ACTIVE FILTER CIRCUITS

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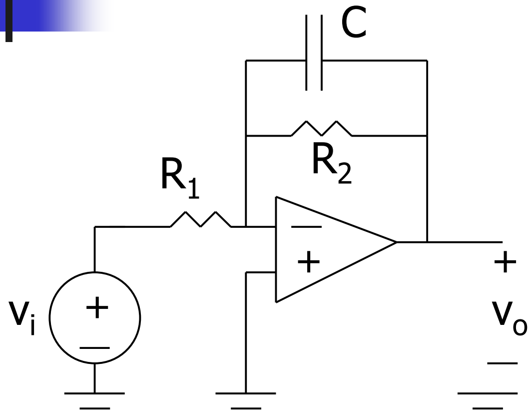
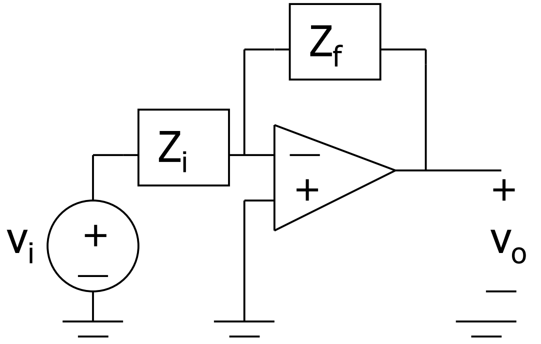
DISADVANTAGES OF PASSIVE FILTER CIRCUITS

- Passive filter circuits consisting of resistors, inductors, and capacitors are incapable of amplification, because the output magnitude does not exceed the input magnitude.
- The cutoff frequency and the passband magnitude of passive filters are altered with the addition of a resistive load at the output of the filter.
- In this section, filters using op amps will be examined. These op amp circuits overcome the disadvantages of passive filter circuits.

FIRST-ORDER LOW-PASS FILTER





$$H(s)=\frac{-Z_f}{Z_i}=\frac{-R_2 \parallel \left(\frac{1}{sC}\right)}{R_1}=-K\frac{\omega_c}{s+\omega_c}$$

$$K=\frac{R_2}{R_1} \qquad \omega_c=\frac{1}{R_2 C}$$

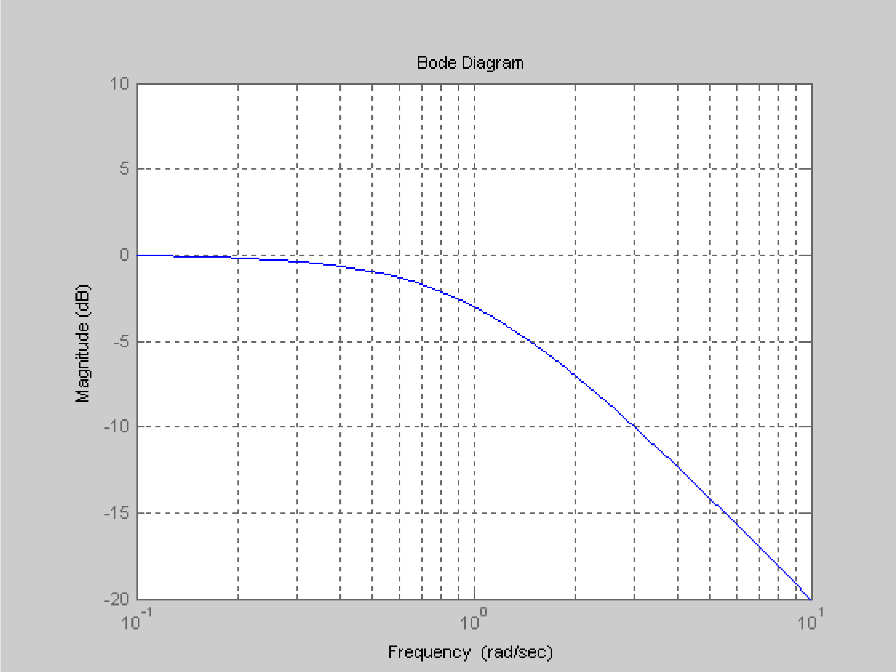
PROTOTYPE LOW-PASS FIRST-ORDER OP AMP FILTER

Design a low-pass first-order filter with R_1=1Ω, having a passband gain of 1 and a cutoff frequency of 1 rad/s.

