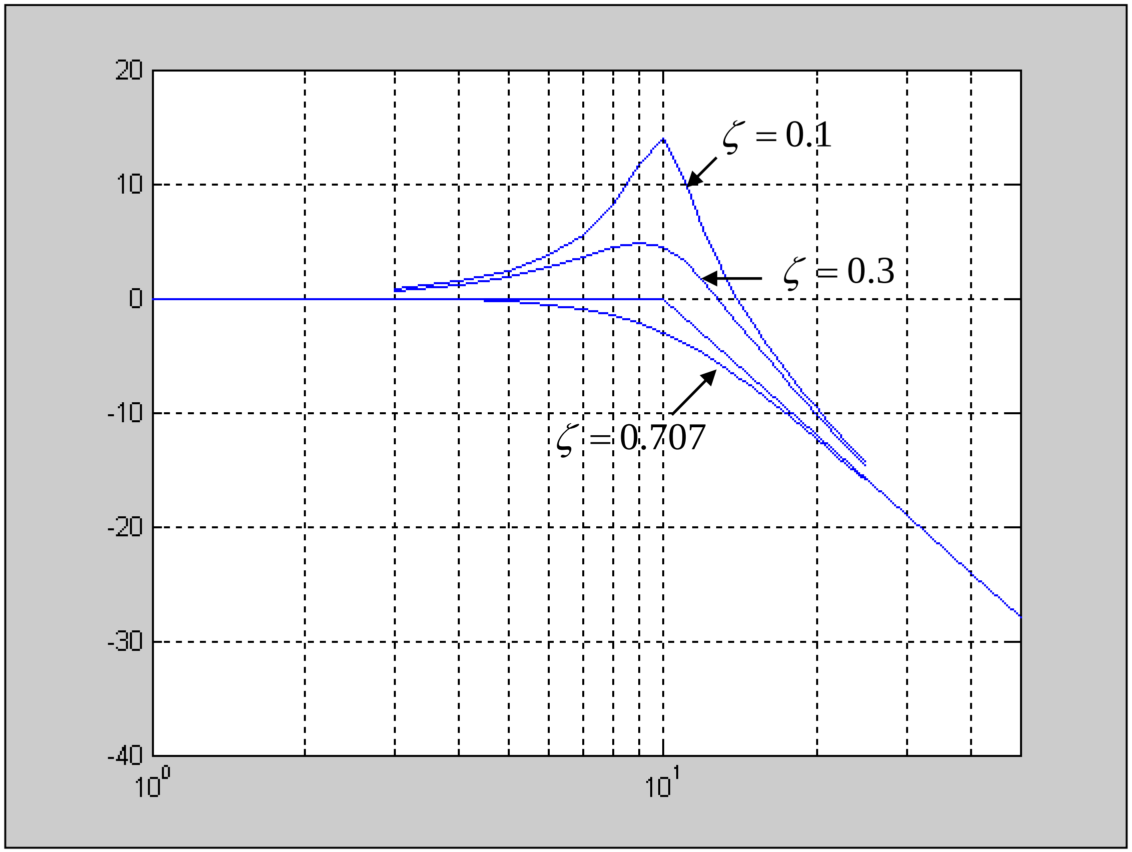
Correcting Straight-Line Amplitude Plots

The straight-line amplitude plot can be corrected by locating four points on the actual curve.

1. One half the corner frequency: At this frequency, the actual amplitude is $A_{dB}(\omega_n/2) = -10\log_{10}(\zeta^2 + 0.5625)$
2. The frequency at which the amplitude reaches its peak value. The amplitude peaks at $\omega_p = \omega_n\sqrt{1-2\zeta^2}$ and it has a peak amplitude $A_{dB} = -10\log_{10}\left[4\zeta^2(1-\zeta^2)\right]$
3. At the corner frequency, $A_{dB}(\omega_n) = -20\log_{10} 2\zeta$
4. The corrected amplitude plot crosses the 0 dB axis at

