Sinusoidal Steady-state Power Calculations

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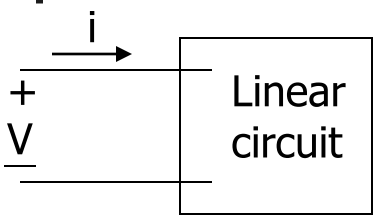
INSTANTANEOUS POWER



Here, v and i are steady-state sinusoidal signals. The power at any instant of time is p=v.i

If the reference direction of the current is in the direction of the voltage rise, the equation must be written with a minus sign.

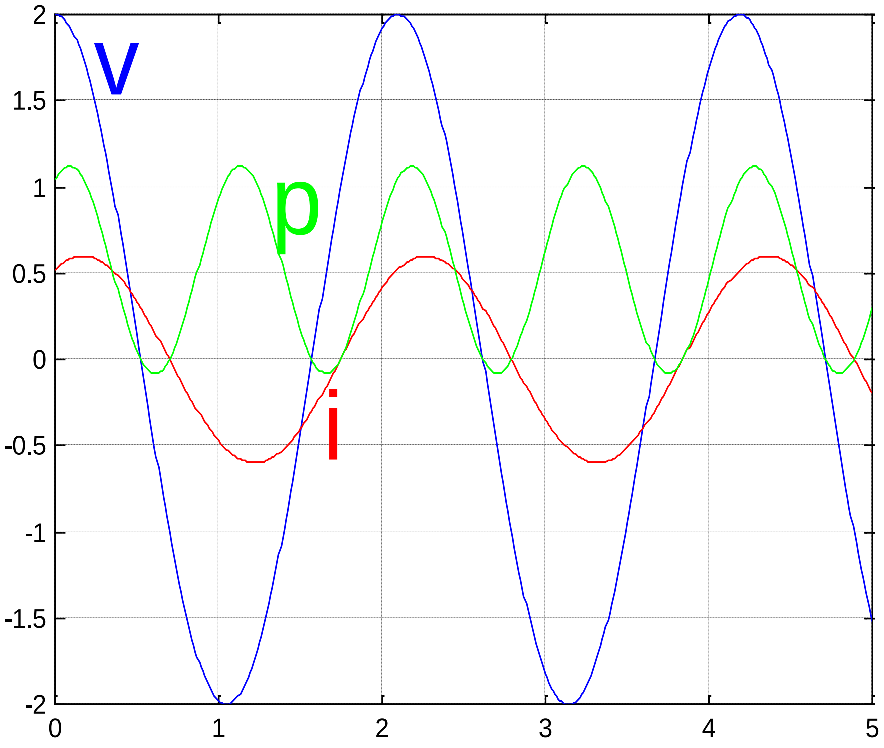
$$v = V_m \cos(\omega t + \theta_v)$$

$$i = I_m \cos(\omega t + \theta_i)$$

$$p = V_m \cos(\omega t + \theta_v) I_m \cos(\omega t + \theta_i)$$

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

```
t=0:0.01:5;
v=2*cos(3*t);
i=0.6*cos(3*t-pi/6);
p=v.*i;
figure(1)
plot(t,v)
hold on
plot(t,i,'r')
plot(t,p,'g')
grid
```



AVERAGE AND REACTIVE POWER

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos 2\omega t - \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin 2\omega t$$

$$p = P + P\cos 2\omega t - Q\sin 2\omega t$$

P is called the **average (real) power**, and Q is called the **reactive power**.

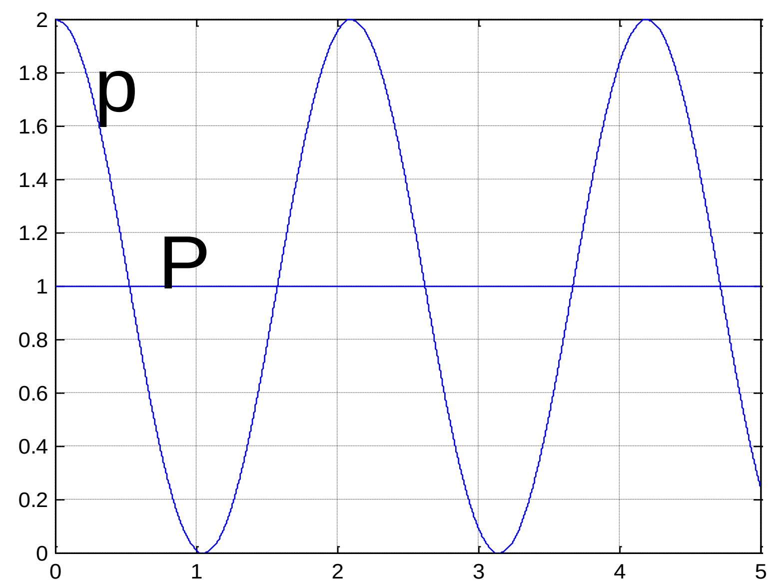
P is the average power associated with sinusoidal signals
The average of the instantaneous power over one period is:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} p\,dt$$

Purely resistive circuits

If the circuit between the terminals is purely resistive, the voltage and current are in phase, $\theta_v = \theta_i$, then $p = P + P\cos 2\omega t$

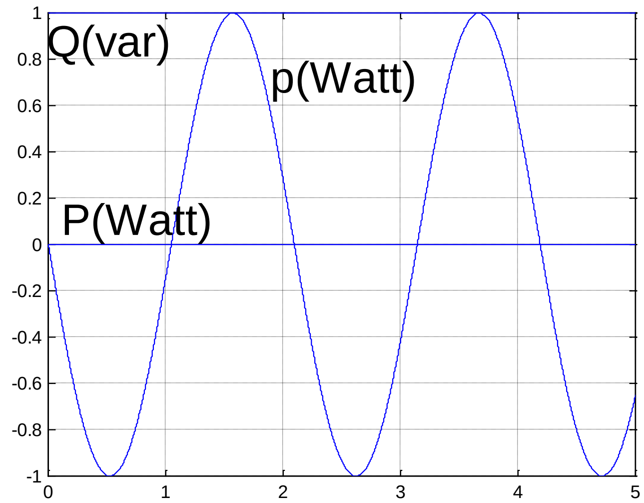
Instantaneous real power for a resistor can never be negative. Therefore, power cannot be extracted from a purely resistive circuit.



