1/ $y = ae^{x} + be^{2x} + x$ iki parametreli eğri ailesinin diferansiyel denklemini bulunuz.

Çözüm: $2/ \; y = ae^{x} + be^{2x} + x$

$-3/ \; y' = ae^{x} + 2be^{2x} + 1$

$+ \quad y'' = ae^{x} + 4be^{2x}$

$$y'' - 3y' + 2y = 2x - 3$$

2/ $16y^{(4)} + 24y'' + 9y = 0$ dif. denkleminin genel çözümünü bulunuz.

Çözüm: Dif. denklemin karakteristik denklemi

$$16\lambda^4 + 24\lambda^2 + 9 = 0$$

$$(4\lambda^2 + 3)^2 = 0 \Rightarrow 4\lambda^2 + 3 = 0$$

$$\lambda_{1,2} = \mp \frac{i\sqrt{3}}{2}, \quad \lambda_{3,4} = \pm \frac{i\sqrt{3}}{2}$$

$$y = (c_1 + c_2 x)\cos\left(\frac{\sqrt{3}x}{2}\right) + (c_3 + c_4 x)\sin\left(\frac{\sqrt{3}x}{2}\right)$$

3/ $y''' + 12y'' + 36y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = -7$ başlangıç değer probleminin genel çözümünü bulunuz.

Çözüm: Dif. denklemin karakteristik denklemi

$$\lambda^3 + 12\lambda^2 + 36\lambda = 0$$

$$\lambda(\lambda^2 + 12\lambda + 36) = 0$$

$$\lambda(\lambda + 6)^2 = 0 \Rightarrow \lambda_1 = 0, \; \lambda_2 = \lambda_3 = 6$$

$y = c_1 + c_2 e^{-6x} + c_3 x e^{-6x}$, $y' = -6c_2 e^{-6x} - 6c_3 x e^{-6x} + c_3 e^{-6x}$

$y'' = 36c_2 e^{-6x} + 36c_3 x e^{-6x} - 12c_3 e^{-6x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y'(0) = 1 \Rightarrow -6c_2 + c_3 = 1$

$y''(0) = -7 \Rightarrow 36c_2 - 12c_3 = 7$

$c_2 = \frac{-5}{36}$, $c_3 = \frac{11}{6}$, $c_1 = \frac{5}{36}$ $\Rightarrow y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{11}{6}xe^{-6x}$

4/ Karakteristik denklemin kökleri

$\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = 2$, $\lambda_4 = 3+2i$, $\lambda_5 = 3-2i$

olan dif denklemin genel çözümünü yazınız.

Çözüm: (-1) kökü 2-katlı, diğer kökleri tek katlı ve $\lambda_4 - \lambda_5$ kompleks eşlenikli köklerdir.

Bu durumda

$$y = (c_1 + c_2 x)e^{-x} + c_3 e^{2x} + e^{3x}\left(c_4 \cos 2x + c_5 \sin 2x\right) \; //$$

c_1, c_2, c_3, c_4, c_5 parametrelerdir.

5/ $y'' + 4y = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & x > \pi/2 \end{cases}$, $y(0) = 1$, $y'(0) = 2$

başlangıç değer problemini çözünüz.

Çözüm: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \mp 2i$

$y_h = c_1 \cos 2x + c_2 \sin 2x$

Aynı zamanda, $x > \pi/2$ için $y = c_1 \cos 2x + c_2 \sin 2x$

$0 \le x \le \pi/2$ için $UC = \{\sin x, \cos x\}$ olup y_h dan lineer bağımsız olduğundan

$$\left.\begin{aligned} y_p &= A\sin x + B\cos x \\ y_p' &= A\cos x - B\sin x \\ y_p'' &= -A\sin x - B\cos x \end{aligned}\right\}$$

$-A\sin x - B\cos x + 4A\sin x + 4B\cos x = \sin x$

$3A\sin x + 3B\cos x = \sin x$

$A = 1/3$, $B = 0$

$y_p = \frac{1}{3}\sin x \to 0 \le x \le \pi/2$

$$y = \begin{cases} c_1 \cos 2x + c_2 \sin 2x + 1/3 \sin x, & 0 \le x \le \pi/2 \\ c_1 \cos 2x + c_2 \sin 2x, & x > \pi/2 \end{cases}$$