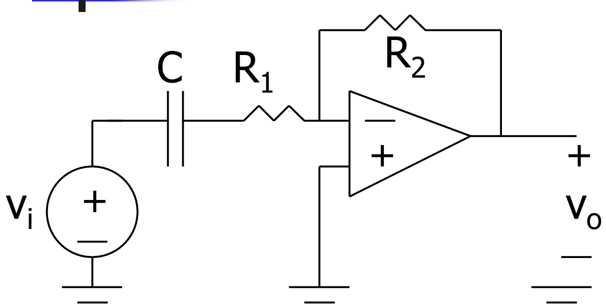
$$R_2 = KR_1 = 1\Omega$$

$$C = \frac{1}{R_2\omega_c} = \frac{1}{(1)(1)} = 1F$$

$$H(s) = -K\frac{\omega_c}{s+\omega_c} = -\frac{1}{s+1}$$

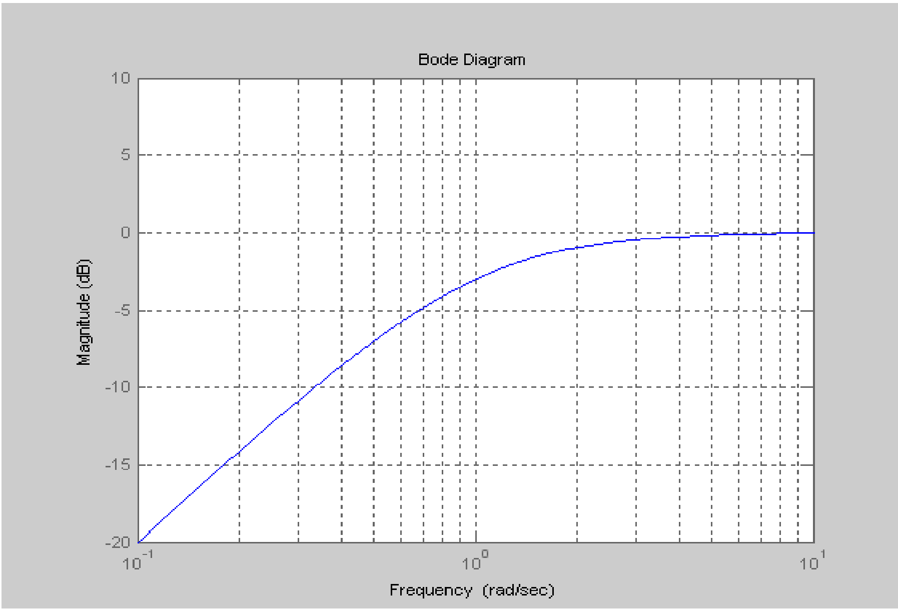


FIRST-ORDER HIGH-PASS FILTER



$$H(s) = -\frac{Z_f}{Z_i} = -\frac{R_2}{R_1 + \frac{1}{sC}} = -K\frac{s}{s+\omega_c}$$

$$K = \frac{R_2}{R_1} \qquad \omega_c = \frac{1}{R_1 C}$$



Prototype high-pass filter with $R_1=R_2=1\Omega$ and C=1F. The cutoff frequency is 1 rad/s. The magnitude at the passband is 1.

EXAMPLE

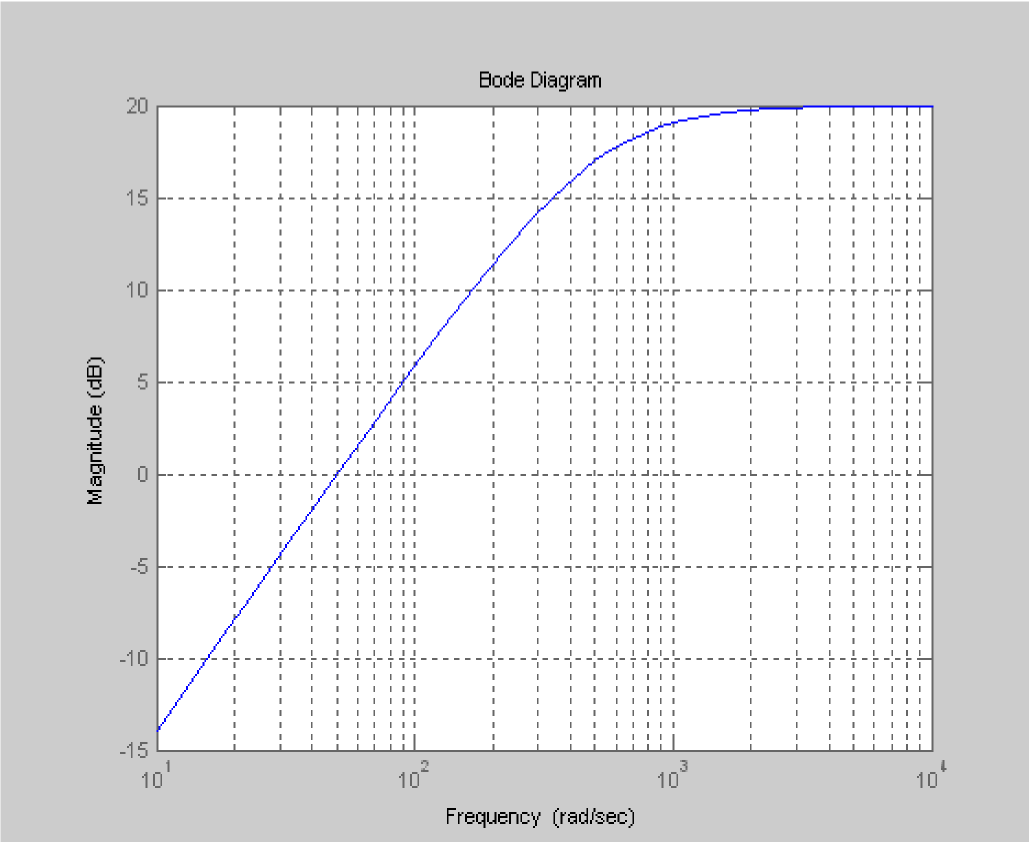


Figure shows the Bode magnitude plot of a high-pass filter. Using the active high-pass filter circuit, determine values of R_1 and R_2. Use a 0.1μF capacitor.

If a 10 KΩ load resistor is added to the filter, how will the magnitude response change?

Notice that the gain in the passband is 20dB, therefore, K=10. Also note the the 3 dB point is 500 Hz. Then, the transfer function for the high-pass filter is

$$H(s) = \frac{-10s}{s+500} = \frac{-\frac{R_2}{R_1}s}{s+\frac{1}{R_1 C}}$$

$$10 = \frac{R_2}{R_1} \qquad 500 = \frac{1}{R_1 C}$$

$$R_1 = 20K\Omega \qquad R_2 = 200K\Omega$$

Because the op amp in the circuit is ideal, the addition of any load resistor has no effect on the behavior of the op amp. Thus, the magnitude response of the high-pass filter will remain the same when a load resistor is connected.

SCALING

In the design of both passive and active filters, working with element values such as 1 Ω, 1 H, and 1 F is convenient. After making computations using convenient values of R, L, and C, the designer can transform the circuit to a realistic one using the process known as **scaling**. There are two types of scaling: **magnitude** and **frequency**.

A circuit is scaled in magnitude by multiplying the impedance at a given frequency by the scale factor k_m. Thus, the scaled values of resistor, inductor, and capacitor become

$$R' = k_m R \quad L' = k_m L \text{ and } C' = C / k_m$$

where the primed values are the scaled ones.