Purely Inductive Circuits

If the circuit between the terminals is purely inductive,the voltage and current are out of phase by 90^0 and the current lags the voltage. Therefore, $\theta_v - \theta_i = +90^0$ and $p = -Q\sin 2\omega t$

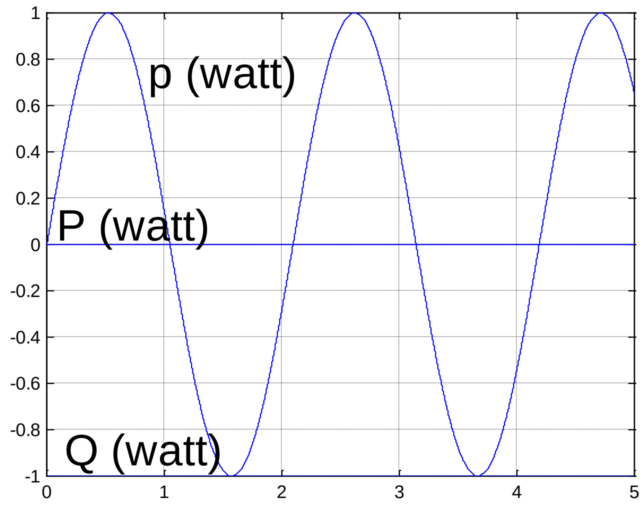
The average power is zero.
p is exchanged between the circuit and the source driving the circuit. When p is positive, energy is being stored in the inductor, and when p is negative energy is extracted from the inductor.
Q is the reactive power and its unit is volt-ampere reactive (VAR)



Purely Capacitive Circuits

If the circuit between the terminals is purely capacitive, the voltage and current are out of phase by 90^0 and the current leads the voltage. Therefore, $\theta_v - \theta_i = -90^0$ and $p = Q\sin 2\omega t$

Note that Q is positive for inductors and negative for capacitors. Therefore, we say that inductors demand (or absorb) magnetizing VARs and capacitors furnish (deliver) magnetizing VARs.



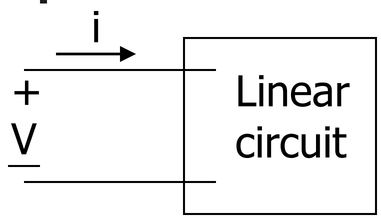
The Power Factor

The angle $\theta_v - \theta_i$ plays a role in the computation of both average and reactive power and is referred to as the **power factor angle.** The cosine of this angle is called the **power factor** (pf), and the sine of this angle is called the **reactive factor** (rf)

$$\text{pf} = \cos(\theta_v - \theta_i) \qquad \text{rf} = \sin(\theta_v - \theta_i)$$

Lagging power factor: Current lags voltage, hence inductive load.
Leading power factor: Current leads voltage, hence capacitive load.

Example



$$v = 100\cos(\omega t + 15^0)V,\quad i = 4\sin(\omega t - 15^0)A$$

Calculate the average and the reactive power.
Is this circuit absorbs or delivers average power and magnetizing VARs?

$$i = 4\cos(\omega t - 105^0)A$$

$$P = \frac{1}{2}(100)(4)\cos(15-(-105)) = -100W$$

$$Q = \frac{1}{2}(100)(4)\sin(15-(-105)) = 173.21VAR$$

Since P is negative, the circuit delivers average power to the terminals.
Q is positive (inductive) and the circuit absorbs magnetizing VARs at its terminals

THE RMS VALUE AND POWER CALCULATIONS

Assume that a sinusoidal voltage $v = V_m \cos(\omega t + \theta_v)$ is applied to the terminals of a resistor. Determine the average power delivered to the resistor

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{V_m^2 \cos^2(\omega t + \theta_v)}{R} dt$$

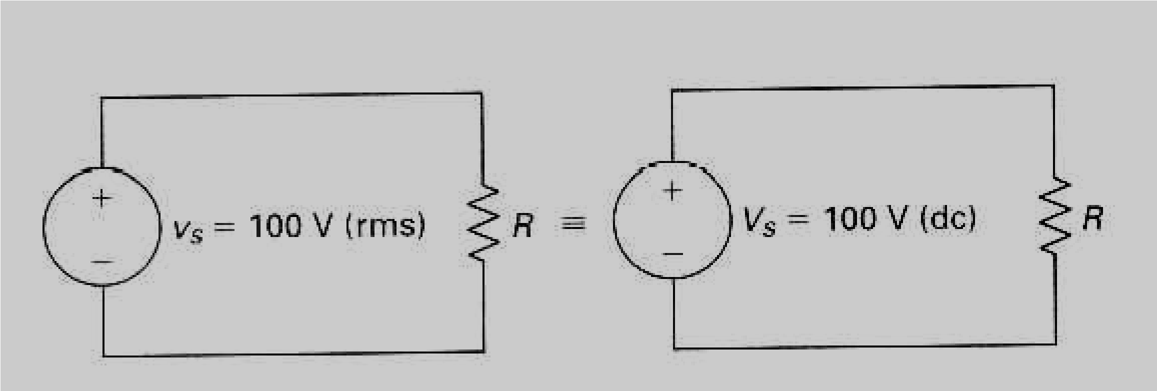
$$= \frac{1}{R}\left[\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \theta_v) dt\right] = \frac{V_{rms}^2}{R}$$

If the resistor is carrying a sinusoidal current $i = I_m \cos(\omega t + \theta_i)$

$$P = I_{rms}^2 R$$

EFFECTIVE VALUE

The rms value is also referred to as the **effective value** of the sinusoidal signal. Given a resistive load, R, and a sinusoidal source applied to this load, the rms (effective) value of the source delivers the same energy to R as does a dc source of the same value.



The average power and the reactive power can be written in terms of the reactive values:

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$

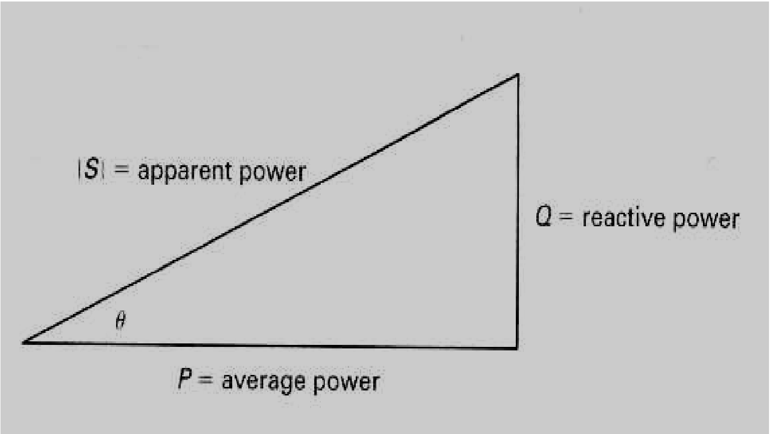
$$= \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i) = V_{eff} I_{eff} \cos(\theta_v - \theta_i)$$

$$Q = V_{eff} I_{eff} \sin(\theta_v - \theta_i)$$

COMPLEX POWER

Complex power is the complex sum of real power and reactive power

$S = P + jQ$ We can compute the complex power directly from the voltage and current phasors, then P=Re(S) and Q=Im(S).
Unit of the complex power is **volt-amps** (VA).