$y(0) = 1 \Rightarrow y = \begin{cases} c_1 = 1, & 0 \le x \le \pi/2 \\ c_1 = 1, & x > \pi/2 \end{cases}$ /

$$y' = \begin{cases} -2c_1 \sin 2x + 2c_2 \cos 2x - \frac{1}{3}\cos x, & 0 \le x \le \pi/2 \\ -2c_1 \sin 2x + 2c_2 \cos 2x, & x > \pi/2 \end{cases}$$

$$y'(0) = 2 \Rightarrow \begin{cases} 2c_2 - 1/3 = 2, & 0 \le x \le \pi/2 \\ 2c_2 = 2, & x > \pi/2 \end{cases} = \begin{cases} c_2 = 7/6, & 0 \le x \le \pi/2 \\ c_2 = 1, & x > \pi/2 \end{cases}$$

$$\Rightarrow y = \begin{cases} \cos 2x + 7/6 \sin 2x - 1/3 \sin x, & 0 \le x \le \pi/2 \\ \cos 2x + \sin 2x, & x > \pi/2. \end{cases}$$

6/ $y''' - 3y'' + 3y' - y = xe^x$ diferansiyel denkleminin genel çözümünü bulunuz.

Çözüm: $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda - 1)^3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1$

$$y_h = (c_1 + c_2 x + c_3 x^2) e^x$$

$$UC = \{xe^x, e^x\} \longrightarrow \{x^2e^x, xe^x\} \longrightarrow \{x^3e^x, x^2e^x\} \longrightarrow \{x^4e^x, x^3e^x\}$$

$$y_p = (Ax^4 + Bx^3)e^x$$

$$y_p' = (4Ax^3 + 3Bx^2)e^x + (Ax^4 + Bx^3)e^x$$

$$y_p'' = (12Ax^2 + 6Bx)e^x + 2(4Ax^3 + 3Bx^2)e^x + (Ax^4 + Bx^3)e^x$$

$$y_p''' = (24Ax + 6B)e^x + 3(12Ax^3 + 6Bx)e^x + 3(4Ax^3 + 3Bx^2)e^x + (Ax^4 + Bx^3)e^x$$

$$(24Ax + 6B)e^x + 3\cancel{(12Ax^3 + 6Bx)}e^x + 3(4A\cancel{x^3 + 3Bx^2})e^x + (Ax^4 + \cancel{Bx^3})e^x - 3\cancel{(12Ax^2 + 6Bx)e^x}$$
$$- 6(4A\cancel{x^3 + 3Bx^2})e^x - 3(A\cancel{x^4 + Bx^3})e^x + 3(4A\cancel{x^3 + 3Bx^2})e^x + 3(A\cancel{x^4 + Bx^3})e^x - (A\cancel{x^4 + Bx^3})e^x$$
$$= xe^x$$

$24A = 1 \Rightarrow A = 1/24, \quad B = 0$

$$y_p = 1/24\, x^4 e^x$$

$$y = (c_1 + c_2 x + c_3 x^2)e^x + 1/24\, x^4 e^x$$

7/ $y'' + 2y' + y = e^{-x}\ln x$ diferansiyel denkleminin genel çözümünü bulunuz.

Çözüm: $\lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$

$$y_h = c_1 e^{-x} + c_2 x e^{-x}$$

$$y = v_1 e^{-x} + v_2 x e^{-x}$$

$$\begin{cases} v_1' e^{-x} + v_2' x e^{-x} = 0 \\ -v_1' e^{-x} + v_2'(1-x)e^{-x} = e^{-x}\ln x \end{cases}$$

$$v_1' = \frac{\begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln x & (1-x)e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}} = \frac{-\cancel{e^{-2x}}\, x\ln x}{\cancel{e^{-2x}}(1-x+x)} = -x\ln x$$

$$v_1 = -\int x\ln x\,dx = -\frac{x^2}{2}\ln x + \frac{x^2}{4} + c_1$$

$$v_2' = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\ln x \end{vmatrix}}{e^{-2x}} = \frac{e^{-2x}\ln x}{e^{-2x}} = \ln x$$

$$v_2 = \int \ln x\,dx = x\ln x - x + c_2$$

$$y = c_1 e^{-x} + e^{-x}\left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right) + c_2 x e^{-x} + xe^{-x}(x\ln x - x) \quad //$$

8/ $y'' + y = \dfrac{1}{1+\sin x}$ diferansiyel denklemini çözünüz.