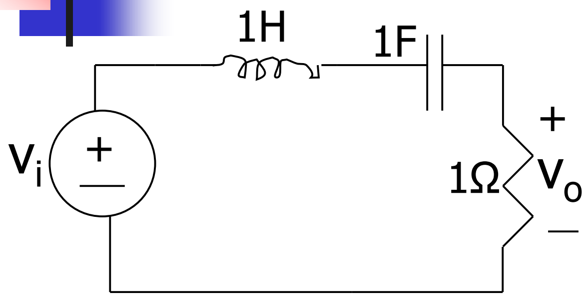
In frequency scaling, we change the circuit parameter so that at the new frequency, the impedance of each element is the same as it was at the original frequency. Let k_f denote the frequency scale factor, then

$$R' = R \qquad L' = L / k_f \quad \text{and} \quad C' = C / k_f$$

A circuit can be scaled simultaneously in both magnitude and frequency. The scaled values in terms of the original values are

$$R' = k_m R \qquad L' = \frac{k_m}{k_f} L \quad \text{and} \quad C' = \frac{1}{k_f k_m} C$$

EXAMPLE



This circuit has a center frequency of 1 rad/s, a bandwidth of 1 rad/s, and a quality factor of 1. Use scaling to compute the values of R and L that yield a circuit with the same quality factor but with a center frequency of 500 rad/s. Use a 2 μF capacitor.

The frequency scaling factor is: $k_f = \dfrac{2\pi(500)}{1} = 3141.59$

The magnitude scaling factor is: $k_m = \dfrac{1}{k_f}\dfrac{C}{C'} = \dfrac{1}{3141.59}\dfrac{1}{2\times10^{-6}} = 159.155$

$$R' = k_m R = 159.155\Omega \qquad L' = \frac{k_m}{k_f} L = 50.66mH$$

$$\omega_o' = \sqrt{1/L'C'} = 3141.61\,\text{rad/s or } 500\text{ Hz.}$$

$$\beta' = R'/L' = 3141.61\,\text{rad/s or } 500\text{ Hz.}$$

$$\text{Q}' = \omega_\text{o}'/\beta' = 1$$

EXAMPLE

Use the prototype low-pass op amp filter and scaling to compute the resistor values for a low-pass filter with a gain of 5, a cutoff frequency of 1000 Hz, and a feedback capacitor of 0.01 μF.

$$k_f = \omega_c' / \omega_c = 2\pi(1000)/1 = 6283.185$$

$$k_m = \frac{1}{k_f}\frac{C}{C'} = \frac{1}{(6283.15)(10^{-8})} = 15915.5$$

$$R_1' = R_2' = k_m R = 15915.5\Omega$$

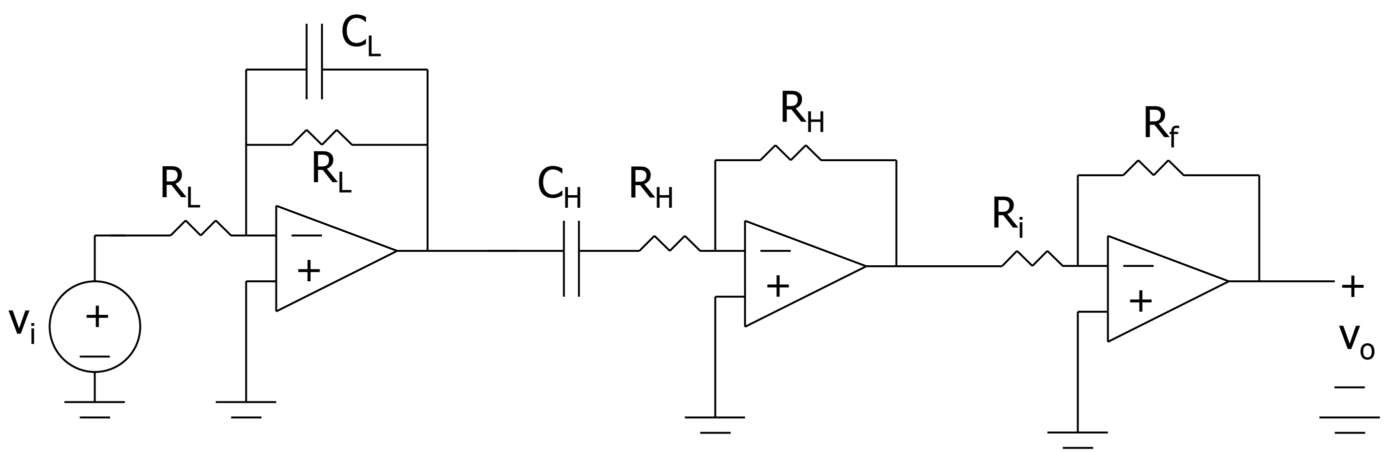
To meet the gain specification, we can adjust one of the resistor values. But, changing the value of R_2 will change the cutoff frequency. Therefore, we can adjust the value of R_1 as $R_1=R_2/5=3183.1\ \Omega$.

OP AMP BANDPASS FILTERS

A bandpass filter consists of three separate components

1. A unity-gain low-pass filter whose cutoff frequency is w_{c2}, the larger of the two cutoff frequencies
2. A unity-gain high-pass filter whose cutoff frequency is w_{c1}, the smaller of the two cutoff frequencies
3. A gain component to provide the desired level of gain in the passband.

These three components are cascaded in series. The resulting filter is called a **broadband** bandpass filter, because the band of frequencies passed is wide.



$$H(s)=\frac{V_o}{V_i}=\left(\frac{-\omega_{c2}}{s+\omega_{c2}}\right)\left(\frac{-s}{s+\omega_{c1}}\right)\left(\frac{-R_f}{R_i}\right)$$

$$=\frac{-K\omega_{c2}s}{s^2+(\omega_{c1}+\omega_{c2})s+\omega_{c1}\omega_{c2}}$$

Standard form for the transfer function of a bandpass filter is

$$H_{BP} = \frac{\beta s}{s^2 + \beta s + \omega_o^2}$$

In order to convert H(s) into the standard form, it is required that $\omega_{c2} >> \omega_{c1}$. If this condition holds, $(\omega_{c1} + \omega_{c2}) \approx \omega_{c2}$