Think of P, Q, and $|S|$ as the sides of a right triangle. Then $\theta = \tan^{-1}\frac{Q}{P} = \theta_v - \theta_i$

is the power factor angle. The magnitude of the complex power is referred to as **apparent power**

$$|S| = \sqrt{P^2 + Q^2}$$

EXAMPLE

An electrical load operates at 240Vrms. The load absorbs an average power of 8 kW at a lagging power factor of 0.8. Calculate the complex power and the impedance of the load.

$$P = |S|\cos\theta \Rightarrow |S| = \frac{8000}{8} = 10kVA$$

$$Q = |S|\sin\theta = 10 * \sin(\cos^{-1} 0.8) = 6kVAR$$

$$S = 8 + j6kVA$$

$$P = V_{eff} I_{eff} \cos\theta \Rightarrow I_{eff} = \frac{8000}{(240)0.8} = 41.67 A_{rms}$$

$$\theta = \cos^{-1} 0.8 = 36.87^0$$

$$|Z| = \frac{V_{eff}}{I_{eff}} = \frac{240}{41.67} = 5.76$$

$$Z = 5.76\angle 36.87^0 = 4.608 + j3.456\Omega$$

POWER CALCULATIONS

$$S = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + j\frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

$$= \frac{V_m I_m}{2}\left[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right]$$

$$= \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} = \frac{1}{2} V_m I_m \angle(\theta_v - \theta_i)$$

$$= V_{eff} I_{eff} \angle(\theta_v - \theta_i) = V_{eff} I_{eff} e^{j(\theta_v - \theta_i)}$$

$$= V_{eff} e^{j\theta_v} I_{eff} e^{-j\theta_i} = \mathbf{V}_{eff} \mathbf{I}_{eff}^{*}$$

ALTERNATE FORMS FOR THE COMPLEX POWER

$$\mathbf{V}_{eff} = Z\mathbf{I}_{eff} \Rightarrow S = Z\mathbf{I}_{eff}\mathbf{I}_{eff}^{*} = \left|\mathbf{I}_{eff}\right|^{2} Z$$

$$S = \left|\mathbf{I}_{eff}\right|^{2}(R + jX) = \left|\mathbf{I}_{eff}\right|^{2} R + j\left|\mathbf{I}_{eff}\right|^{2} X = P + jQ$$

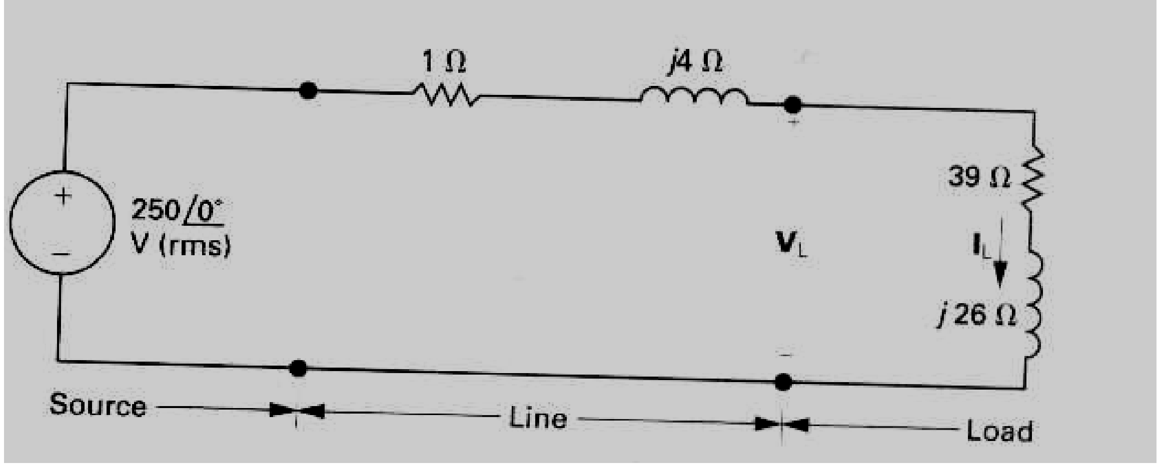
$$P = \left|\mathbf{I}_{eff}\right|^{2} R = \frac{1}{2} I_m^2 R$$

$$Q = \left|\mathbf{I}_{eff}\right|^{2} X = \frac{1}{2} I_m^2 X$$

Or $$S = \mathbf{V}_{eff}\left(\frac{\mathbf{V}_{eff}}{Z}\right)^{*} = \frac{\left|\mathbf{V}_{eff}\right|^{2}}{Z^{*}} = P + jQ$$

$$P = \frac{\left|\mathbf{V}_{eff}\right|^{2}}{R} \qquad Q = \frac{\left|\mathbf{V}_{eff}\right|^{2}}{X}$$

EXAMPLE



$$\mathbf{I}_L = \frac{250}{40 + j30} = 4 - j3 = 5\ \angle{-36.87^0}\ A_{rms}$$

$$\mathbf{V}_L = (39 + j26)\mathbf{I}_L = 234 - j13 = 234.36\ \angle{-3.18^0} V_{rms}$$

$$S = \mathbf{V}_L \mathbf{I}_L^* = (234 - j13)(4 + j3) = 975 + j650VA$$

$$P_{line} = (5)^2 1 = 25W$$

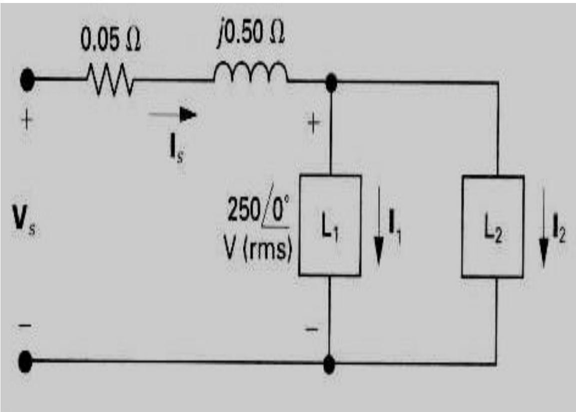
$$Q_{line} = (5)^2 4 = 100VAR$$

$$S_s = 25 + j100 + 975 + j650$$

$$= 1000 + j750VA$$

EXAMPLE

Load 1 absorbs an average power of 8 kW at a leading power factor of 0.8. Load 2 absorbs 20 kVA at a lagging power factor of 0.6.