Çözüm: $\lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \mp i$

$$y_h = c_1 \cos x + c_2 \sin x$$

$$y = u_1 \cos x + u_2 \sin x$$

$$\begin{cases} u_1' \cos x + u_2' \sin x = 0 \\ -u_1' \sin x + u_2' \cos x = \dfrac{1}{1+\sin x} \end{cases}$$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ 1/1+\sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\dfrac{\sin x}{1+\sin x}}{1} = -\frac{\sin x}{1+\sin x}$$

$$u_1 = -\int \frac{\sin x}{1+\sin x} dx = -x - \frac{\cos x}{1+\sin x} + c_1$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & 1/1+\sin x \end{vmatrix}}{1} = \frac{\cos x}{1+\sin x}$$

$$u_2 = \int \frac{\cos x}{1+\sin x} = \ln|1+\sin x| + c_2$$

$$y = c_1 \cos x + c_2 \sin x + \sin x \ln(1+\sin x) - x\cos x - \frac{\cos^2 x}{1+\sin x}$$

9/ $y^{(5)} + y'' = x + 2e^x \sin x$ diferansiyel denkleminin genel çözümünü bulunuz.

Çözüm: $\lambda^5 + \lambda^2 = 0$

$\lambda^2(\lambda^3+1) = 0 \Rightarrow \lambda_{1,2} = 0, \quad \lambda_{3,4,5} = -1$

$y_h = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2) e^{-x}$

$UC_1 = \{x, 1\} \longrightarrow \{x^2, x\} \longrightarrow \{x^3, x^2\} \Rightarrow y_{p_1} = Ax^3 + Bx^2$

$y_{p_1}' = 3Ax^2 + 2Bx, \quad y_{p_1}'' = 6Ax + 2B, \quad y_{p_1}''' = 6A, \quad y_{p_1}^{iv} = 0, \quad y_{p_1}^{v} = 0.$

$6Ax + 2B = x \Rightarrow B = 0, \quad A = 1/6$

$y_{p_1} = 1/6\, x^3$

$UC_2 = \{e^x \sin x, e^x \cos x\} \Rightarrow y_{p_2} = De^x \sin x + F e^x \cos x$

$D = -2/5, \quad F = 1/5$

$\Rightarrow y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2) e^{-x} + 1/6\, x^3 + e^x\left(-2/5 \sin x + 1/5 \cos x\right)$

10/ $ty'' + (1-2t)y' + (t-1)y = te^t$ diferansiyel denkleminin lineer bağımsız homojen çözümleri $y_1 = e^t$, $y_2 = e^t \ln t$ olduğuna göre genel çözümünü bulunuz.

Çözüm;

$$y_h = c_1 e^t + c_2 e^t \ln t$$

$$y = u_1 e^t + u_2 e^t \ln t$$

$$\begin{cases} u_1' e^t + u_2' e^t \ln t = 0 \\ u_1' e^t + u_2' e^t (\ln t + 1/t) = \dfrac{te^t}{t} = e^t \end{cases}$$

$$u_1' = \frac{\begin{vmatrix} 0 & e^t \ln t \\ e^t & e^t(\ln t + 1/t) \end{vmatrix}}{\begin{vmatrix} e^t & e^t \ln t \\ e^t & e^t(\ln t + 1/t) \end{vmatrix}} = -\frac{\cancel{e^{2t}} \ln t}{\cancel{e^{2t}}(\cancel{\ln t} + 1/t - \cancel{\ln t})} = -\frac{\ln t}{1/t} = t \ln t$$

$$u_1 = -\int t \ln t \, dt = -\left(\frac{t^2}{2} \ln t - \frac{t^2}{4}\right) + c_1$$

$$u_2' = \frac{\begin{vmatrix} e^t & 0 \\ e^t & e^t \end{vmatrix}}{e^{2t}/t} = \frac{e^{2t}}{e^{2t}/t} = t$$

$$u_2 = \int t \, dt = \frac{t^2}{2} + c_2$$

$$y = c_1 e^t + e^t\left(-\frac{t^2}{2} \ln t + \frac{t^2}{4}\right) + c_2 e^t \ln t + \frac{t^2}{2} e^t \ln t$$