Then the transfer function for the bandpass filter becomes

$$H(s) = \frac{-K\omega_{c2}s}{s^2 + \omega_{c2}s + \omega_{c1}\omega_{c2}}$$

Compute the values of R_L and C_L to give us the desired cutoff frequency

$$\omega_{c2} = \frac{1}{R_L C_L}$$

Compute the values of R_H and C_H to give us the desired cutoff frequency

$$\omega_{c1} = \frac{1}{R_H C_H}$$

To compute the values of R_i and R_f, consider the magnitude of the transfer function at the center frequency w_o

$$|H(j\omega_o)| = \left| \frac{-K\omega_{c2}(j\omega_o)}{(j\omega_o)^2 + \omega_{c2}(j\omega_o) + \omega_{c2}\omega_{c1}} \right| = K = \frac{R_f}{R_i}$$

EXAMPLE

Design a bandpass filter to provide an amplification of 2 within the band of frequencies between 100 and 10000 Hz. Use 0.2 μF capacitors.

$$\omega_{c2} = \frac{1}{R_L C_L} = (2\pi)10000 \Rightarrow R_L = \frac{1}{[2\pi(10000)](0.2\times10^{-6})} \approx 80\Omega$$

$$\omega_{c1} = \frac{1}{R_H C_H} = (2\pi)100 \Rightarrow R_H = \frac{1}{[2\pi(100)](0.2\times10^{-6})} \approx 7958\Omega$$

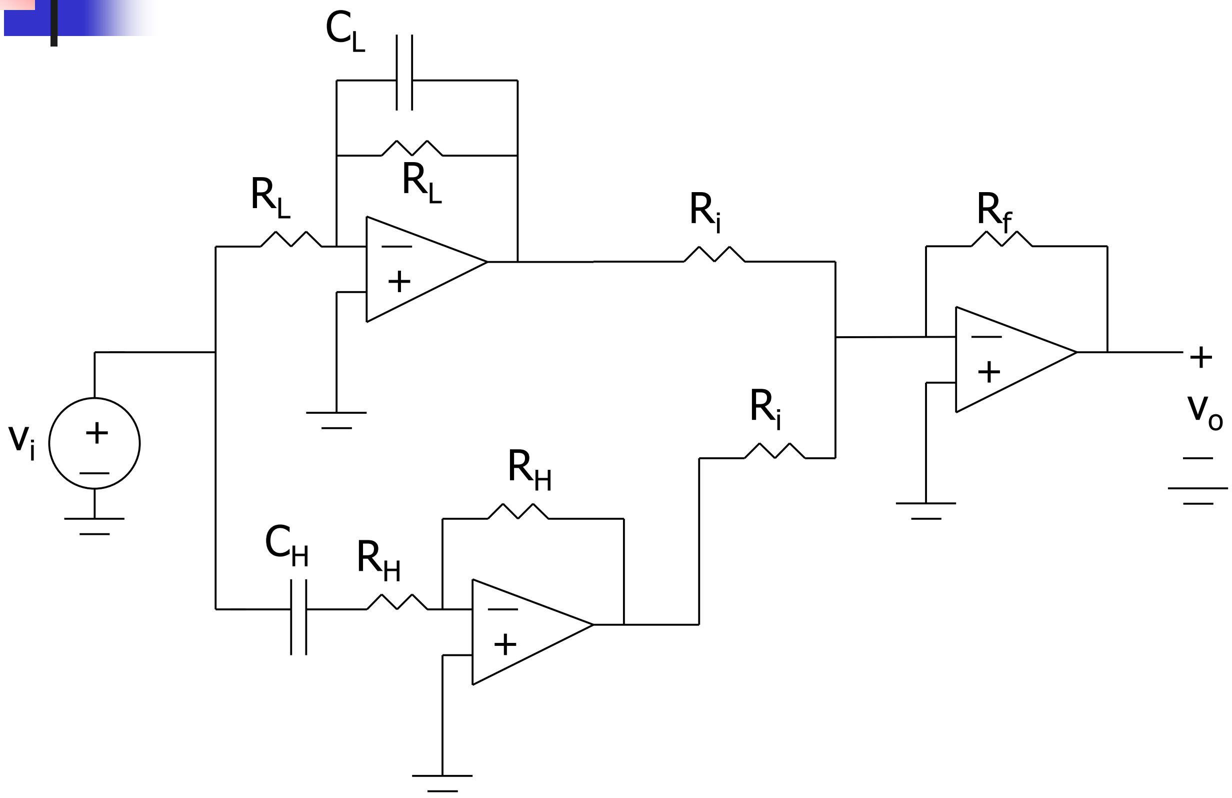
Arbitrarily select R_i=1 kΩ, then R_f=2R_i=2 KΩ

OP AMP BANDREJECT FILTERS

Like the bandpass filters, the bandreject filter consists of three separate components

- The unity-gain low-pass filter has a cutoff frequency of w_{c1}, which is the smaller of the two cutoff frequencies.
- The unity-gain high-pass filter has a cutoff frequency of w_{c2}, which is the larger of the two cutoff frequencies.
- The gain component provides the desired level of gain in the passbands.

The most important difference is that these components are connected in parallel and using a summing amplifier.



C_L
R_L
R_L
R_i
R_f
+
v_o
—
R_i
v_i
+
—
R_H
C_H
R_H

$$H(s)=\left(-\frac{R_f}{R_i}\right)\left[\frac{-\omega_{c1}}{s+\omega_{c1}}+\frac{-s}{s+\omega_{c2}}\right]$$

$$=\frac{R_f}{R_i}\left(\frac{s^2+2\omega_{c1}s+\omega_{c1}\omega_{c2}}{(s+\omega_{c1})(s+\omega_{c2})}\right)$$

$$\omega_{c1}=\frac{1}{R_L C_L} \qquad \omega_{c2}=\frac{1}{R_H C_H} \qquad K=\frac{R_f}{R_i}$$

The magnitude of the transfer function at the center frequency

$$|H(j\omega_o)|=\left|\frac{R_f}{R_i}\left(\frac{(j\omega_o)^2+2\omega_{c1}(j\omega_o)+\omega_{c1}\omega_{c2}}{(j\omega_o)^2+(\omega_{c1}+\omega_{c2})(j\omega_o)+\omega_{c1}\omega_{c2}}\right)\right|$$