a) Determine the pf of the parallel loads
b) Determine the apparent power supplied to the loads, magnitude of $\mathbf{I}_s$ and P_{line}
c) Determine the value of C that would correct the pf to 1 if placed in parallel to the loads. Calculate the values in b for the load with the corrected pf.

$$S = (250)\mathbf{I}_s^* = (250)(\mathbf{I}_1 + \mathbf{I}_2)^* = S_1 + S_2$$

$$S_1 = 8000 - j\frac{8000(0.6)}{0.8} = 8000 - j6000VA$$

$$S_2 = 20000(0.6) + j20000(0.8) = 12000 + j16000VA$$

$$S = 20000 + j10000VA \Rightarrow \mathbf{I}_s^* = \frac{20000 + j10000}{250} = 80 + j40A$$

$$\mathbf{I}_s = 80 - j40 = 89.44\angle -26.57^0 A$$

$$\text{pf} = \cos(0 + 26.57^0) = 0.8944\text{Lagging}$$

b)

$$|S| = |20 + j10| = 22.36kVA$$

$$|\mathbf{I}_s| = |80 - j40| = 89.44A$$

$$P_{line} = |\mathbf{I}_s|^2 R = (89.44)^2(0.05) = 400W$$

c) We can correct the pf to 1 if we place a capacitor in parallel with the existing loads such that the capacitor supplies 10kVAR.

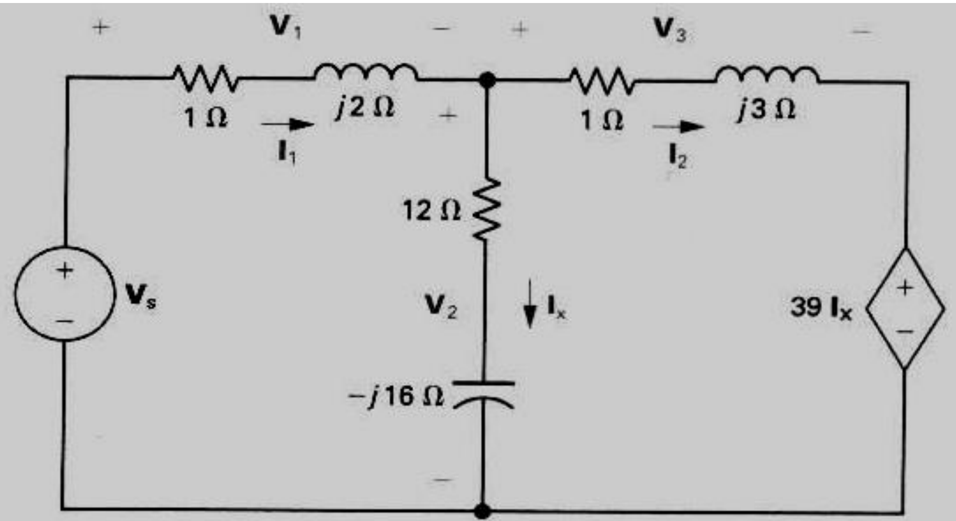
$$X = \frac{|V_{eff}|^2}{Q} = \frac{250^2}{-10000} = -6.25\Omega \qquad C = \frac{-1}{\omega X} = \frac{-1}{(2\pi 60)(-6.25)} = 424.4\mu F$$

After correcting the pf to 1.

$$|S| = P = 20kVA \qquad \text{and} \qquad |\mathbf{I}_s| = \frac{20000}{250} = 80A$$

$$P_{line} = |\mathbf{I}_s|^2 R = (80)^2(0.05) = 320W$$

EXAMPLE



$\mathbf{V}_s = 150\angle 0^\circ$ V

$\mathbf{V}_1 = (78 - j104)$ V

$\mathbf{V}_2 = (72 + j104)$ V

$\mathbf{V}_3 = (150 - j130)$ V

$\mathbf{I}_1 = (-26 - j52)$ A

$\mathbf{I}_x = (-2 + j6)$ A

$\mathbf{I}_2 = (-24 - j58)$ A

Calculate the average and reactive power of each element in the circuit. Verify consumption of power

$$S_1 = \frac{1}{2}\mathbf{V}_1\mathbf{I}_1^* = P_1 + jQ_1$$

$$= \frac{1}{2}(78 - j104)(-26 + j52)$$

$$= 1690 + j3380VA$$

1+j2 absorbs both average power and magnetizing VARs

$$S_2 = \frac{1}{2}\mathbf{V}_2\mathbf{I}_x^* = P_2 + jQ_2$$
$$= \frac{1}{2}(72 + j104)(-2 - j6)$$
$$= 240 - j320VA$$

12-j16Ω absorbs average power and delivers magnetizing VARS.

$$S_s = \frac{1}{2}\mathbf{V}_s\mathbf{I}_1^* = P_s + jQ_s$$
$$= \frac{1}{2}(150)(-26 + j52)$$
$$= 1950 - j3900VA$$

$$S_3 = \frac{1}{2}\mathbf{V}_3\mathbf{I}_2^* = P_3 + jQ_3$$
$$= \frac{1}{2}(150 - j130)(-24 + j58)$$
$$= 1970 + j5910VA$$

1+j3Ω absorbs average power and magnetizing VARS.

$$S_x = \frac{1}{2}(39\mathbf{I}_x)\mathbf{I}_2^* = P_x + jQ_x$$
$$= \frac{1}{2}(-78 + j234)(-24 + j58)$$
$$= -5850 - j5070VA$$

Independent voltage source is absorbing average power and delivering magnetizing VARs. Dependent source delivers average power and magnetizing VARs.