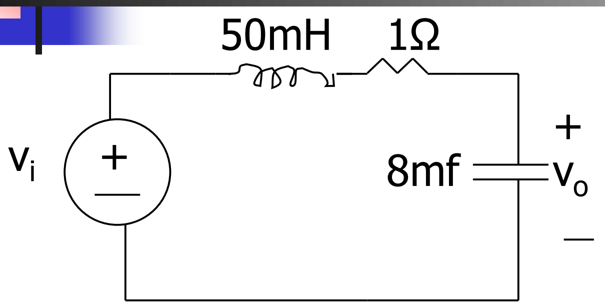
$$\omega_0 = \omega_n\sqrt{2(1-2\zeta^2)} = \sqrt{2}\omega_p$$


10
5
0
-5
-10
-15
-20
1
2
3
4
wn/2
wp
wn
w0



When ζ>1/√2, the corrected amplitude plot lies below the straight line approximation. As ζ becomes very small, a large peak in the amplitude occurs around the corner frequency.

EXAMPLE



$$H(s)=\frac{\frac{1}{LC}}{s^2+\left(\frac{R}{L}\right)s+\frac{1}{LC}}=\frac{2500}{s^2+20s+2500}$$

$$\omega_n^2=2500\Rightarrow\omega_n=50\text{ rad/s}$$

$$K_0=2500/\omega_n^2=1,\quad \zeta=\frac{20}{2\omega_n}=0.2$$

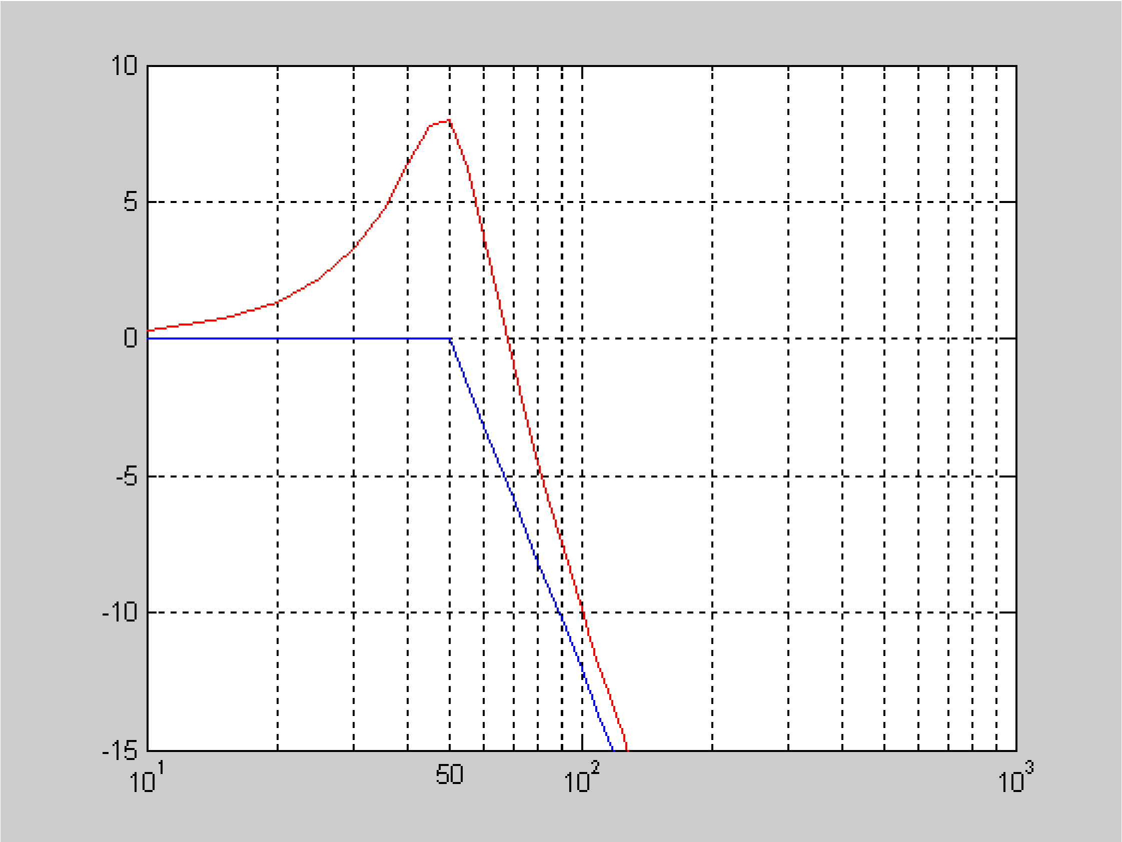
$$A_{dB}(25)=-10\log_{10}(0.2^2+0.5625)=2.2dB$$