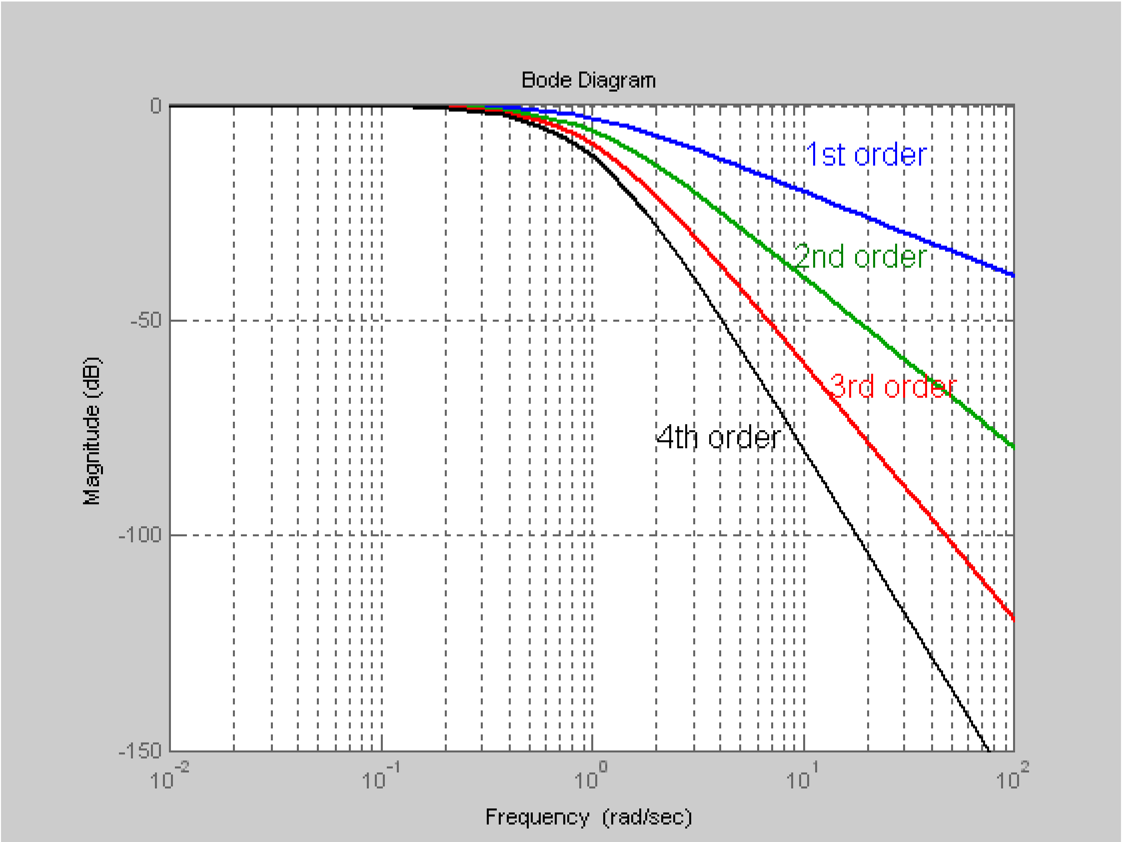
$$=\frac{R_f}{R_i}\frac{2\omega_{c1}}{\omega_{c1}+\omega_{c2}}\approx\frac{R_f}{R_i}\frac{2\omega_{c1}}{\omega_{c2}}$$

HIGHER ORDER OP AMP FILTERS

All of the filters considered so far are nonideal and have a slow transition between the stopband and passband. To obtain a sharper transition, we may connect identical filters in cascade.

For example connecting two first-order low-pass identical filters in cascade will result in -40 dB/decade slope in the transition region. Three filters will give -60 dB/decade slope, and four filters should have -80 db/decade slope. For a cascaded of n protoptype low-pass filters, the transfer function is

$$H(s)=\left(\frac{-1}{s+1}\right)\left(\frac{-1}{s+1}\right)\cdots\left(\frac{-1}{s+1}\right)=\left(\frac{-1}{s+1}\right)^{n}$$



Bode Diagram
0
-50
-100
-150
Magnitude (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
1st order
2nd order
3rd order
4th order

But, there is a problem with this approach. As the order of the low-pass is increased, the cutoff frequency changes. As long as we are able to calculate the cutoff frequency of the higher-order filters, we can use frequency scaling to calculate the component values that move the cutoff frequency to its specified location. For an nth-order low-pass filter with n prototype low-pass filters

$$\left|H(j\omega_{cn})\right| = \left|\frac{1}{(j\omega_{cn}+1)^n}\right| = \frac{1}{\sqrt{2}}$$

$$\frac{1}{\left(\sqrt{\omega_{cn}^2+1}\right)^n} = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\omega_{cn}^2+1} = \left(\frac{1}{\sqrt{2}}\right)^{2/n}$$

$$\sqrt[n]{2} = \omega_{cn}^2 + 1 \Rightarrow \omega_{cn} = \sqrt{\sqrt[n]{2}-1}$$