$$P_{abs} = P_1 + P_2 + P_3 + P_s = 5850W$$

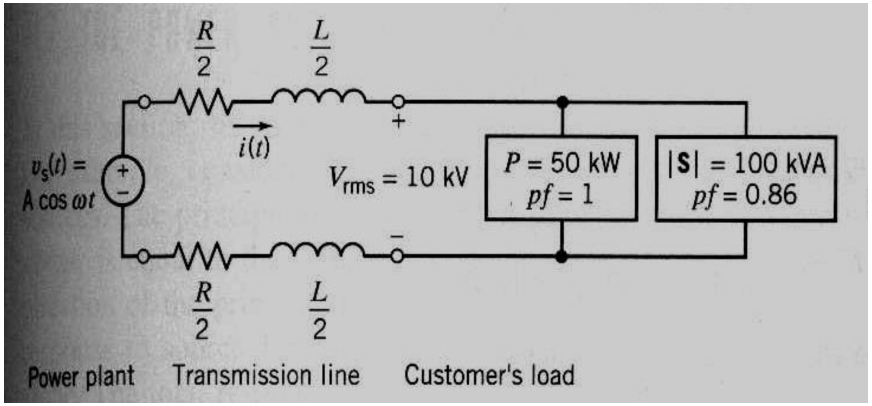
$$P_{del} = 5850W$$

$$Q_{abs} = Q_1 + Q_3 = 9290VAR$$

$$Q_{del} = Q_2 + Q_s + Q_x = 9290VAR$$

EXAMPLE

A customer's plant has two parallel loads connected to the power utility's distribution lines. The first load consists of 50 kW of heating and is resistive. The second load is a set of motors that operate at 0.86 lagging power factor. The motor's load is 100 kVA. Power is supplied to the plant at 10000 Vrms. Determine the total current flowing from the utility's lines into the plant and the plant's overall power factor.



$$S_1 = P_1 = 50kW$$

$$\theta_2 = \cos^{-1}(0.86) = 30.7^0$$

$$S_2 = |S_2| \angle \theta_2 = 100\angle 30.7^0 kVA$$

$$= 86 + j51 kVA$$

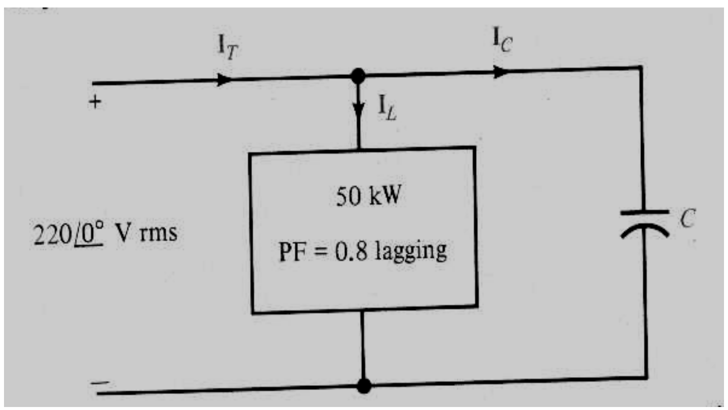
$$S = S_1 + S_2 = 136 + j51 = 145.2\angle 20.6^0 kVA$$

$$\text{Pf} = \cos(20.6^0) = 0.94 \text{ lagging}$$

$$I_{rms} = \frac{|S|}{V_{rms}} = \frac{145200}{10000} = 14.52 Arms$$

POWER FACTOR CORRECTION

An industrial load consisting of a bank of induction motors consumes 50 kW at a Pf of 0.8 lagging from a 220-Vrms, 60-Hz line. We wish to raise the Pf to 0.95 lagging by placing a bank of capacitors in parallel to the load.



$$\theta_{old} = \cos^{-1} 0.8 = 36.87^0$$

$$Q_{old} = |S_{old}| \sin(\theta_{old}) = \frac{50}{0.8} \sin(36.87^0) = 37.5kVAR$$

$$S_{old} = 50000 + j37500$$

$$\theta_{new} = \cos^{-1} 0.95 = 18.19^0$$

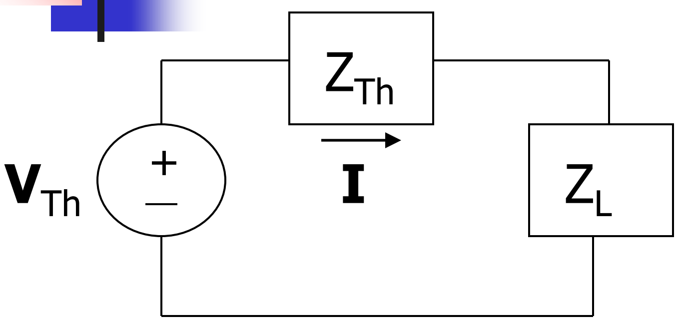
$$Q_{new} = |S_{new}| \sin(\theta_{new}) = \frac{50}{0.95} \sin(18.19^0) = 16.43kVAR$$

$$S_{new} = 50000 + j16430$$

$$S_{cap} = S_{new} - S_{old} = -j21070$$

$$C = \frac{21070}{(377)(220)^2} = 1155\mu F$$

MAXIMUM POWER TRANSFER



Determine the value of Z_L for maximum average power transfer

$$Z_{Th} = R_{Th} + jX_{Th} \qquad Z_L = R_L + jX_L$$

$$\mathbf{I} = \frac{\mathbf{V}_{Th}}{(R_{Th} + R_L) + j(X_{Th} + X_L)} \qquad P = |\mathbf{I}|^2 R_L = \frac{|\mathbf{V}_{Th}|^2}{(R_{Th} + R_L)^2 + (X_{Th} + X_L)^2}$$

$$\frac{\partial P}{\partial X_L} = \frac{-|\mathbf{V}_{Th}|^2 2R_L(X_L + X_{Th})}{\left[(R_L + R_{Th})^2 + (X_L + X_{Th})^2\right]^2}$$

$$\frac{\partial P}{\partial R_L} = \frac{|\mathbf{V}_{Th}|^2\left[(R_L + R_{Th})^2 + (X_L + X_{Th})^2 - 2R_L(R_L + R_{Th})\right]}{\left[(R_L + R_{Th})^2 + (X_L + X_{Th})^2\right]^2}$$

$$\frac{\partial P}{\partial X_L} = 0 \quad \text{when} \quad X_L = -X_{Th}$$

$$\frac{\partial P}{\partial R_L} = 0 \quad \text{when} \quad R_L = \sqrt{R_{Th}^2 + (X_L + X_{Th})^2}$$