$$\omega_p=50\sqrt{1-2(0.2)^2}=47.96\text{ rad/s}$$

$$A_{dB}(47.96)=-10\log_{10}[4(0.2)^2(1-0.2^2)]=8.14dB$$

$$A_{dB}(50)=-20\log_{10}(2\cdot0.2)=2.96dB$$

$$\omega_0=\sqrt{2}(47.96)=67.82\text{ rad/s}$$


10
5
0
-5
-10
-15
10^1
50
10^2
10^3

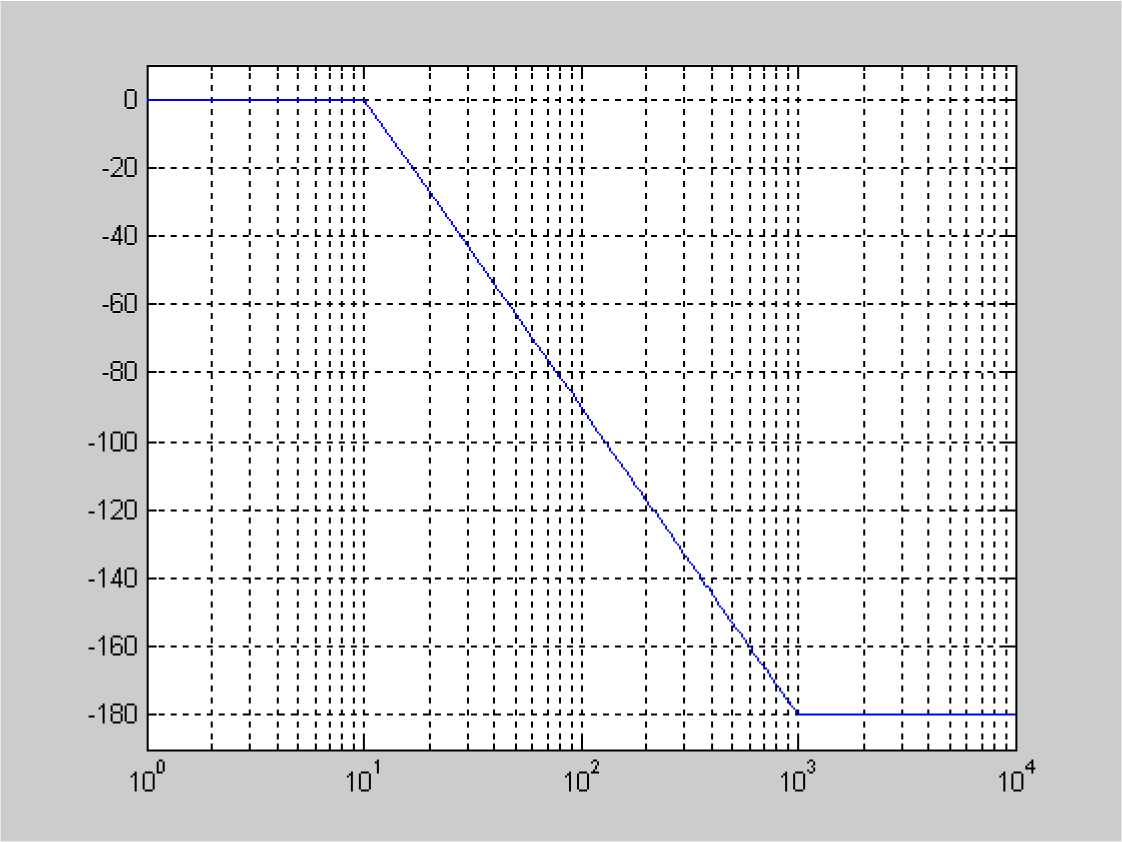
PHASE ANGLE PLOTS

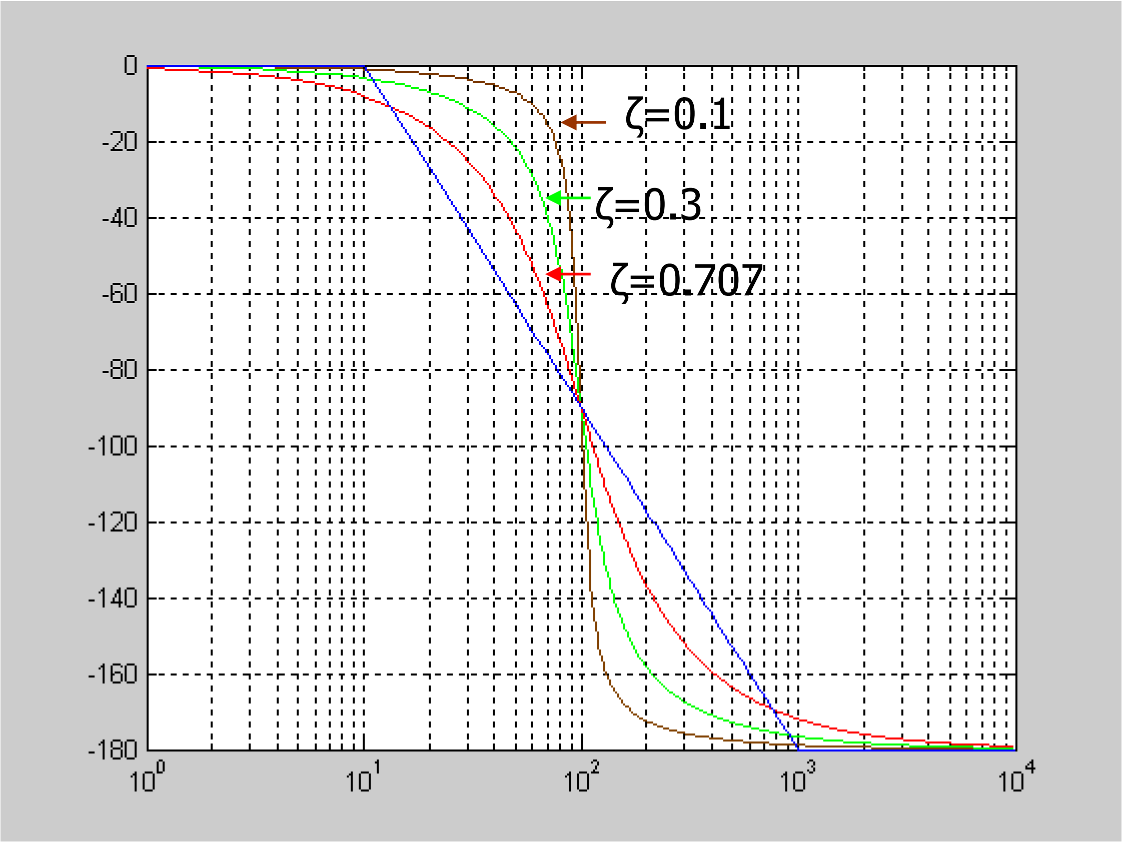
For a second-order zero or pole not at the origin,

For frequencies less than one tenth the corner frequency, the phase angle is assumed to be zero.

- For frequencies greater than 10 times the corner frequency, the phase angle is assumed to be $\pm 180^0$.
- Between these frequencies the plot is a straight line that goes from 0^0 to $\pm 180^0$ with a slope of $\pm 90^0$/decade.

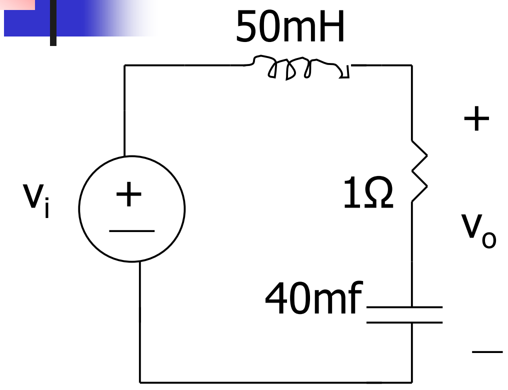
As in the case of the amplitude plot, ζ is important in determining the exact shape of the phase angle plot. For small values of ζ, the phase angle changes rapidly in the vicinity of the corner frequency.