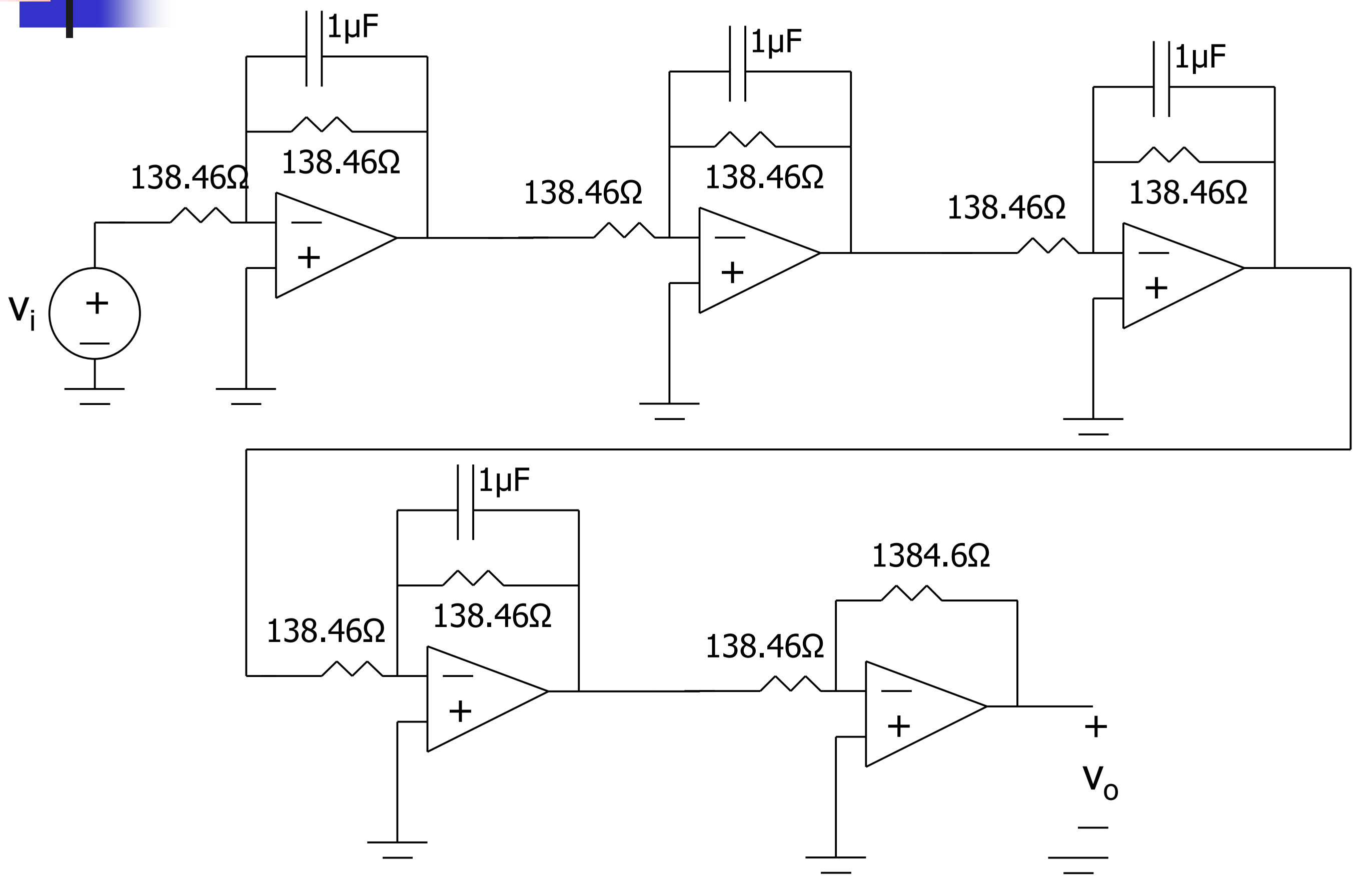
EXAMPLE

Design a fourth-order low-pass filter with a cutoff frequency of 500 rad/s and a passband gain of 10. Use 1 μF capacitors.

$$\omega_{c4} = \sqrt{\sqrt[4]{2} - 1} = 0.435 \text{ rad/s} \Rightarrow k_f = \frac{2\pi(500)}{0.435} = 7222.39$$

$$k_m = \frac{1}{7222.39(1 \times 10^{-6})} = 138.46$$

Thus, R=138.46Ω and C=1 μF. To set the passband gain to 10, choose Rf/Ri=10. For example Rf=1384.6 Ω and R_i =138.46 Ω.


1μF
138.46Ω
138.46Ω
1μF
138.46Ω
138.46Ω
1μF
138.46Ω
138.46Ω
v_i
1μF
138.46Ω
138.46Ω
1384.6Ω
138.46Ω
v_o

By cascading identical prototype filters, we can increase the asymptotic slope in the transition and control the location of the cutoff frequency. But the gain of the filter is not constant between zero and the cutoff frequency. Now, consider the magnitude of the transfer function for a unity-gain low-pass nth order cascade.

$$H(s) = \frac{\omega_{cn}^n}{(s+\omega_{cn})^n}$$

$$|H(j\omega)| = \frac{\omega_{cn}^n}{\left(\sqrt{\omega^2+\omega_{cn}^2}\right)^n} = \frac{1}{\left(\sqrt{(\omega/\omega_{cn})+1}\right)^n}$$

BUTTERWORTH FILTERS

A unity-gain Butterworth low-pass filter has a transfer function whose magnitude is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_c)^{2n}}}$$

1. The cutoff frequency is w_c for all values of n.
2. If n is large enough, the denominator is always close to unity when $w<w_c$.
3. In the expression for |H(jw)|, the exponent of w/w_c is always even.

Given an equation for the magnitude of the transfer function, how do we find H(s)? To find H(s), note that if N is a complex quantity, the $|N|^2=NN^*$. Then,

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega) = H(s)H(-s)$$

$$\text{since} \quad s^2 = -\omega^2$$

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}} = \frac{1}{1+(\omega^2)^n} = \frac{1}{1+(-s^2)^n}$$

$$H(s)H(-s) = \frac{1}{1+(-1)^n s^{2n}}$$

The procedure for finding H(s) for a given n is:

1. Find the roots of the polynomial $1+(-1)^n s^{2n}=0$
2. Assign the left-half plane roots to H(s) and the right-half plane roots to H(-s)
3. Combine terms in the denominator of H(s) to form first- and second-order factors

EXAMPLE

Find the Butterworth transfer function for n=2.