Both derivatives are zero when $Z_L = Z_{Th}^*$

$$P_{\max} = \frac{1}{4}\frac{|\mathbf{V}_{Th}|^2}{R_L}$$

If the Thevenin voltage is expressed in terms of its maximum value rather than its rms amplitude,

$$P_{\max} = \frac{1}{8}\frac{V_m^2}{R_L}$$

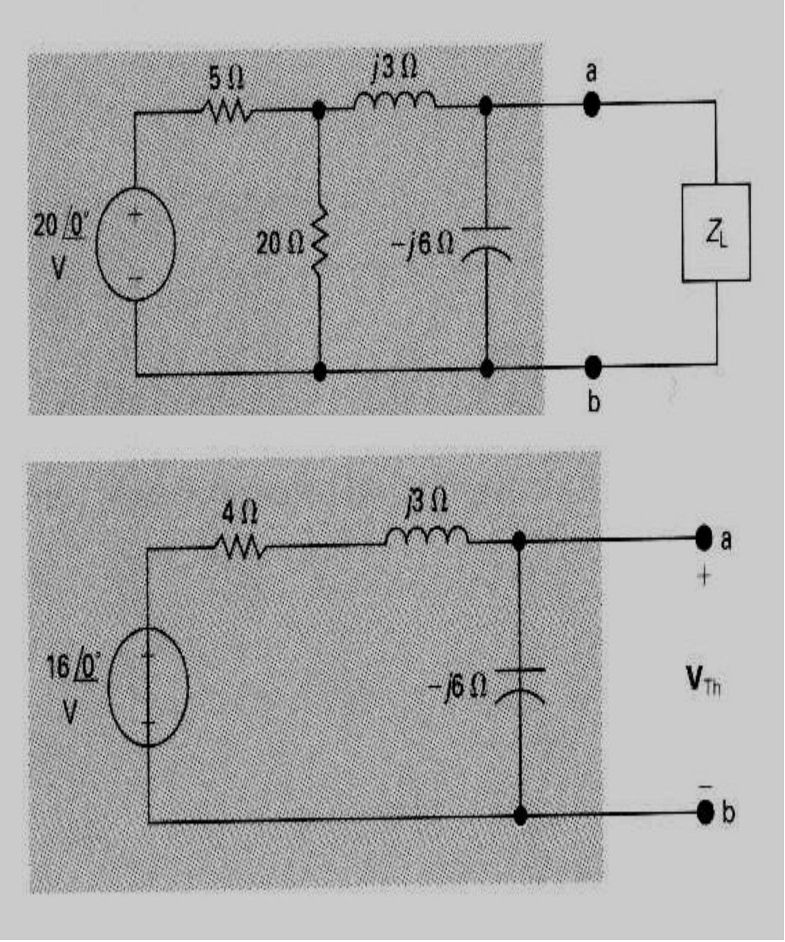
MAXIMUM POWER TRANSFER WHEN Z IS RESTRICTED

When it is not possible to set Z_L equal to the conjugate of Z_{Th}:

I) R_L and X_L may be restricted to a limited range of values. In this case, the optimum condition for R_L and X_L is to adjust X_L as near to $-X_{Th}$ as possible and then adjust R_L as close to $\sqrt{R_{Th}^2 + (X_L + X_{Th})^2}$ as possible

II) A second type of restriction occurs when the magnitude of Z_L can be varied but its phase angle cannot. Under this restriction, the greatest amount of power is transferred to the load when the magnitude of Z_L is set equal to the magnitude of Z_{Th}. $|Z_L|=|Z_{Th}|$

Example



Determine the impedance Z_L that results in maximum average power transfer to Z_L.

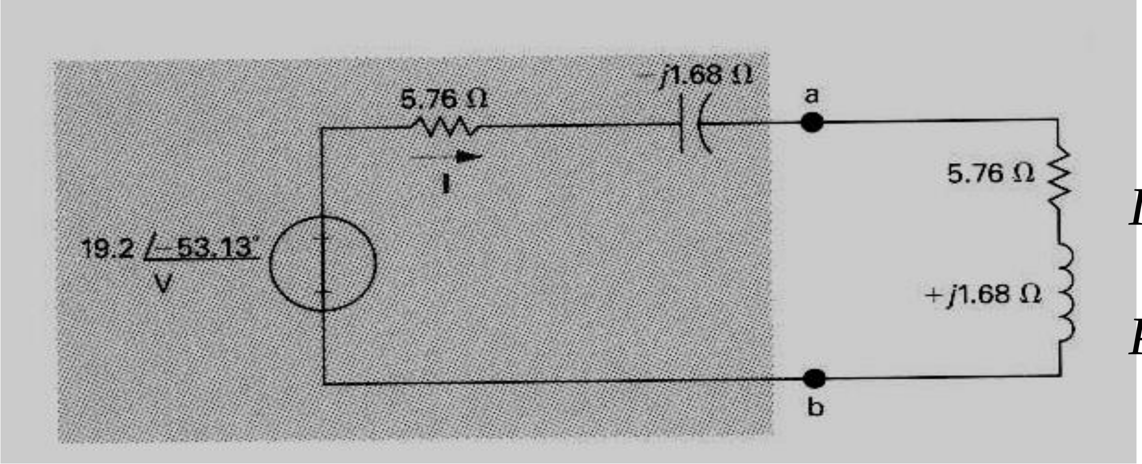
We need to determine the Thevenin equivalent seen at the terminals a-b. Using 2 successive source transformations, circuit becomes and

$$\mathbf{V}_{Th} = \frac{16}{4 + j3 - j6}(-j6)$$

$$= 19.2\angle -53.13^0 = 11.52 - j15.36V$$

$$Z_{Th} = \frac{(-j6)(4+j3)}{4+j3-j6} = 5.76 - j1.68\Omega$$

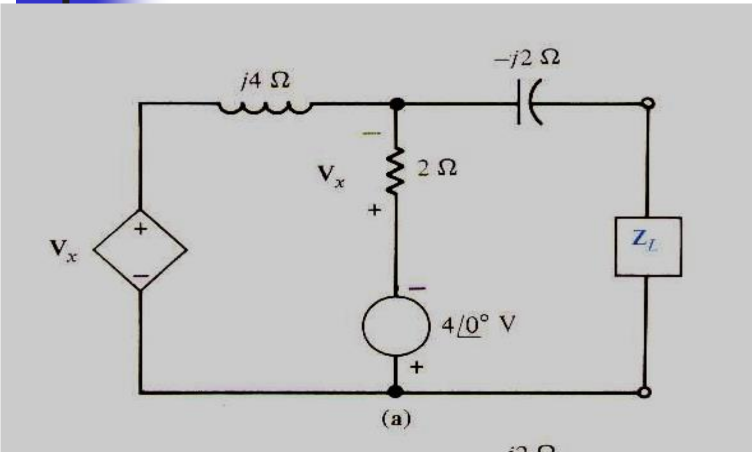
$$Z_L = Z_{Th}^{*} = 5.76 + j1.68\Omega$$