0
-20
-40
-60
-80
-100
-120
-140
-160
-180
10^0
10^1
10^2
10^3
10^4



0
-20
-40
-60
-80
-100
-120
-140
-160
-180
10^0
10^1
10^2
10^3
10^4
ζ=0.1
ζ=0.3
ζ=0.707

EXAMPLE



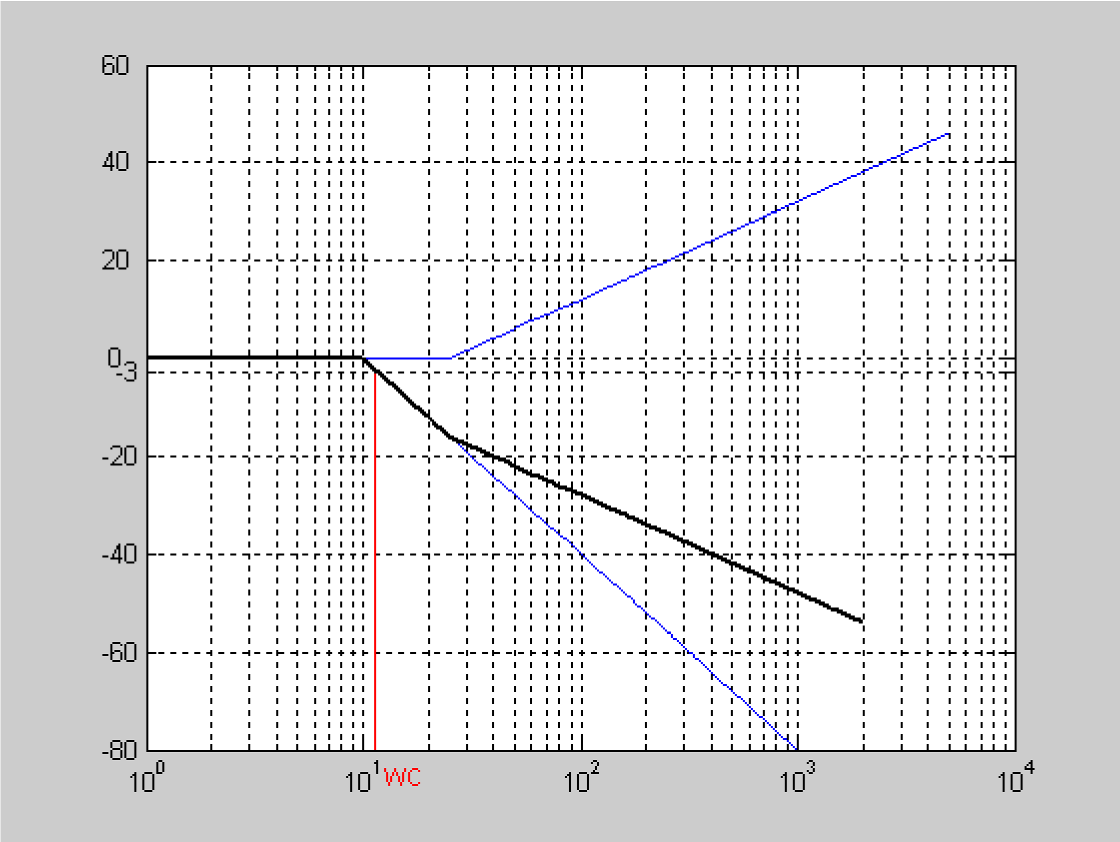
$$H(s)=\frac{\frac{R}{L}s+\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}=\frac{4(s+25)}{s^2+4s+100}$$

$$H(s)=\frac{s/25+1}{1+(s/10)^2+0.4(s/10)}$$

$$H(j\omega)=\frac{|1+j\omega/25|\angle\psi_1}{|1-(\omega/10)^2+j0.4(\omega/10)|\angle\beta_1}$$

$$A_{dB}=20\log_{10}|1+j\omega/25|-20\log_{10}|1-(\omega/10)^2+j0.4(\omega/10)|$$

$$\theta(\omega)=\psi_1-\beta_1$$



60
40
20
0
-3
-20
-40
-60
-80
10^0
10^1
WC
10^2
10^3
10^4

From the straight-line plot, this circuit acts as a low-pass filter. At the cutoff frequency, the amplitude of H(jw) is 3 dB less than the amplitude in the passband. From the plot, the cutoff frecuency is predicted approximately as 13 rad/s.