For n=2, $1+(-1)^2 s^4=0$, then $s^4=-1=1\angle 180^0$

$$s_1 = 1\angle 45^0 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} \qquad s_2 = 1\angle 135^0 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$s_3 = 1\angle 225^0 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} \qquad s_4 = 1\angle 315^0 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

Roots s_2 and s_3 are in the left-half plane. Thus,

$$H(s) = \frac{1}{\left(s + 1/\sqrt{2} - j/\sqrt{2}\right)\left(s + 1/\sqrt{2} + j/\sqrt{2}\right)}$$

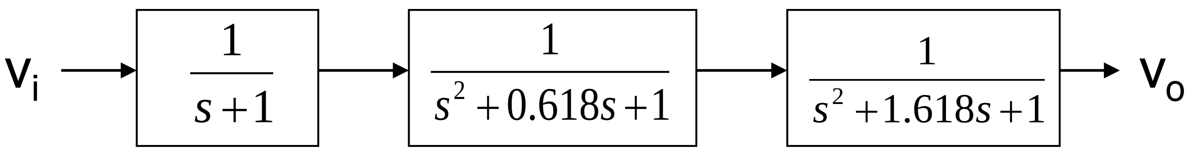
$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Normalized Butterworth Polynomials

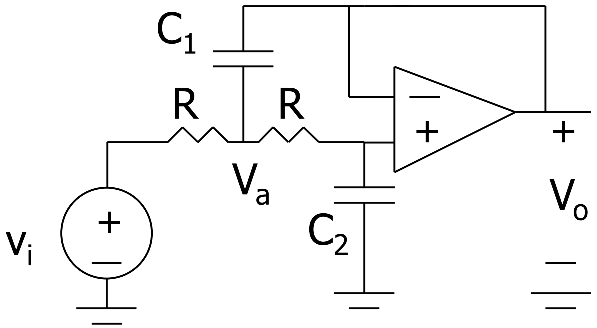
1	$(s+1)$
2	$(s^2+\sqrt{2}s+1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.765s+1)(s^2+1.848s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$
6	$(s^2+0.518s+1)(s^2+\sqrt{2}s+1)(s^2+1.932s+1)$

BUTTERWORTH FILTER CIRCUITS

To construct a Butterworth filter circuit, we cascade first- and second-order op amp circuits using the polynomials given in the table. A fifth-order prototype Butterworth filter is shown in the following figure:



All odd-order Butterworth polynomials include the factor (s+1), so all odd-order BUtterworth filters must include a subcircuit to implement this term. Then we need to find a circuit that provides a transfer function of the form $H(s) = \dfrac{1}{s^2 + b_1 s + 1}$



$$\frac{V_a - V_i}{R} + (V_a - V_o)sC_1 + \frac{V_a - V_o}{R} = 0$$

$$V_o sC_2 + \frac{V_o - V_a}{R} = 0$$

$$(2 + RC_1 s)V_a - (1 + RC_1 s)V_o = V_i$$

$$-V_a + (1 + RC_2 s)V_o = 0$$

$$V_o = \frac{1}{R^2 C_1 C_2 s^2 + 2RC_2 s + 1} V_i$$

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{1}{R^2 C_1 C_2}}{s^2 + \frac{2}{C_1} s + \frac{1}{C_1 C_2}}$$

$$b_1 = \frac{2}{C_1} \qquad 1 = \frac{1}{C_1 C_2}$$

EXAMPLE

Design a fourth-order low-pass filter with a cutoff frequency of 500 Hz and a passband gain of 10. Use as many 1 KΩ resistor as possible.

From table, the fourth-order Butterworth polynomial is

$$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$$

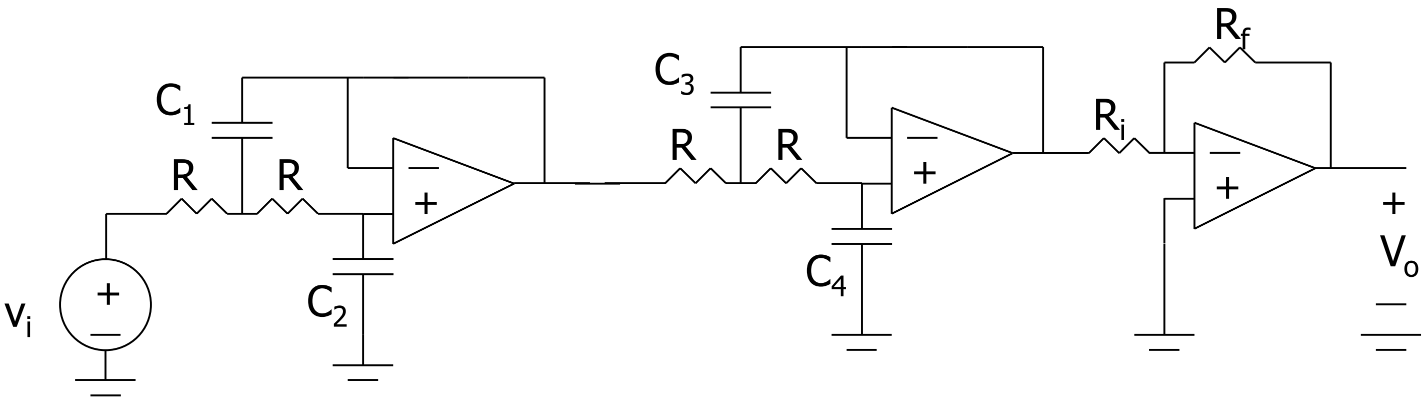
For the first stage: C_1=2/0.765=2.61 F, C_2=1/2.61=0.38F

For the second stage: C_3=2/1.848=1.08 F, C_4=1/1.08=0.924F

These values along with 1-Ω resistors will yield a fourth-order Butterworth filter with a cutoff frequency of 1 rad/s.

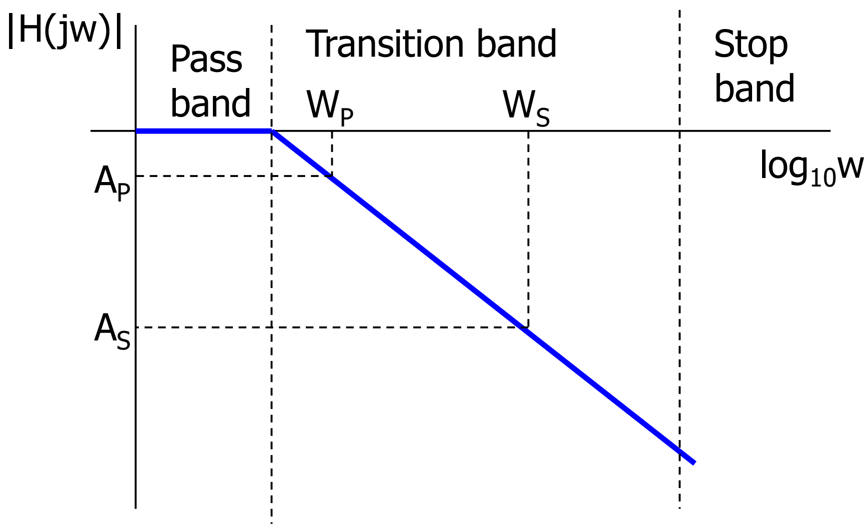
A frequency scale factor of k_f=3141.6 will move the cutoff frequency to 500 Hz. A magnitude scale factor k_m=1000 will permit the use of 1 kΩ resistors. Then,

R=1 kΩ, C_1=831 nF, C_2=121 nF, C_3= 344 nF, C_4=294 nF, $R_{f=}$ 10 kΩ.