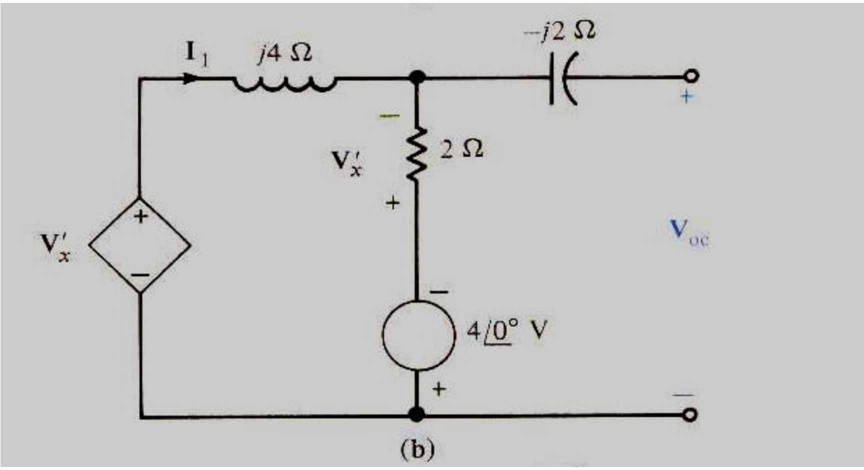


$$I_{eff} = \frac{19.2/\sqrt{2}}{2(5.76)} = 1.1785A$$

$$P = I_{eff}^{2}(5.76) = 8W$$

Example





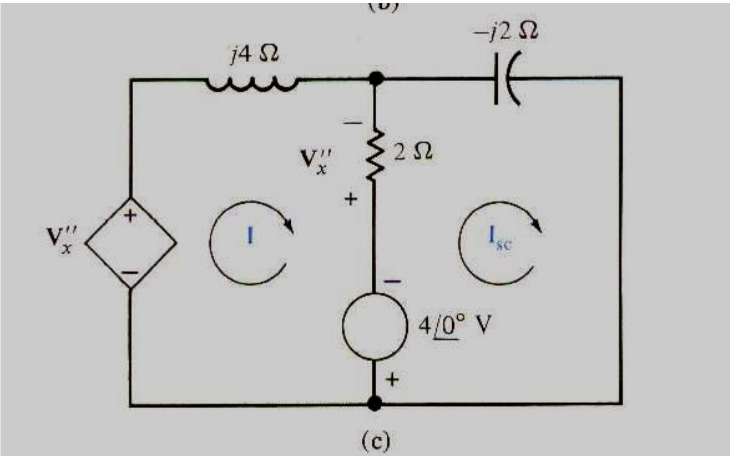
Find the value of Z_L for maximum power transfer.

$$\mathbf{V}'_x + 4 = (2 + j4)\mathbf{I}_1 \qquad \mathbf{V}'_x = -2\mathbf{I}_1$$

$$\mathbf{I}_1 = \frac{1\angle{-45^0}}{\sqrt{2}}$$

$$\mathbf{V}_{oc} = 2\mathbf{I}_1 - 4\angle 0^0 = \sqrt{2}\angle{-45^0} - 4\angle 0^0$$

$$= -3 - j1 = -3.16\angle 18.43^0 V$$



$$\mathbf{V}''_x + 4 = (2 + j4)\mathbf{I} - 2\mathbf{I}_{sc}$$

$$-4 = -2\mathbf{I} + (2 - j2)\mathbf{I}_{sc}$$

$$\mathbf{V}''_x = -2(\mathbf{I} - \mathbf{I}_{sc})$$

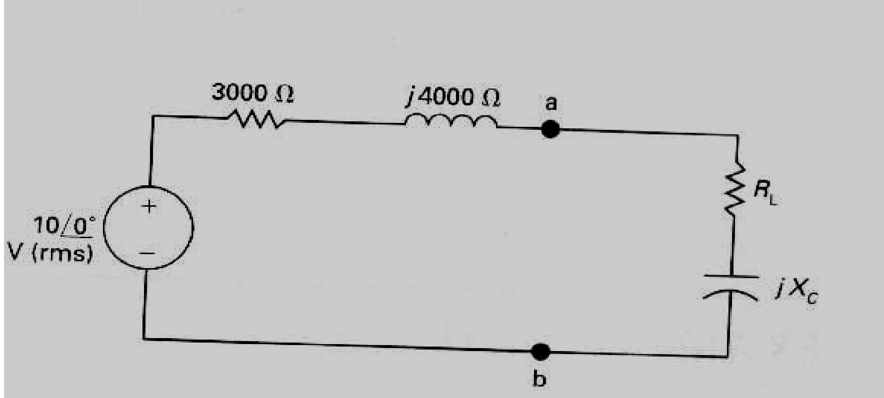
$$\mathbf{I}_{sc} = -(1 + j2)A \Rightarrow Z_{Th} = \frac{\mathbf{V}_{oc}}{\mathbf{I}_{sc}} = \frac{3 + j1}{1 + j2} = 1 - j1\Omega$$

$$Z_L = Z_{Th}^* = 1 + j1\Omega$$

$$\mathbf{I}_L = \frac{\mathbf{V}_{oc}}{Z_L + Z_{Th}} = \frac{-3 - j1}{2} = -1.58\angle 18.43^0\, A$$

$$P_{LMax} = \frac{1}{2}(1.58)^2(1) = 1.25W$$

EXAMPLE



Assume that the load resistance can be varied between 0 and 4000 and that the capacitive reactance of the load can be varied between 0 and -2000. What settings of R_L and X_L transfer the most average power to the load?