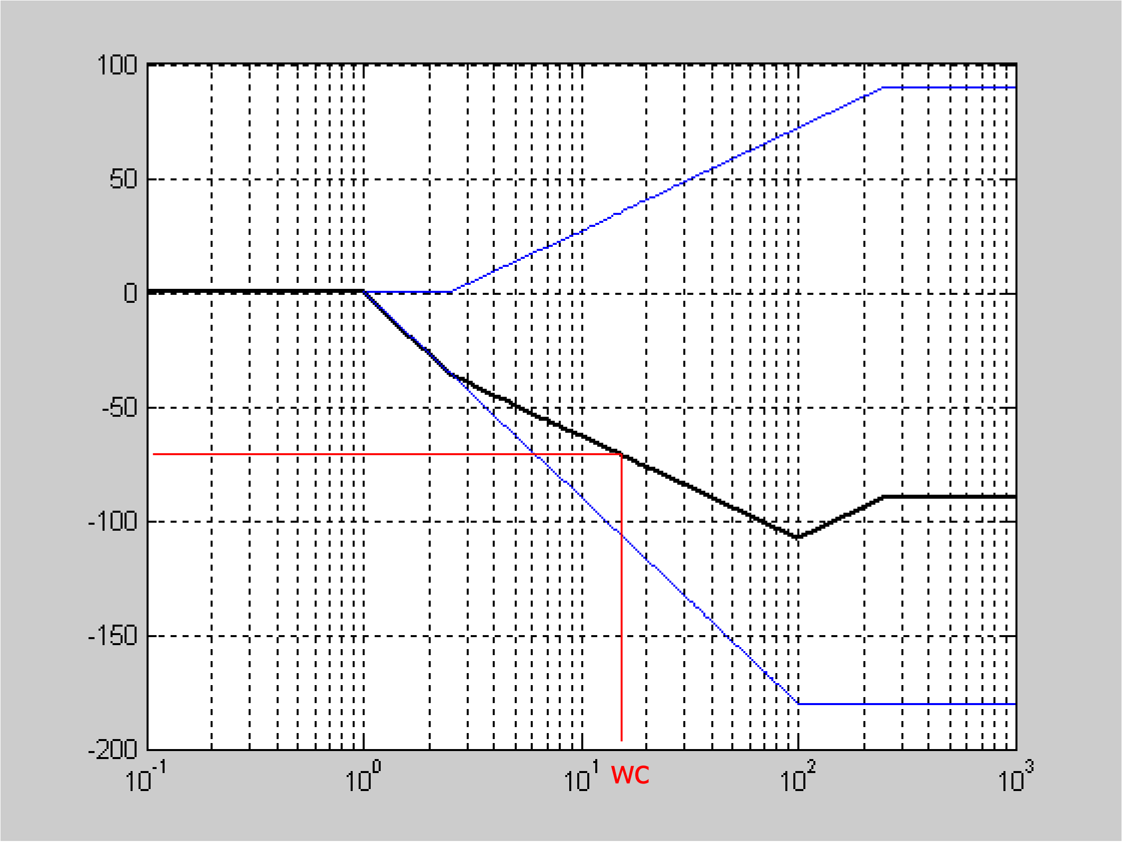
To solve tha actual cutoff frequency, follow the procedure as:

$$H_{\max} = 1 \Rightarrow |H(j\omega_c)| = \frac{1}{\sqrt{2}}$$

$$H(j\omega) = \frac{4(j\omega) + 100}{(j\omega)^2 + 4(j\omega) + 100}$$

$$|H(j\omega_c)| = \frac{\sqrt{(4\omega_c)^2 + 100^2}}{\sqrt{(100 - \omega_c^2)^2 + (4\omega_c)^2}} = \frac{1}{\sqrt{2}}$$

$$\omega_c = 16 \text{ rad/s}$$


100
50
0
-50
-100
-150
-200
10^{-1}
10^{0}
10^{1}
WC
10^{2}
10^{3}

From the phase plot, the phase angle at the cutoff frequency is estimated to be -65^0.

The exact phase angle at the cutoff frequency can be calculated as

$$H(j16) = \frac{4(j16+25)}{(j16)^2 + 4(j16) + 100}$$

$$\theta(j16) = \tan^{-1}(16/25) - \tan^{-1}(64/(100-16^2)) = -125^0$$

Note the large error in the predicted error. In general, straight-line phase angle plots do not give satisfactory results in the frequency band where the phase angle is changing.