The Order of a Butterworth Filter

As the order of the Butterworth filter increases, the magnitude characteristic comes closer to that of an ideal low-pass filter. Therefore, it is important to determine the smallest value of n that will meet the filtering specifications.



$$A_p = 20\log_{10}\frac{1}{\sqrt{1+\omega_p^{2n}}}$$

$$= -10\log_{10}(1+\omega_p^{2n})$$

$$A_s = 20\log_{10}\frac{1}{\sqrt{1+\omega_s^{2n}}}$$

$$= -10\log_{10}(1+\omega_s^{2n})$$

$$10^{-0.1A_p} = 1 + \omega_p^{2n} \qquad 10^{-0.1A_s} = 1 + \omega_s^{2n}$$

$$\left(\frac{\omega_s}{\omega_p}\right)^n = \frac{\sqrt{10^{-0.1A_s} - 1}}{\sqrt{10^{-0.1A_p} - 1}} = \frac{\sigma_s}{\sigma_p}$$

$$n \log_{10}(\omega_s / \omega_p) = \log_{10}(\sigma_s / \sigma_p)$$

$$n = \frac{\log_{10}(\sigma_s / \sigma_p)}{\log_{10}(\omega_s / \omega_p)}$$

If w_p is the cutoff frequency, then

$$A_p = -20\log_{10}\sqrt{2} \quad \text{and} \quad \sigma_p = 1$$

$$n = \frac{\log_{10}\sigma_s}{\log_{10}(\omega_s/\omega_p)}$$

For a steep transition region, $10^{-0.1A_s} >> 1$ Thus,

$$\sigma_s \approx 10^{-0.05A_s} \Rightarrow \log_{10}\sigma_s \approx -0.05A_s$$

$$n = \frac{-0.05A_s}{\log_{10}(\omega_s/\omega_p)}$$

EXAMPLE

Determine the order of a Butterworth filter that has a cutoff frequency of 1000 Hz and a gain of no more than -50 dB at 6000 Hz. What is the actual gain in dB at 6000 Hz?

Because the cutoff frequency is given, $\sigma_p = 1$ and $10^{-0.1(-50)} >> 1$

$$n = \frac{-0.05(-50)}{\log_{10}(6000/1000)} = 3.21$$

Therefore, we need a fourth-order Butterworth filter. The actual gain at 6000 Hz is

$$K = 20\log_{10}\left(\frac{1}{\sqrt{1+6^8}}\right) = -62.25\,\mathrm{dB}$$

EXAMPLE

Determine the order of a Butterworth filter whose magnitude is 10 dB less than the passband magnitude at 500 Hz and at least 60 dB less than the passband magnitude at 5000 Hz.

$$\sigma_p = \sqrt{10^{-0.1(-10)} - 1} = 3, \qquad \sigma_s = \sqrt{10^{-0.1(-60)} - 1} \approx 1000$$

$$\omega_s / \omega_p = f_s / f_p = 5000/500 = 10$$

$$n = \frac{\log_{10}(1000/3)}{\log_{10}(10)} = 2.52$$

Thus we need a third-order filter.

Determine the cutoff frequency.

$$-10\log_{10}[1 + (\omega/\omega_c)^6] = -10 \Rightarrow 1 + (\omega/\omega_c) = 10$$

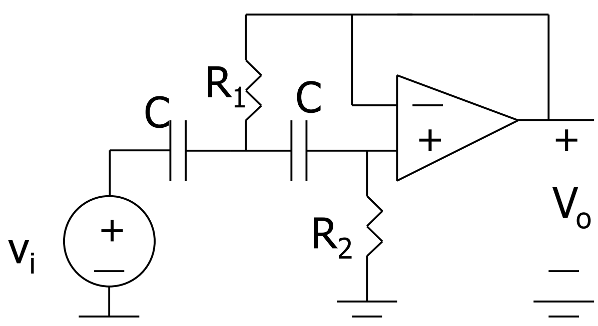
$$\omega_c = \frac{\omega}{\sqrt[6]{9}} = \frac{1000\pi}{\sqrt[6]{9}} = 2178.26 \ \text{rad/s}$$

BUTTERWORTH HIGH-PASS FILTERS

To produce the second-order factors in the Butterworth polynomial, we need a circuit with a transfer function of

$$H(s) = \frac{s^2}{s^2 + b_1 s + 1}$$

$$H(s) = \frac{V_o}{V_i} = \frac{s^2}{s^2 + \frac{2}{R_2 C} s + \frac{1}{R_1 R_2 C^2}}$$



Setting C= 1F

$$H(s) = \frac{V_o}{V_i} = \frac{s^2}{s^2 + \frac{2}{R_2} s + \frac{1}{R_1 R_2}}$$

$$b_1 = \frac{2}{R_2} \qquad 1 = \frac{1}{R_1 R_2}$$

NARROWBAND BANDPASS AND BANDREJECT FILTERS

The cascade or parallel component designs from simpler low-pass and high-pass filters will result in low-Q filters. Consider the transfer function

$$H(s) = \left(\frac{-\omega_c}{s+\omega_c}\right)\left(\frac{-s}{s+\omega_c}\right) = \frac{s\omega_c}{s^2 + 2\omega_c + \omega_c^2}$$

$$= \frac{0.5\beta s}{s^2 + \beta s + \omega_c^2}$$

$$\beta = 2\omega_c\,, \quad \omega_o^2 = \omega_c^2 \Rightarrow Q = \frac{\omega_o}{\beta} = \frac{1}{2}$$

Thus with discrete real poles, the highest quality factor bandpass filter we can achieve has Q=1/2