Set X_L=-2000Ω. Then set R_L as close as possible to

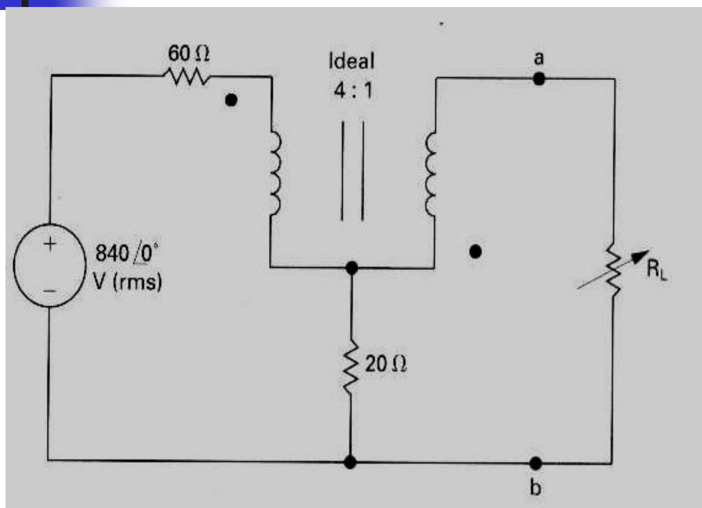
$$R_L = \sqrt{R_{Th}^2 + (X_L + X_{Th})^2} = \sqrt{3000^2 + (-2000 + 4000)^2} = 3605.55\Omega$$

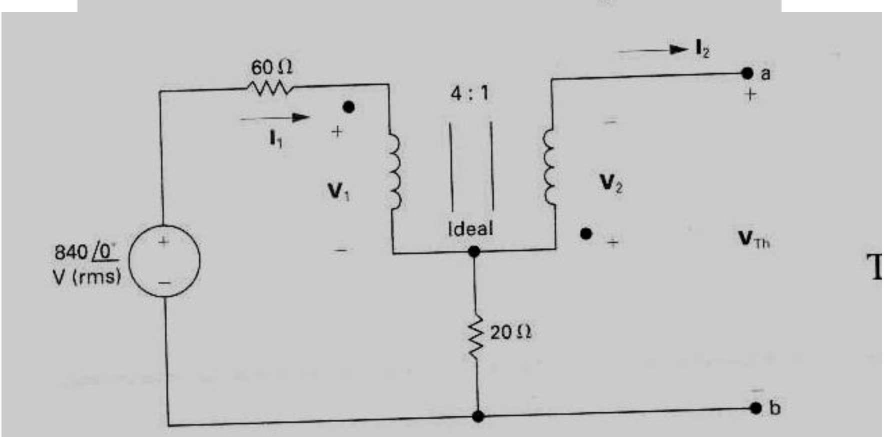
Using R_L=3605.55 Ω, Z_L=3605.55-j2000 Ω

$$\mathbf{I}_{eff} = \frac{10}{6605.55 + j2000} = 1.4489 \angle -16.85^0 mA$$

$$P = (1.4489 \times 10^{-3})^2 (3605.55) = 7.57 mW.$$

Example





The variable resistor in the circuit is adjusted until maximum average power is delivered to R_L.

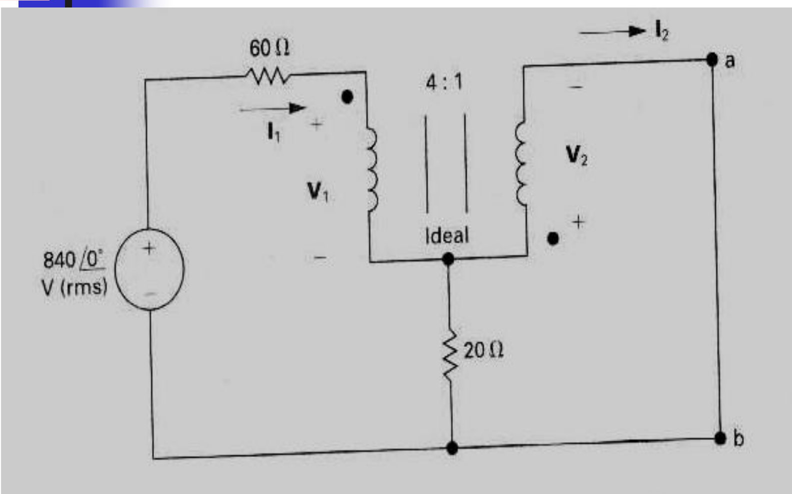
a) What is the value of R_L?

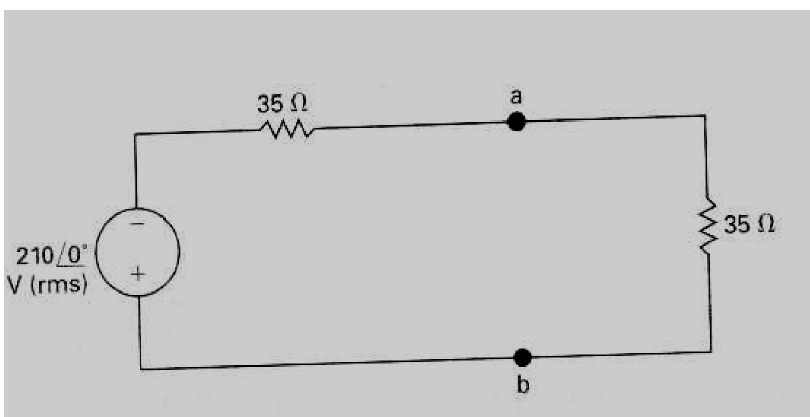
b) What is the maximum average power delivered to R_L?

$$\mathbf{V}_2 = \frac{1}{4}\mathbf{V}_1\,, \quad \mathbf{I}_1 = -\frac{1}{4}\mathbf{I}_2$$

Since $\mathbf{I}_2$ is zero $\mathbf{I}_1$ is zero and

$$\mathbf{V}_{Th} = -\mathbf{V}_2 = -210\angle 0^0$$





$$840\angle 0^0 = 80\mathbf{I}_1 - 20\mathbf{I}_2 + \mathbf{V}_1 = -40\mathbf{I}_2 + \mathbf{V}_1$$

$$0 = 20\mathbf{I}_2 - 20\mathbf{I}_1 + \mathbf{V}_2 = 25\mathbf{I}_2 + \frac{\mathbf{V}_1}{4}$$

$$\mathbf{I}_2 = -6A = \mathbf{I}_{sc}$$

$$R_{Th} = \frac{-210}{-6} = 35\Omega$$

$$P_{\max} = \left(\tfrac{-210}{70}\right)^2 35 = 315W$$